1 Gibbons 2.3 (p.131)

Let us consider the three-period game first.

1.1 The Three-Period Game

The structure of the game as was described in section 2.1D (pp.68-71). Let us solve the game backwards.

1.1.1 Stage (2b)

Player 1 accepts player 2’s proposal \((s_2, 1 - s_2)\) if and only if the following condition is satisfied:

\[ s_2 \geq \delta_1 s \]

1.1.2 Stage (2a)

Conditional on player 1 accepting the offer, player 2 maximises his/her payoff by offering \(s_2 = \delta_1 s\). Then, player 2 gets \(\delta_2(1 - s_2) = \delta_2(1 - \delta_1 s)\).

Any rejected offer leads 2 to get a payoff of \(\delta_2^2(1 - s)\). Player 2 is better off with a proposal that is accepted if and only if:

\[
\begin{align*}
  d_2 &= \delta_2(1 - \delta_1 s) - \delta_2^2(1 - s) \\
  &= \delta_2[1 - \delta_2 + s(\delta_2 - \delta_1)] \\
  &\geq 0
\end{align*}
\]

If \(\delta_2 \geq \delta_1\), we have \(d_2 \geq \delta_2[1 - \delta_2] > 0\) since \(0 < \delta_2 < 1\).

If \(\delta_2 < \delta_1\), we have \(d_2 \geq \delta_2[1 - \delta_1] > 0\) since \(0 < \delta_1, \delta_2 < 1\).

Either way, we have that player 2 prefers to have his/her proposal accepted, and offers \(s_2 = \delta_1 s\)
1.1.3 Stage (1b)

Player 2 accepts player 1’s proposal \((s_1, 1 - s_1)\) if and only if:

\[
1 - s_1 \geq \delta_2 (1 - s_2) \\
1 - s_1 \leq 1 - \delta_2 [1 - \delta_1 s]
\]

1.1.4 Stage (1a)

Conditional on player 2 accepting the offer, player 1 maximises his/her payoff by offering \(s_1 = 1 - \delta_2 [1 - \delta_1 s]\).

Any rejected offer leads 1 to get a payoff of \(\delta_1 s_2 = \delta_2^2 s\). Player 1 is better off with a proposal that is accepted if and only if:

\[
d_1 = 1 - \delta_2 [1 - \delta_1 s] - \delta_2^2 s \\
= 1 - \delta_2 + s_1 \delta_2 - \delta_1 \delta_2 \geq 0
\]

If \(\delta_2 \geq \delta_1\), we have \(d_1 \geq 1 - \delta_2 > 0\) since \(0 < \delta_2 < 1\).

If \(\delta_2 < \delta_1\), we have \(d \geq (1 - \delta_2 + \delta_1)(1 - \delta_1) > 0\) since \(0 < \delta_1, \delta_2 < 1\).

Either way, we have that player 1 prefers to have his/her proposal accepted, and offers \(s_1 = 1 - \delta_2 [1 - \delta_1 s]\)

The outcome of the game is that players 1 and 2 agree on the distribution \((s^*_1, 1 - s^*_1) = (1 - \delta_2 [1 - \delta_1 s], \delta_2 [1 - \delta_1 s])\).

1.2 The Infinite-Horizon Game

Let \(s\) be a payoff that player 1 can get in a backwards-induction of the game as a whole, and \(s_H\) the maximum value of \(s\). Imagine using \(s\) as the third-payoff to Player 1. Player 1’s first-period payoff is a function of \(s\), namely \(f(s) = 1 - \delta_2 [1 - \delta_1 s]\). Since this function is increasing in \(s\), \(f(s_H)\) is the highest possible first-period payoff, so \(f(s_H) = s_H\). Then

\[
1 - \delta_2 [1 - \delta_1 s_H] = s_H \\
\iff s_H = \frac{1 - \delta_2}{1 - \delta_1 \delta_2}
\]

A parallel argument shows that \(f(s_L) = s_L\), where \(f(s_L)\) is the lowest payoff that player 1 can achieve in any backwards-induction of the game as a whole. Therefore, the only value of \(s\) that satisfies \(f(s) = s\) is \(\frac{1 - \delta_2}{1 - \delta_1 \delta_2}\). Thus \(s_H = s_L = s^*\), so there is a unique backwards-induction outcome of the game as a whole: A distribution

\[
(s^*, 1 - s^*) = \left\{ \frac{1 - \delta_2}{1 - \delta_1 \delta_2}, \frac{\delta_2 (1 - \delta_1)}{1 - \delta_1 \delta_2} \right\}
\]