1. Connection in Rindler spacetime

The spacetime for an accelerated observer that we derived on Pset 2,

\[ ds^2 = -(1 + g\bar{x})^2 d\bar{t}^2 + d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2 \]  

is known as “Rindler spacetime”. Compute all non-zero Christoffel symbols for this spacetime. (Carroll problem 3.3 will help you quite a bit here.)

2. Relativistic Euler equation

(a) Starting from the stress-energy tensor for a perfect fluid, \( T = \rho \bar{U} \otimes \bar{U} + P \bar{h} \), where \( \bar{h} = g^{-1} + \bar{U} \otimes \bar{U} \), using local energy momentum conservation, \( \nabla \cdot T = 0 \), derive the relativistic Euler equation,

\[ (\rho + P) \nabla \cdot \bar{U} = -\bar{h} \cdot \nabla P \, . \]  

(Note: Because both \( T \) and \( h \) are symmetric tensors, there is no ambiguity in the dot products that appear in this problem.)

(b) For a nonrelativistic fluid \( (\rho \gg P, v^t \gg v^i) \) and a cartesian basis, show that this equation reduces to the Euler equation,

\[ \frac{\partial v_i}{\partial t} + v_k \partial_k v_i = -\frac{1}{\rho} \partial_i P \, . \]  

(\( i, k \) are spatial indices running from 1 to 3.) What extra terms are present if the connection is non-zero (e.g., spherical coordinates)?

(c) Apply the relativistic Euler equation to Rindler spacetime for hydrostatic equilibrium. Hydrostatic equilibrium means that the fluid is at rest in the \( \bar{x} \) coordinates, i.e. \( U^\bar{x} = 0 \). Suppose that the equation of state (relation between pressure and density) is \( P = w\rho \) where \( w \) is a positive constant. Find the general solution \( \rho(\bar{x}) \) with \( \rho(0) = \rho_0 \).

(d) Suppose now instead that \( w = w_0/(1 + g\bar{x}) \) where \( w_0 \) is a constant. Show that the solution is \( \rho(\bar{x}) = \rho_0 \exp(-\bar{x}/L) \). Find \( L \), the density scale height, in terms of \( g \) and \( w_0 \). Convert to “normal” units by inserting appropriate factors of \( c \) — \( L \) should be a length.

(e) Compare your solution to the density profile of a nonrelativistic, plane-parallel, isothermal atmosphere (for which \( P = \rho kT/\mu \), where \( T \) is temperature and \( \mu \) is the mean molecular weight) in a constant gravitational field. [Use the nonrelativistic Euler equation with gravity: add a term \(-\partial_i \Phi = g_i \), where \( \Phi \) is Newtonian gravitational potential and \( g_i \) is Newtonian gravitational acceleration, to the right hand side of Eq. (3).] Why does hydrostatic equilibrium in Rindler spacetime — where there is no gravity — give such similar results to hydrostatic equilibrium in a gravitational field?
3. Spherical hydrostatic equilibrium

As we shall derive later in the course, the line element for a spherically symmetric static spacetime may be written

\[ ds^2 = -e^{2\Phi(r)}dt^2 + \left[1 - \frac{2GM(r)}{r}\right]^{-1}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) , \]

where \( \Phi(r) \) and \( M(r) \) are some given functions. In hydrostatic equilibrium, \( U^i = 0 \) for \( i \in [r, \theta, \phi] \). Using the relativistic Euler equation, show that in hydrostatic equilibrium \( p = p(r) \) with

\[ \frac{\partial p}{\partial r} = -(\rho + P) \frac{\partial \Phi}{\partial r} . \]

4. Converting from non-affine to affine parameterization

Suppose \( v^\alpha = dx^\alpha/d\lambda^* \) obeys the geodesic equation in the form

\[ \frac{Dv^\alpha}{d\lambda^*} = \kappa(\lambda^*)v^\alpha . \]

Clearly \( \lambda^* \) is not an affine parameter.

Show that \( u^\alpha = dx^\alpha/d\lambda \) obeys the geodesic equation in the form

\[ \frac{Du^\alpha}{d\lambda} = 0 \]

provided that

\[ \frac{d\lambda}{d\lambda^*} = \exp \left[ \int \kappa(\lambda^*) d\lambda^* \right] . \]

5. Conserved quantities with charge

A particle with electric charge \( e \) moves with 4-velocity \( u^\alpha \) in a spacetime with metric \( g_{\alpha\beta} \) in the presence of a vector potential \( A_\mu \). The equation describing this particle’s motion can be written

\[ u^\beta \nabla_\beta u_\alpha = eF_{\alpha\beta}u^\beta , \]

where

\[ F_{\alpha\beta} = \nabla_\alpha A_\beta - \nabla_\beta A_\alpha . \]

The spacetime admits a Killing vector field \( \xi^\alpha \) such that

\[ \mathcal{L}_\xi g_{\alpha\beta} = 0 , \]

\[ \mathcal{L}_\xi A_\alpha = 0 . \]

Show that the quantity \( (u_\alpha + eA_\alpha)\xi^\alpha \) is constant along the worldline of the particle.