1 Gibbons 2.4

Let us find the best-response for player 2.

If $c_1 \geq R \implies U_2 = V, \forall c_2$
If $c_2 < 0 \implies U_2 = V - c_2^2, \forall c_2 \geq R - c_1$ and $U_2 = 0, \forall c_2 < R - c_1$.

From this, we have

$$BR_2(c_1) = \begin{cases} 
0 & \text{if } c_1 \geq R \text{ or } c_1 < R - \sqrt{V} \\
R - c_1 & \text{if } R - \sqrt{V} < c_1 < R \\
\epsilon\{0, \sqrt{V}\} & \text{if } c_1 = R - \sqrt{V}
\end{cases}$$

Anticipating this response from player 2, player 1 conjectures that his payoff is as follows.

If $c_1 \geq R \implies U_1 = V - c_1^2$
If $R - \sqrt{V} < c_1 < R \implies U_1 = \delta V - c_1^2$

We need to consider different cases here. If $R - \sqrt{V} < 0$, then we have characterised all possible payoffs for $c_1 \geq 0$.

If $R - \sqrt{V} = 0$, then we have that if $c_1 = 0$, $U_1 \epsilon\{\delta V, 0\}$ depending on the decision of player 2 to invest or not.

If $R - \sqrt{V} > 0$, then for $c_1 \epsilon[0, R - \sqrt{V})$, $U_1 = -c_1^2$ and for $c_1 = R - \sqrt{V}$, $U_1 \epsilon\{\delta V - c_1^2, 0\}$ depending on the decision of player 2 to invest or not.

From these observations, we can derive the Nash Equilibrium outcomes (I stress outcomes; I’ll only specify an outcome for player 2; a Nash Equilibrium strategy would write the full best-response correspondence for player 2 as written above).
1.1 $R - \sqrt{V} < 0$

Here, player 1 is choosing between $c_1 = 0$ and $c_1 = R$, with $U_1(0, BR_2(0)) = \delta V$; $U_1(R, BR_2(R)) = V - R^2$

Let us write $c_i^{NEO}$ for the outcome of player i’s choice in a Nash Equilibrium. So, we have $(c_1^{NEO}, c_2^{NEO}) = $

| (0, R) if $R > [(1 - \delta)V]^{1/2}$ |
| (R, 0) if $R < [(1 - \delta)V]^{1/2}$ |
| $(0, R), (R, 0)$ if $R = [(1 - \delta)V]^{1/2}$ |

1.2 $R - \sqrt{V} = 0$

Here, we add the possibility that player 1 plays $R - \sqrt{V}$, where his payoffs depend on player 2’s strategy. Player 2 is indifferent between $c_2 = \sqrt{V}$ and $c_2 = 0$. Let us assume that player 1 is playing the former strategy with probability $p$ and the latter with probability $1 - p$. Then it is easy to see that there is no Nash Equilibrium where $p \neq 1$. Why? Because then player 1 has no best-response. $U_1(R, BR_2(R)) = V - R^2 = 0$, $U_1(\varepsilon, BR_2(\varepsilon)) = \delta V - \varepsilon^2$ and $U_1(0, p*\sqrt{V} + (1 - p)*0) = p\delta V$.

Then, if $p \neq 1$, $U_1(\varepsilon, BR_2(\varepsilon)) > U_1((0, p*\sqrt{V} + (1 - p)*0) \Leftrightarrow \varepsilon < \sqrt{(1 - p)\delta V}$, which can always be satisfied for $\varepsilon$ sufficiently small. But, there is no unique strategy $\varepsilon > 0$ that maximises $U_1(\varepsilon, BR_2(\varepsilon))$. So, we will assume that $p = 1$. Then we have $(c_1^{NEO}, c_2^{NEO}) = (0, R)$

1.3 $R - \sqrt{V} > 0$

In addition to the previous case, we add the possibility that player 1 plays $c_1 < R - \sqrt{V}$, in which case $U_1 = -c_1$. Of course, we need only to retain the value $c_1 = 0$ within that interval. Yet again it is clear that we cannot have a Nash Equilibrium where player 2 is playing $c_2 = 0$ with some probability when $c_1 = R - \sqrt{V}$. So, we have $(c_1^{NEO}, c_2^{NEO}) = $

| (0, 0) if $R > [(1 + \delta)\sqrt{V}$ |
| $(R - \sqrt{V}, \sqrt{V})$ if $R < [(1 + \delta)\sqrt{V}$ |
| $(0, 0), (R - \sqrt{V}, \sqrt{V})$ if $R = [(1 + \delta)\sqrt{V}$ |

1 The choices for 1 boil down to the following possible strategies, with the corresponding payoffs: $U_1(R, BR_2(R)) = V - R^2 < 0$, $U_1(0, BR_2(0)) = 0$ and $U_1(\gamma(R - \sqrt{V}), BR_2(\gamma(R - \sqrt{V}))) = \delta V - (\gamma(R - \sqrt{V}))^2$ where $1 < \gamma < \frac{R}{R - \sqrt{V}}$; $U_1((R - \sqrt{V}), p*\sqrt{V} + (1 - p)*0) = p\delta V - (R - \sqrt{V})^2$. If $p \neq 1$, $U_1(\gamma(R - \sqrt{V}), BR_2(\gamma(R - \sqrt{V}))) = U_1((R - \sqrt{V}), p*\sqrt{V} + (1 - p)*0) \Leftrightarrow (\gamma^2 - 1)/(R - \sqrt{V})^2 < (1 - p)\delta V$ is satisfied for $\gamma$ close enough to 1. But again, there is no unique $\gamma$ that maximises $U_1(\gamma(R - \sqrt{V}), BR_2(\gamma(R - \sqrt{V})))$. (Note: I guess this depends on $\delta$ being large enough- you can always set $c_1 = 0$ and get 0- but I will ignore this at this point).