Problem 1) Matching with Massive Electrons

Consider QED with electrons and photons. For photon momenta $q^\mu$ much less than $m_e$ we can integrate out the electrons.

a) Calculate the one-loop photon vacuum polarization diagram with dimensional regularization and $\overline{\text{MS}}$, and expand $\Pi(q^2)$ in $q^2/m_e^2$.

[Note: The first term in the expansion motivates matching onto a theory without electrons at a scale $\mu \sim m_e$ rather than $\mu \sim 1 \text{ TeV}$ so that a large logarithm does not upset the perturbative expansion in $\alpha$.]

b) Write down a Lagrangian with a gauge invariant photon operator that reproduces the second term in the expansion. Use your calculation from part a) to determine the Wilson coefficient of the operator at this order in $\alpha$.

c) What QED symmetry(s) forbid dimension-6 operators with three field strengths from ever appearing?

d) At dimension-8, operators are generated which give light-by-light scattering. Determine the number of $\alpha$’s in their coefficients. Then use dimensional analysis in the low energy effective theory to numerically estimate the size of the $\gamma\gamma \to \gamma\gamma$ cross section for 10 keV photons.

Problem 2) Right Handed Neutrinos

Consider adding three right-handed singlet neutrinos $N_R$ to the standard model. A Majorana mass term is allowed, so

$$\mathcal{L}_N = \bar{N}_R i \partial \sigma N_R - \frac{1}{2} \bar{N}_R^c M N_R - \frac{1}{2} \bar{N}_R M^* N_R^c,$$

where $N_R^c = C N^T$ is the charge conjugate field, $C = i \gamma_2 \gamma_0$ (in the Dirac representation), and $M$ is a complex symmetric Majorana mass matrix.

a) Use gauge symmetry to determine the most general dimension-4 operators that couple $N_R$ to the other fields in the standard model.

b) Starting with Eq. (1) transform the $N_R$ fields to three Majorana mass eigenstates that satisfy $N_i = N_i^c$, $i = 1, 2, 3$ with real masses $M_i$. For the diagonalization of the Majorana mass matrix feel free to simply quote the relevant linear algebra theorem.
c) Count the total number of physical parameters in $M$ and the coefficients of the operators in part a).

[Hint: Consider the $G = U(3) \times U(3) \times U(3)$ flavor symmetry of the free $L_L, e_R,$ and $N_R$ kinetic terms. This symmetry is broken by the mass and Yukawa matrices, so the number of physical parameters can be obtained by subtracting the number of parameters in $G$ from the number in the original Yukawa and Majorana matrices. For the ambitious, repeat the counting for $n$ families of light leptons and $n'$ right-handed neutrinos. How many of the parameters are CP-odd phases? The case $n = 3, n' = 2$ should agree with the 14 parameters mentioned in class. In this case 3 parameters are CP-odd phases.]

d) Take the masses $M_i$ large compared to the electroweak scale and integrate out the right handed neutrinos at tree level. Show that the leading term reduces to the form of the dimension-5 standard model operator we discussed in class.

Problem 3) X-Decay

You observe a very heavy ($m > 1$ TeV) particle $X$ of unknown origin which decays to well known light hadrons and/or leptons. You observe it decaying to two light particles $X \to Y_1 Y_2$ and to three $X \to Y_1 Y_2 Y_3$. Assume you have a model where the couplings for these two transitions are the same size. What can you estimate for the size of the ratio of decay rates $R = \Gamma(X \to Y_1 Y_2 Y_3)/\Gamma(X \to Y_1 Y_2)$?