Problem Set 2  
Post date: Thursday, February 16th  
Due date: Thursday, February 23rd

1. Show that the number density of dust measured by an observer whose 4-velocity is $\vec{U}$ is given by $n = -\vec{N} \cdot \vec{U}$, where $\vec{N}$ is the matter current 4-vector.

2. Take the limit of the continuity equation for $|v| \ll 1$ to get $\partial n/\partial t + \partial (nv^i)/\partial x^i = 0$.

3. In an inertial frame $\mathcal{O}$, calculate the components of the stress-energy tensors of the following systems:
   (a) A group of particles all moving with the same 3-velocity $v = \beta \vec{e}_x$ as seen in $\mathcal{O}$. Let the rest-mass density of these particles be $\rho_0$, as measured in their own rest frame. Assume a sufficiently high density of particles to enable treating them as a continuum.
   (b) A ring of $N$ similar particles of rest mass $m$ rotating counter-clockwise in the $x - y$ plane about the origin of $\mathcal{O}$, at a radius $a$ from this point, with an angular velocity $\omega$. The ring is a torus of circular cross-section $\delta a \ll a$, within which the particles are uniformly distributed with a high enough density for the continuum approximation to apply. Do not include the stress-energy of whatever forces keep them in orbit. Part of this calculation will relate $\rho_0$ of part (a) to $N$, $a$, $\delta a$, and $\omega$.
   (c) Two such rings of particles, one rotating clockwise and the other counter-clockwise, at the same radius $a$. The particles do not collide or otherwise interact in any way.

4. Use the identity $\partial_{\nu} T^{\mu\nu} = 0$ to prove the following results for a bounded system (i.e., a system for which $T^{\mu\nu} = 0$ beyond some bounded region of space):
   (a) $\partial_i \int T^{00} d^3x = 0$. This expresses conservation of energy and momentum.
   (b) $\partial_i \int T^{00} x^i x^j d^3x = 2 \int T^{ij} d^3x$. This result is a version of the virial theorem; it will come in quite handy when we derive the quadrupole formula for gravitational radiation.
   (c) $\partial_i \int T^{00} (x^i x_i)^2 d^3x = 4 \int T^{ij} x^i x^j d^3x + 8 \int T^{ij} x_i x_j d^3x$. No pithy wisdom for this one.
5. The vector potential $\mathbf{A} \equiv (A^0, \mathbf{A})$ generates the electromagnetic field tensor via

$$F_{\mu \nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}.$$ 

(a) Show that the electric and magnetic fields in a specific Lorentz frame are given by

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla A^0,$$
$$\mathbf{B} = \nabla \times \mathbf{A}.$$ 

Here, $\nabla$ is taken to be the normal gradient operator in Euclidean space.

(b) Show that Maxwell’s equations hold if and only if

$$\partial_{\mu} \partial^{\mu} A^{\alpha} - \partial^{\alpha} \partial_{\mu} A^{\mu} = -4\pi J^{\alpha}.$$ 

(c) Show that a gauge transformation of the form

$$A_{\mu}^{\text{new}} = A_{\mu}^{\text{old}} + \partial_{\mu} \phi$$

leaves the field tensor unchanged.

(d) Show that one can adjust the gauge so that

$$\partial_{\mu} A^{\mu} = 0.$$ 

Show that Maxwell’s equations take on a particularly simple form with this gauge choice. Use the operator $\square \equiv \partial_{\mu} \partial^{\mu}$ to simplify your result.
6. An astronaut has acceleration \( g \) in the \( x \) direction (in other words, the magnitude of his 4-acceleration, \( \sqrt{\mathbf{a} \cdot \mathbf{a}} \), is \( g \)). This astronaut assigns coordinates \((\bar{t}, \bar{x}, \bar{y}, \bar{z})\) to spacetime as follows:

First, the astronaut defines spatial coordinates to be \((\bar{x}, \bar{y}, \bar{z})\), and sets the time coordinate \( \bar{t} \) to be his own proper time.

Second, at \( \bar{t} = 0 \), the astronaut assigns \((\bar{x}, \bar{y}, \bar{z})\) to coincide with the Euclidean coordinates \((x, y, z)\) of the inertial reference frame that momentarily coincides with his motion. (In other words, though the astronaut is not inertial — he is accelerating — there is an inertial frame that, at \( \bar{t} = 0 \), is momentarily at rest with respect to him. This is the frame used to assign \((\bar{x}, \bar{y}, \bar{z})\) at \( \bar{t} = 0 \).) Observers who remain at fixed values of the spatial coordinates \((\bar{x}, \bar{y}, \bar{z})\) are called coordinate-stationary observers (CSOs). Note that proper time for these observers is not necessarily \( \bar{t} \) — we cannot assume that the CSOs’ clocks remain synchronized with the clocks of the astronaut.

Assume that some function \( A \) converts between coordinate time \( \bar{t} \) and proper time at the location of a CSO:

\[
A = \frac{d\bar{t}}{d\tau}
\]

The function \( A \) is evaluated at a CSO’s location and thus can in principle depend on all four coordinates \( \bar{t}, \bar{x}, \bar{y}, \bar{z} \).

Finally, the astronaut requires that the worldlines of CSOs must be orthogonal to the hypersurfaces \( \bar{t} = \text{constant} \), and that for each \( \bar{t} \) there exists an inertial frame, momentarily at rest with respect to the astronaut, in which all events with \( \bar{t} = \text{constant} \) are simultaneous.

It is easy to see that \( \bar{y} = y \) and \( \bar{z} = z \); henceforth we drop this coordinates from the problem.

(a) What is the 4-velocity of the astronaut, as a function of \( \bar{t} \), in the initial inertial frame [the frame that uses coordinates \((t, x, y, z)\)]? (Hint: by considering the conditions on \( \mathbf{u} \cdot \mathbf{u}, \mathbf{u} \cdot \mathbf{a}, \) and \( \mathbf{a} \cdot \mathbf{a} \), you should be able to find simple forms for \( u^t \) and \( u^x \).)

(b) Imagine that each coordinate-stationary observer carries a clock. What is the 4-velocity of each clock in the initial inertial frame?

(c) Explain why \( A(\bar{x}, \bar{t}) \) cannot depend on time. In other words, why can we put \( A(\bar{x}, \bar{t}) = A(\bar{x}) \)? (Hint: consider the coordinate system that a different CSO may set up.)

(d) Find an explicit solution for the coordinate transformation \( x(\bar{t}, \bar{x}) \) and \( t(\bar{t}, \bar{x}) \).

(e) Show that the line element \( ds^2 = d\bar{x} \cdot d\bar{x} \) in the new coordinates takes the form

\[
 ds^2 = -dt^2 + dx^2 = -(1 + g\bar{x})^2 d\bar{t}^2 + d\bar{x}^2 .
\]