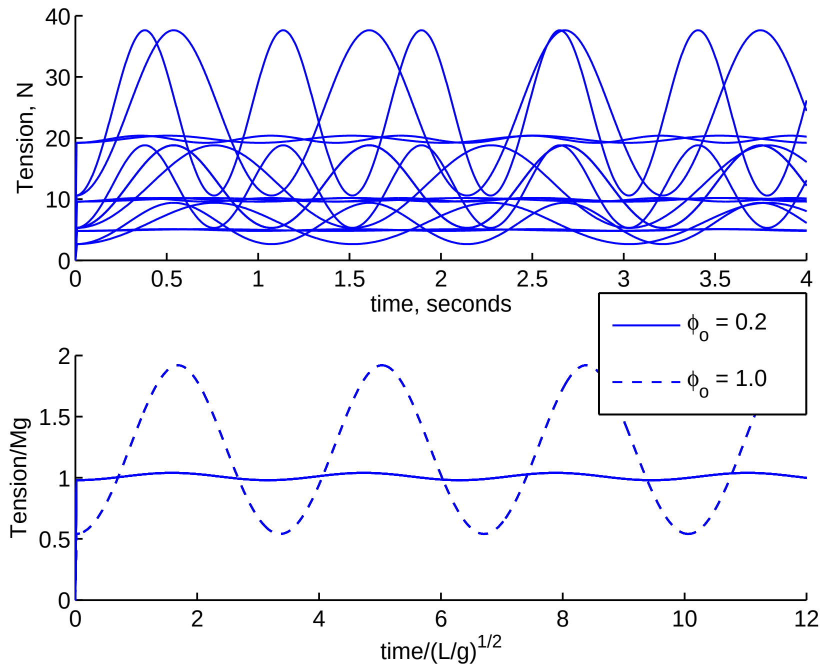
Dimensional Analysis of Models and Data Sets: Similarity Solutions and Scaling Analysis



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Summary: This essay describes a three-step procedure of dimensional analysis that can be applied to all quantitative models and data sets. In rare but important cases the result of dimensional analysis will be a solution; more often the result is an efficient way to display a large or complex data set.

The first step of an analysis is to define an appropriate physical model, which is nothing more than a list of the dependent variable and all of the independent variables and parameters that are thought to be significant. The premise of dimensional analysis is that a complete equation made from this list of variables will be independent of the choice of units. This leads to the second step, calculation of a null space basis of the corresponding dimensional matrix (a computer code is made available for this calculation). To each vector

of the null space basis there corresponds a nondimensional variable, the number of which is less than the number of dimensional variables. The nondimensional variables are themselves a basis set and in most cases their form is not determined by dimensional analysis alone.

The third and in some respects most interesting step is to choose an optimal form for the basis set. One very useful strategy is to nondimensionalize the dependent variable by a physically motivated 'zero order' solution. When carried to completion, this leads naturally to a scaling analysis in which the nondimensional variables of a model equation are O(1) in some relevant limit. Scaling analysis is often an essential first step in an approximation method. The remaining nondimensional variables can then be formed in ways that define the geometry of the problem or that correspond to the ratios of terms in a model equation, e.g., the Reynolds number or Froude number often arise in models of geophysical fluid dynamics.

Cover page graphic: The tension in the line of a simple pendulum computed by a model for 16 different combinations of length, mass, gravitational acceleration and initial displacement, the angle, ϕ_o. The upper graph shows the tension as a function of time in dimensional units, and the lower graph is the same data shown in non-dimensional units. The 16 separate solutions collapse to just two, depending upon ϕ_o and time alone. This kind of data compression illustrates an important advantage to the use of nondimensional variables and dimensional analysis that is described further in Section 4.3.

Contents

1 About dimensional analysis

Dimensional analysis is a remarkable tool in so far as it can be applied to any and every quantitative model or data set; recent applications include topics from donuts to dinosaurs and the most fundamental theories of physics.[1,2] The results of dimensional analysis can be of greater or lesser value. It is most useful, indeed almost indispensable, for problems having no solvable theory. Dimensional analysis can always make a little progress towards a solution, and some of these, the universal spectrum of inertial-range turbulence and the log-layer profile of a turbulent boundary layer, are landmarks in fluid mechanics. More often the result of dimensional analysis is a broad hint at the form of a solution or a more efficient way to display or correlate a large data set. These kinds of results, though seldom complete if taken alone, are nevertheless an essential building block of many investigations.

1.1 The goal and the plan

The goal of this essay is to demonstrate some of the advantages of dimensional analysis and to present a systematic and partially automated method of dimensional analysis. The plan is to demonstrate dimensional analysis on several familiar problems from classical physics; the simple pendulum in Sections 2-4, the simple viscous pendulum in Section 5, diffusion in one dimension, Section 6, and projectile motion in variable gravity, Section 7. Dimensional analysis has all the makings of a full mathematical analysis, though in a very compressed format. The first and most important step is to define a problem having one dependent variable. The physical model for this problem is represented by nothing more than a list of all of the independent variables and parameters that are thought to be important to determining the outcome of the dependent variable (Section 2). That something useful could follow from such a minimal specification is at the heart of what makes dimensional analysis so widely applicable and also a bit mysterious. Physical models as first written are likely to be rather general. Anything that can be done to hone the physical model or add physical constraints is likely to make the subsequent analysis much more useful. The premise of dimensional analysis is that complete equations can be written in a form that is independent of the choice of units. The consequence is that the variables that appear in a complete equation must appear in combinations that are nondimensional. The second and mathematical step is to find the nondimensional form that the variables must take (Sections 3 and 4). The usual method of finding the nondimensional variables relies upon the Buckingham Pi theorem to define the number of required nondimensional variables, followed by trial and error.[3] Here the nondimensional variables are computed using a method from linear algebra that, while not necessarily

[1] Footnotes provide references, extensions or qualifications of material discussed in the main text, along with a few homework assignments. They may be skipped on first reading.

[2] E. Thurairajasingam, E. Shayan, and S. Masood, "Modelling of a continuous food pressing process by dimensional analysis," Computer and Industrial Engineering, in press; J. R. Hutchinson and M. Garcia, "Tyrannosaurus was not a fast runner," Nature **415**, 1018–1022 (2002); F. Wilczek, "Getting its from bits," Nature **397**, 303–306 (1999).

[3] An introduction to dimensional analysis can be found in most comprehensive fluid mechanics textbooks. Recent examples include P. K. Kundu and I. C. Cohen, *Fluid Mechanics* (Academic Press, 2001), B. R. Munson, D. F. Young, and T. H. Okiishi, *Fundamentals of Fluid Mechanics* (John Wiley and Sons, NY, 1998), 3rd ed., D. C. Wilcox, *Basic Fluid Mechanics* (DCW Industries, La Canada,

simpler to derive (Sections 4.1 and 4.2), is nevertheless readily automated[4] and so is readily applied (Sections 4.3 and later). The third and final step of a dimensional analysis is to reassemble the initial basis set of nondimensional variables into an optimum form (examples in Sections 5 and 7). This requires some sense of the intent and possible uses for the analysis. When combined with a zero-order solution for the dependent variable, dimensional analysis develops naturally into a scaling analysis (Section 7). This essay will emphasize the interpretive aspects of a dimensional analysis — specification of an appropriate physical model and the choice of the basis set — once the purely mathematical second step has been set aside in Section 4.

1.2 About this essay

This essay is an introduction to dimensional analysis at about the level appropriate for a first course in fluid dynamics. It is intended to supplement the discussions of dimensional analysis that can be found in most comprehensive fluid dynamics and applied mathematics text books.[3] The method of dimensional analysis described in Section 4 is believed to be somewhat novel, but it is impossible to know all of the literature on a topic that has roots more than a century deep in many, many different fields. References are cited where they are known, but it is emphasized that this essay is intended to be educational, rather than a report research findings. This essay may be used for any personal, educational purpose and may be freely copied. Comments and questions are welcomed.[5]

2 Models of a simple pendulum

The first problem we take up in some depth is that of a simple pendulum.[6] Consider a pendulum that can be made and observed usefully with very inexpensive tools and materials; a small lead fishing sinker having a mass of a few tens of grams suspended on a thin monofilament line a few meters in length. The motion of such a pendulum is only lightly damped by drag with the surrounding air and can be characterized by two distinct time scales – a regular, fast time-scale oscillation having a period, P, and a slow, more-or-less exponential decay with a time-scale, Γ^{-1}. Our specific goal in Sections 2-5 will be to learn how these time scales and some other variables, e.g., the tension in the line, depend upon the line length, the mass of the bob,

CA, 2000), and F. M. White, *Fluid Mechanics* (McGraw-Hill, NY, 1994), 3rd ed. An older but very useful reference is by H. Rouse, *Elementary Mechanics of Fluids* (Dover Publications, NY, 1946). A particularly good discussion of the relationship between dimensional analysis and other analysis methods is by C. C. Lin and L. A. Segel, *Mathematics Applied to Deterministic Problems in the Natural Sciences* (MacMillan Pub., 1974).

[4]An algorithm for computing nondimensional variables using this method has been implemented in Matlab and can be downloaded from the author's web page, <http://www.whoi.edu/science/PO/people/jprice/misc/Danalysis.m>or from the Matlab File Central archive (the file name is Danalysis.m).

[5]A condensed version of this essay, roughly Sections 2-5 and 8, has been published, J. F. Price, 'Dimensional analysis of models and data sets', *Am. J. Phys.*, **71**(5), 437-447.

[6]The simple pendulum is the starting point for most discussion of dimensional analysis including the classic text by P. W. Bridgman, *Dimensional Analysis* (Yale Univ. Press, New Haven, CT, 1937), 2nd ed., which is an excellent introduction to the topic, and the more advanced treatment by L. I. Sedov, *Similarity and Dimensional Methods in Mechanics* (Academic Press, NY, 1959). Still more advanced is G. I. Barenblatt, *Scaling, Self-Similarity and Intermediate Asymptotics* Cambridge Univ. Press, Cambridge, 1996).

etc. If the simple pendulum is already quite familiar to you, then skip ahead to Section 3; if the use of nondimensional variables is also familiar, skip ahead to Section 4.

2.1 A physical model

To analyze the motion of this pendulum we begin by listing the variables that are presumed to be relevant to the aspect of the motion that is of interest. To start, consider the fast time-scale, oscillatory motion. The line will be idealized as rigid, so that the bob must swing along a constant radius. The motion of the bob is then defined by the angle of the line to the vertical, $\phi(t)$, and its time derivatives; the angle ϕ is the dependent variable of this physical model and the time, t, is the only independent variable. Several properties of the pendulum would seem to be of importance — the mass of the bob, M, the length of the supporting line, L, and the acceleration of gravity, g. To account for why there is motion at all, the initial angle, ϕ_0, or an initial angular velocity must also be included; for later comparison to experimental data it is preferable to take the initial angular velocity to be zero. This list of relevant variables constitutes

- A physical model for the oscillatory motion of a simple, inviscid pendulum:

1. the angle of the line, $\phi \doteq$ nond, the dependent variable,
2. time, $t \doteq m^0 l^0 t^1$, the only independent variable,
3. mass of the bob, $M \doteq m^1 l^0 t^0$, a parameter,
4. length of the line, $L \doteq m^0 l^1 t^0$, a parameter,
5. acceleration of gravity, $g \doteq m^0 l^1 t^{-2}$, a parameter,
6. the initial angle, $\phi_0 \doteq$ nond, a parameter.

The notation $X \doteq m^a l^b t^c$ indicates the dimensions mass, length and time (or nond if the variable is nondimensional). Parameters are variables that are constant during a particular realization — M, L, g and ϕ_0 in this list — but that vary over some range that defines the family of pendulums and environments that are of interest.

2.2 A mathematical model

Dimensional analysis is most useful in the case that a mathematical model is not known. Mathematical models of the simple pendulum are well-known and we will use them to generate numerical data and to show how dimensional analysis can be applied to a mathematical model. For an inviscid pendulum the rate of change of angular momentum of the bob is due solely to the torque associated with the downward force of gravity acting on the bob,

$$L^2 M \frac{d^2\phi}{dt^2} = -LMg \sin\phi. \tag{1}$$

If we divide by $L^2 M$, the equation of motion becomes

$$\frac{d^2\phi}{dt^2} = -\frac{g}{L} \sin\phi. \tag{2}$$

For experimental purposes it is preferable to start from a state of rest and so the initial conditions at $t = 0$ are taken to be

$$\phi = \phi_0 \quad \text{and} \quad \frac{d\phi}{dt} = 0. \tag{3}$$

It may also be of interest to compute the tension in the line, T, from the radial equation of motion, $dr/dt = 0$, and thus

$$T = gM \cos\phi + LM(\frac{d\phi}{dt})^2. \tag{4}$$

The appropriate solution method to Eqs. (2) and (3) depends upon the initial angle, ϕ_0. If ϕ_0 is restricted to values less than about 0.1 radian, then sin ϕ in Eq. (2) can be approximated well by ϕ and the resulting linear model has the well-known solution

$$\phi = \phi_0 \cos(\frac{t}{\sqrt{L/g}}) . \tag{5}$$

In the general case where ϕ_0 may take any value from $-\pi$ to π, Eq. (2) is nonlinear and a solution cannot be given in elementary functions. Numerical integration of the nonlinear model Eqs. (2) and (3) is straightforward, however, and yields numerical data (Fig. 1) that we will treat as an intermediary between experiment and theory; we know exactly the physical model that underlies numerical solutions, assuming that the numerical errors are negligible, but we do not know not the parameter dependence of the model.

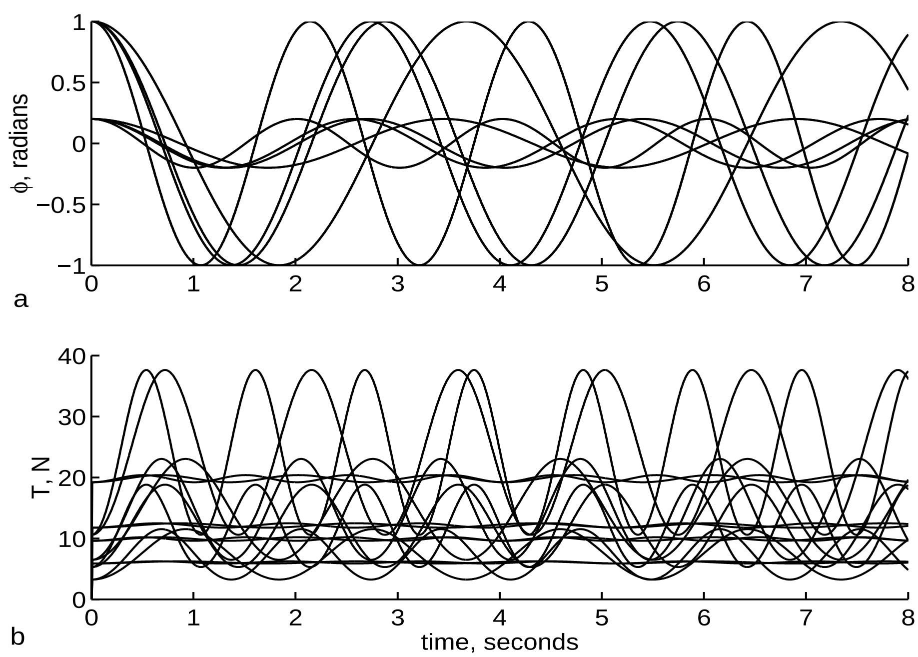

Figure 1: Numerical solutions of the simple, inviscid pendulum for two values each of L $(1, 1.8)$ m, M $(1, 2)$ kg, g $(9.8, 6)$ m^2/s and ϕ_0 $(0.2, 1.0)$ radians or 16 solutions in total. (a) The angle, ϕ. The mathematical model Eqs. (2) and (3) does not depend upon M, and so there are eight distinct solutions here. (b) Tension (Newtons) for the same set of solutions. Here there are 16 distinct solutions, though some are difficult to distinguish. As these data were acquired it was noticed that the maximum tension did not vary with L.

2.3 Models generally

Model equations are a relation between a dependent variable, the angle ϕ or the tension T, and the independent variables and parameters that make up the physical model. Even if we had no idea of the mathematical model, we could still assert that a complete physical model could be used to define a relation

$$\phi = F(t, g, L, M, \phi_0) , \tag{6}$$

where F will be used to indicate an unknown function. If our goal was to solve for the period of the oscillation, then we would evaluate the time at some (arbitrary) repeated value of ϕ to find

$$P = F(g, L, M, \phi_0) . \tag{7}$$

For the tension, T, and the maximum tension during an oscillation, T_{max}, we could similarly write

$$T = F(t, g, L, M, \phi_0) , \tag{8}$$

and

$$T_{\mathrm{max}} = F(g, L, M, \phi_0) . \tag{9}$$

It will often happen that the list of variables for the physical model will include one or more parameters that do not appear in the mathematical model. If we compare Eqs. (6) and (2), the physical model includes the mass, M, while in the mathematical model, the mass appeared as a coefficient in the gravitational force (right side of Eq. 1) and in the inertial force (left side of Eq. 1) and cancels. In this regard, the mathematical model, Eq. (2), is a considerable advance over the physical model, Eq. (6). Note too that the angular velocity, $d\phi/dt$, appears in the mathematical model for the tension, Eq. (4), although not in the physical model Eq. (8). Even if we were aware that the mathematical model of tension depended upon $d\phi/dt$, we should still omit this second dependent variable from Eq. (8) because $d\phi/dt$ must itself depend upon t, g, L, M and ϕ_0 and should not be written into the physical model again.

The relations (6)–(9) could be written in one of several forms, for example,

$$\phi/F(t, g, L, M, \phi_0) = 1, \tag{10}$$

or reusing F yet again,

$$F(\phi, t, g, L, M, \phi_0) = 1. \tag{11}$$

What is most important is the assertion that the physical model is complete, meaning that it includes all of the variables required to construct a mathematical model that could in principle yield a unique solution. If we do not know the corresponding mathematical model, then an assertion of completeness can only be a hypothesis.

While it is essential that the physical model be complete, it is also highly desirable that the physical model be as concise as possible, i.e., that it includes only those variables that have a significant effect upon the dependent variable. The selection of variables for the physical model thus requires considerable judgment.

3 An informal dimensional analysis

3.1 Invariance to a change of units

We take it for granted that every equation must be dimensionally consistent, or homogeneous.[5,7] But how about the units used to measure length, time, etc.? The premise of dimensional analysis is that the physical relationship expressed by a complete equation is not dependent upon the choice of units, that is, whether SI, British engineering, or any other. Invariance to the choice of units implies a constraint on the form that the dimensional variables can take in a complete equation, and dimensional analysis is a systematic procedure for learning what that form is.

Angles are an interesting and relevant case. An angle is the ratio of two lengths, an arc length and a radius, and is thus inherently nondimensional. (Angles may be specified in units of radians or degrees, among others.) If we compute an angle ϕ by measurements of arc length and radius in units of meters, we will get a certain number. If we then use feet to measure these same lengths, we will get precisely the same number, that is, the same angle. Thus the left side of Eq. (6) is invariant to a change in the units of length. How about the right side of Eq. (6)? For invariance to the choice of units to hold, the length and the acceleration of gravity must appear in the ratio g/L (or any power of the ratio, for example, $\sqrt{L/g}$), and not as g or L separately, because the latter would imply a change of F with a change in the units of length. Thus, we already know something about the invariant form of Eq. (6). Consider the mass, M. A change in the units of mass should also leave F unchanged, and yet it is impossible to see how that could hold since M is the only variable in the physical model having dimensions of mass. Informal analysis leads to the conclusion that an equation for ϕ that is invariant to a change of units cannot depend upon the mass of the bob alone. This conclusion is an obvious result of the mathematical model, Eqs. (2) and (3), but can be deduced by dimensional analysis in the absence of the mathematical model. A similar consideration of the units used to measure time indicates that t and g must also appear together in a nondimensional variable, say $t/\sqrt{L/g}$. Again, any power of this variable is possible, but we might as well leave the independent variable t to the first power. The upshot is that the variables that appear in a form of Eq. (6) that is invariant to a change of units can only appear in a nondimensional form; the simplest, but not the only form is

$$\phi = F(\frac{t}{\sqrt{L/g}}, \phi_0) \,. \tag{12}$$

In place of a dependence upon one independent dimensional variable and four parameters, as in the original Eq. (6), we now have a dependence upon one nondimensional independent variable, $t/\sqrt{L/g}$, and one nondimensional parameter. When the data of Fig. 1 are plotted using this nondimensional format, Fig. 2, there is a very significant reduction in the volume of data required to display and define the data set, an important benefit of dimensional analysis applied to a presentation of data.

[7] An excellent discussion of physical measurement and much else that is relevant to dimensional analysis is given by A. A. Sonin, 'The physical basis of dimensional analysis', 2001. This manuscript is available from `<http://me.mit.edu/people/sonin/html>`.

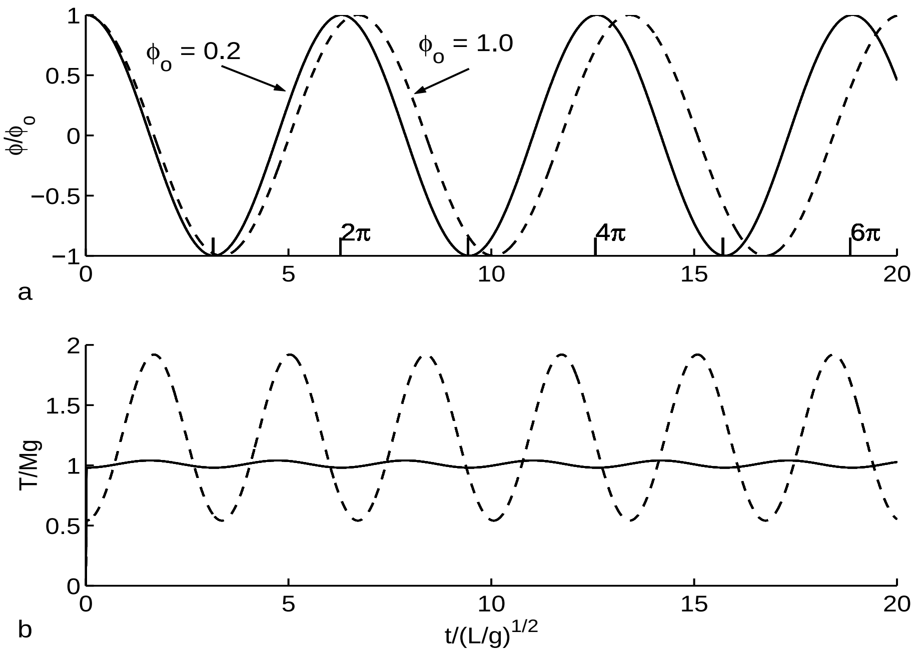

Figure 2: The numerical solutions of Fig. 1 (two values each of L, M, g, and ϕ_0) plotted in a nondimensional format. The time is nondimensionalized by $\sqrt{L/g}$. In (a) the angle ϕ is normalized by the initial angle, ϕ_0. This normalization helps us to compare the period of the two solutions, but obscures the important difference in amplitude. The eight distinct solutions of Fig. 1a collapse to just two curves that correspond to the cases $\phi_0 = 0.2$ (solid curve) and $\phi_0 = 1.0$ (dashed curve). In (b) the tension is nondimensionalized by Mg. The 16 separate curves of Fig. 1b collapse to just two curves that have the ϕ_0 as in (a).

The period of the motion can be written in a way analogous to Eq. (7) as

$$\frac{P}{\sqrt{L/g}} = F(\phi_0). \tag{13}$$

If ϕ_0 is small, say less than about 0.1 radian, the dependence on ϕ_0 is found experimentally to be negligible (Fig. 3a), and $F(\phi_0 \ll 1) = \text{constant}$. The period of a simple pendulum undergoing small amplitude oscillations thus increases in proportion to the square root of the length of the supporting line divided by the local acceleration of gravity, g. This famous result is often attributed to Galileo Galilei, who observed the motion of (inadvertent) pendulums in a Pisa cathedral. The measurement of the period of just one linear pendulum is sufficient to fix the constant, $F(\phi_0 \ll 1) = 2\pi$, for all such pendulums. If ϕ_0 is not small, then from dimensional analysis and Eq. (13) it is evident that the nondimensional period will depend upon the single parameter ϕ_0. The function $F(\phi_0)$, often referred to as a similarity law,[7] might be determined by experiment (assuming that viscous effects are negligible), by the analysis of numerical simulations, or from theory (Fig. 3a).

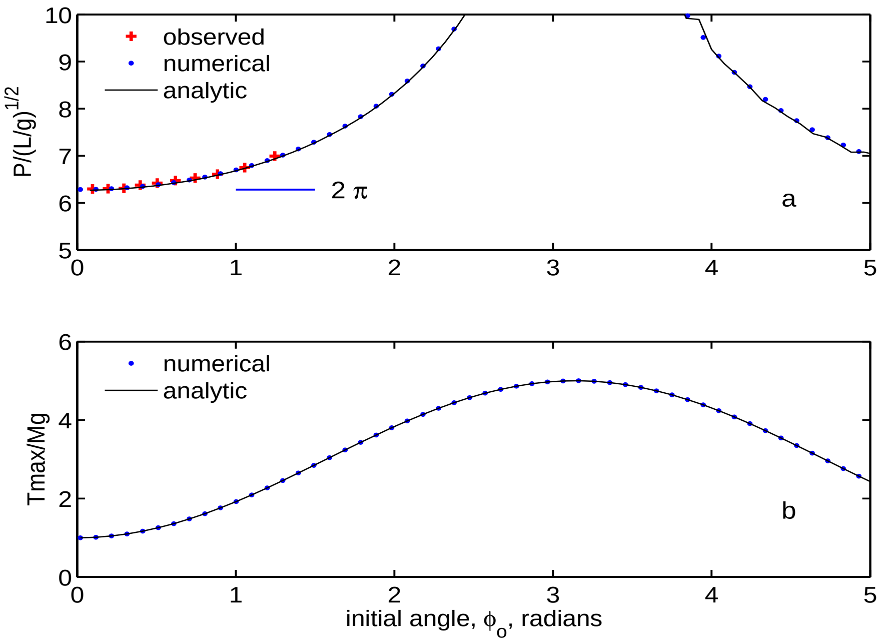

Figure 3: (a) The period of a simple pendulum as diagnosed from a series of numerical solutions (dots), as computed from theory that yields an elliptic integral that is also evaluated numerically (solid line), and as observed (crosses). The observations were acquired by measuring the time required for ten oscillations of a nearly conservative pendulum using an electronic stopwatch. Tthe observed period is accurate to about 0.3%. The flexible line of this pendulum and the initial condition $d\phi/dt = 0$ limit the initial angle to about $-\pi/2 < \phi_0 < \pi/2$. The period goes to infinity as $\phi_0 \rightarrow \pi$ because the initial condition is $\frac{d\phi}{dt}(t = 0) = 0$. From dimensional analysis we expect that this $F(\phi_0)$ holds for all simple, inviscid pendulums. (b) The maximum tension, T_{max}, diagnosed from a series of numerical solutions (dots) and as computed from energy conservation and the radial equation of motion (solid line); we had no way to measure this variable.

3.2 Natural units

A complementary way to come to the same result is to consider the units used to measure time in the mathematical model, Eqs. (2) and (3). There is no compelling physical reason to use seconds, but there is, of course, the practical convenience that clocks are calibrated in seconds. But suppose that our aim was to simplify the mathematical model by choosing a unit of time that is natural to the problem itself. The natural time scale of the pendulum is, of course, the (linear) period, which can be used to define a nondimensional time, omitting the factor 2π,

$$t^* = \frac{t}{P/2\pi} = \frac{t}{\sqrt{L/g}}. \tag{14}$$

The variable t^* is a pure number that has the same numerical value regardless of the units used to measure t, g, and L, a hint that there might be something useful here.

Nondimensional time may sound a little esoteric, but amounts to nothing more than counting time in units of the linear period while taking explicit account of the $\sqrt{L/g}$ dependence of the period. If we were to consider only one pendulum, then the whole exercise would amount to dividing the time by a constant. But if we consider all possible pendulums, i.e., all possible L and g, then there is real merit to counting time in these units. To see why, let's follow through by rewriting the equation of motion, Eq. (2), using the nondimensional time, t^*. Time derivatives transform as $dt = dt^* \sqrt{(}L/g)$ and the equation of motion becomes

$$\frac{d^2\phi}{dt^{*2}} = -\sin\phi\,, \tag{15}$$

with the initial conditions as before. The solution will be of the form

$$\phi = F(t^*, \phi_0)\,, \tag{16}$$

which is just like Eq. (12). If the amplitude of the motion is small, then the linear solution of Eq. (15) is just

$$\phi = \phi_0 \cos t^*. \tag{17}$$

The dependence upon L and g has not been omitted, but is rather subsumed into the nondimensional time, t^*, so that Eq. (17) suffices for all L and g.

Recall that the linear pendulum has the solution $\phi = \phi_0 \cos(t/\sqrt{L/g})$, and note that the argument of that cosine function is the nondimensional time — it was there all along! (since the arguments of trigonometric and exponential functions are nondimensional). The difference between Eqs. (5) and (17) is evidently in how you look at them; do you see the dimensional time, t, as the independent variable, or do you see instead the nondimensional time, $t^* = t/\sqrt{L/g}$? The answer will probably depend on the stage of an investigation (and no doubt upon our familiarity with dimensional analysis); experimental data is almost always recorded in dimensional units, and it may be helpful to carry out a numerical integration using dimensional units (assuming that these are chosen to avoid overflow). But when it comes time to report a mass of data from many experiments or integrations, there is often a great advantage to the use of nondimensional variables defined by dimensional analysis.

3.3 Extra and omitted variables

Dimensional analysis revealed that the period of a simple, inviscid pendulum did not depend upon the mass of the bob, M. This result might suggest that the inclusion of extra or superfluous variables in a physical model will not spoil the result. However, in most cases an extra variable will not be detected by dimensional analysis alone, and will lead to an extra nondimensional variable. For example, if we had included the bob diameter, D_b, in the physical model of the inviscid pendulum, it would have been carried through to a nondimensional variable, D_b/L. If we had access to an experiment, we would soon find that D_b/L was of no significance in determining the period of a nearly conservative pendulum, and would drop it from the final result.[8]

It may occur to ask whether the omission of a relevant variable would be detected. The answer is yes, rarely, if the omission makes it impossible to nondimensionalize the dependent variable. For example, if we analyzed tension under the assumption that the mass would be irrelevant as it was for the period, then it would not be possible to find a nondimensional tension. That would be a clear signal that something important had

[8] What would be the result if the acceleration of gravity, g, was omitted, that is, what phenomenon would that entail? What if g were omitted, but an initial angular velocity $\frac{d\phi}{dt}$ was included? What if in place of g we used the acceleration due to the Earth's rotation, $\frac{\Omega^2}{R}$? ($\Omega \doteq m^0 l^0 t^{-1}$ is the rotation rate of the Earth and R is the distance normal to the rotation axis.)

been omitted from the physical model. However, if the dependent variable can be nondimensionalized with the variables that are included, and in practice this is much more likely, then the purely formal procedure of dimensional analysis is not able to identify an incomplete model.

4 A basis set of nondimensional variables

Once a preliminary physical model has been defined, the second and mathematical step of a dimensional analysis is to find a complete set of nondimensional variables for that model. With a little experience and for small problems such as the simple, inviscid pendulum, the nondimensional variables can be chosen by inspection. For larger problems it may be helpful to use the following technique that relies upon the matrix methods of linear algebra. Elements of linear algebra are commonly used in dimensional analysis, and an exhaustive exposition of matrix methods can be found Szirtes (1997).[9] Brückner and colleagues[10] show how matrix methods can be applied to very large problems. The following development differs from most others in that it does not rely on the Buckingham Pi theorem, although it comes to the same result, and instead utilizes the null space basis to find a basis set of nondimensional variables.[11]

4.1 The mathematical problem

What can we infer about a function given only that it is invariant to a change of units? An arbitrary change of units for the dimensional variable X_i can be written as

$$X_i' = \alpha_1^{D_{1i}} \alpha_2^{D_{2i}} \ldots \alpha_M^{D_{Ji}} X_i \, , \tag{18}$$

where α_1 is the scale change associated with mass, α_2 the scale change associated with length, α_3 is for time and so on up to J fundamental units. For pendulum problems and for mechanics generally, $J = 3$ (mass, length and time), which is assumed to simplify later expressions. The doubly indexed object, D_{ji}, is the dimensionality of the ith dimensional variable with respect to the jth fundamental unit, and when written as a matrix is called the dimensional matrix, $\mathbb{D}$. We have already listed the elements of $\mathbb{D}$ as part of the physical model. This may look slightly abstract because it is meant to be general, and an example will be helpful. Suppose that the dimensional variable X_5 is a speed in British engineering units, feet/second, and that we wish to compute X_5' in SI units, meters/second. Spped has dimensionality, $D_{15} = 0$ (X_5 does not have units of mass), $D_{25} = 1$ for length, and $D_{35} = -1$ for time. The appropriate scale change factors are $\alpha_1 = 0.435$ (pounds to kilograms for nominal g), $\alpha_2 = 0.3048$ (feet to meters), and $\alpha_3 = 1$ (seconds to seconds). Thus $X_5' = \alpha_1^0 \alpha_2^1 \alpha_3^{-1} X_5 = 0.3048 X_5$, which we could have written down without this formalism.

[9] T. Szirtes, *Applied Dimensional Analysis and Modeling* (McGraw-Hill Publishing, 1997).

[10] S. Brückner and the University of Stuttgart Pi-Group, `<http://www.pi-group.de/pi/index2.html>` is an excellent resource for advanced applications of dimensional analysis.

[11] The calculation of a null space basis is, in effect, what all computational methods accomplish, and see E. A. Bender, *An Introduction to Mathematical Modelling* (Dover Publications, 1977).

Assuming a physical model with I dimensional variables, invariance to the system of units generally (though for $J = 3$) may be written as

$$F(X_1, X_2, \ldots, X_I) = F(\alpha_1^{D_{11}}\alpha_2^{D_{21}}\alpha_3^{D_{31}}X_1,\ \alpha_1^{D_{12}}\alpha_2^{D_{22}}\alpha_3^{D_{32}}X_2\ \ldots\ \alpha_1^{D_{1I}}\alpha_2^{D_{2I}}\alpha_3^{D_{3I}}\ X_I) \tag{19}$$

for all α, i.e., all possible changes of units. Thus for α_j, for example, we can write that

$$\frac{\partial F}{\partial \alpha_j} = \frac{\partial F}{\partial X_1}\frac{\partial X_1}{\partial \alpha_j} + \frac{\partial F}{\partial X_2}\frac{\partial X_2}{\partial \alpha_j} + \ldots\ \frac{\partial F}{\partial X_I}\frac{\partial X_I}{\partial \alpha_j} = 0\,. \tag{20}$$

If we multiply Eq. (20) by α_j/F (assuming F to be non-zero as in Eq. 10), and use

$$D_{ji} = \frac{\alpha_j}{X_i}\frac{\partial X_i}{\partial \alpha_j}\,, \tag{21}$$

which follows from Eq. (18), we obtain $J = 3$ equations, one for each α:

$$D_{11}\frac{X_1}{F}\frac{\partial F}{\partial X_1} \quad + \quad D_{12}\frac{X_2}{F}\frac{\partial F}{\partial X_2} + \ldots\ D_{1I}\frac{X_I}{F}\frac{\partial F}{\partial X_I} = 0, \tag{22}$$

$$D_{21}\frac{X_1}{F}\frac{\partial F}{\partial X_1} \quad + \quad D_{22}\frac{X_2}{F}\frac{\partial F}{\partial X_2} + \ldots\ D_{2I}\frac{X_I}{F}\frac{\partial F}{\partial X_I} = 0, \tag{23}$$

$$D_{31}\frac{X_1}{F}\frac{\partial F}{\partial X_1} \quad + \quad D_{32}\frac{X_2}{F}\frac{\partial F}{\partial X_2} + \ldots\ D_{3I}\frac{X_I}{F}\frac{\partial F}{\partial X_I} = 0. \tag{24}$$

This set of equations is best written and solved in matrix form

$$D_{ji}S_i = 0\,, \tag{25}$$

where $\mathbb{D}$ is a known $J \times I$ matrix, and $\boldsymbol{S}$ is an unknown $I \times 1$ vector of the (logarithmic) derivatives of F with respect to the dimensional variables that we seek to find;

$$S_i = \frac{\partial \log F}{\partial \log X_i}. \tag{26}$$

We will discuss a solution method in the following, but anticipate here that there will usually be several solution vectors denoted by $\mathbf{S_k}$, with $k = 1 \ldots K$ (a bold subscript denotes which vector, not an element of the vector as in Eq. (26)). For example, let's say that there are $I = 4$ dimensional variables and $K = 2$ solution vectors (written in row form) that happened to be $\boldsymbol{S_1} = [\beta_1\ 0\ \beta_2\ 0]$ and $\boldsymbol{S_2} = [0\ \beta_3\ 0\ 0]$, where the β are usually small rational numbers. The first solution vector indicates that

$$\frac{X_1}{F}\frac{\partial F}{\partial X_1} = \beta_1, \quad \frac{X_2}{F}\frac{\partial F}{\partial X_2} = 0, \quad \frac{X_3}{F}\frac{\partial F}{\partial X_3} = \beta_2, \quad \frac{X_4}{F}\frac{\partial F}{\partial X_4} = 0\,. \tag{27}$$

A solution for F is thus

$$F = X_1^{\beta_1}X_3^{\beta_2}\,, \tag{28}$$

where it is useful to term the right hand side a "Pi-variable," that is,

$$\Pi_1 = X_1^{\beta_1}X_3^{\beta_2}\,, \tag{29}$$

with the subscript on $\Pi_{()}$ referring to the subscript on the solution vector $\boldsymbol{S}_{()}$. Any multiple of Π_1 is a solution to Eq. (27), as is any power of Π_1, as is any sum of any power; evidently any function having the argument Π_1 is a solution to Eq. (27). Another solution can be found from the second solution vector $\mathbf{S_2}$ and is some function of $\Pi_2 = X_2^{\beta_3}$. In effect, we have integrated a partial differential equation but without supplying boundary or initial data; thus we learn something about the argument of an otherwise arbitrary function. We find that the dimensional variables can appear in F only in certain combinations that correspond one-to-one with the solution vectors $\boldsymbol{S_k}$,

$$\Pi_k = X_1^{S_{1k}} X_2^{S_{2k}} \ldots X_{\boldsymbol{I}}^{S_{Ik}} = \boldsymbol{X}^{\boldsymbol{S}_k}, \tag{30}$$

where $\mathbf{X} = [X_1 \; X_2 \; \ldots \; X_{\boldsymbol{I}}]$ is a vector of the dimensional variables in the order they were entered into the dimensional matrix, $\mathbb{D}$. As anticipated in Sec. 2, these Pi-variables are nondimensional. The relationship among the Pi-variables can be written as

$$\Pi_1 = F(\Pi_2, \Pi_3 \; \ldots \; \Pi_{\boldsymbol{K}}) \tag{31}$$

with no loss of generality. In the uncommon case that $K = 1$ and there is only one nondimensional variable, the function F must be a constant whose value cannot be determined from dimensional analysis alone. The period of the linear pendulum is an example, and in that case $F = 2\pi$ could be determined by experiment or theory (Fig. 3a). Neither can dimensional analysis determine anything further about the form of F in the much more common case that $K > 1$.

4.2 The null space

Eq. (25) is under determined in the usual case that there are more unknown exponents than there are equations. There will thus be many possible solution vectors that collectively make up the null space of the matrix $\mathbb{D}$. To represent the null space we seek a basis set from which any solution vector can be constructed. The computation of a null space basis is readily automated[4] and so we will not delve into the solution method.[12] It is essential, however, to understand that a given null space basis is generally *not* a unique solution to the underdetermined problem Eq. (25), to which there are often many possible solutions, i.e., many possible null space bases. Indeed, the specific null space basis we first get will depend upon the entirely arbitrary order in which the variables are listed in the dimensional matrix. The following two important properties hold for all null space bases:

P1) *The number of solution vectors, K, is the same for all basis sets and is given by the number of dimensional variables in the physical model minus the rank of the dimensional matrix,* $K = I - R$. K is also the number of nondimensional variables and in that respect all basis sets are equally efficient, i.e., they all achieve the same reduction in the number of variables. Nevertheless, one particular basis set may be more useful than the others, and so it is very often necessary to transform from one basis to another. A transformation is readily accomplished because

P2) *The basis set vectors are orthogonal and given basis set will span the null space.* Thus any vector that is a solution of the homogeneous system Eq. (25) can be found as a linear combination of the vectors in

[12] The null space is developed in most introductory texts on linear algebra, an excellent example being G. Strang, *Introduction to Linear Algebra* (Wellesley-Cambridge Press, Wellesley, MA, 1998).

any basis set. For example, if $\boldsymbol{S_1}$ and $\boldsymbol{S_2}$ are a null space basis, then their linear combination, say $\boldsymbol{S_3} = a_1\boldsymbol{S_1} + a_2\boldsymbol{S_2}$, with a_1 and a_2 any real number, is in the null space and is also a solution. The corresponding nondimensional variable is $\Pi_3 = \Pi_1^{a_1}\Pi_2^{a_2}$. If $\boldsymbol{S_3}$,$\boldsymbol{S_1}$ and Π_3,Π_1 are preferred over, say, $\boldsymbol{S_2}$,Π_2, then a revised basis set can be taken as $\boldsymbol{S_1}$,Π_1, and $\boldsymbol{S_3}$,Π_3 while omitting $\boldsymbol{S_2}$,Π_2. The revised basis set has the same number of vectors as the initial basis set and it too spans the null space. The basis set of nondimensional variables we first compute may thus be transformed to another, preferred basis set simply by multiplying or dividing the Πs in any order (examples are in Secs. 5 and 7).

4.3 A basis set for the simple, inviscid pendulum

An application to the fast time-scale oscillation of the simple, inviscid pendulum may help clarify the use of the null space basis. The dimensional matrix $\mathbb{D}$ can be read directly from the physical model:

$$\mathbb{D} = \begin{matrix} \\ m \\ l \\ t \end{matrix} \begin{matrix} \begin{matrix} \phi & t & M & L & g & \phi_0 \end{matrix} \\ \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & -2 & 0 \end{bmatrix} \end{matrix}, \tag{32}$$

where the first row is the dimensionality for mass, the second row is the dimensionality of length, and the third row is for time. The dependent variable ϕ is represented by the first column, 0 0 0, all zeros because angles are nondimensional; the time t by the second column, 0 0 1; the mass M by the third column, 1 0 0, etc. The order of listing the dimensional variables is important only insofar as the algorithm seeks to make the first few dimensional variables appear in the nondimensional variables with exponents of 1. Hence it makes sense to have ϕ come first, the dependent variable t next, and after that there is no special ordering. The calculation of a null space basis for the $\mathbb{D}$ of Eq. (32) yields three solution vectors, here concatenated into a matrix, $\mathbb{S} = [\mathbf{S_1}; \mathbf{S_2}; \mathbf{S_3}]$,

$$\mathbb{S} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{33}$$

Notice that the dependent variable ϕ appears in the first solution vector only, and to the first power, and that the elements are small rational numbers. The corresponding basis set of nondimensional variables has three elements that are easily constructed from the solution vectors:

$$\Pi_1 = \boldsymbol{X}^{\boldsymbol{S_1}} = \phi^1\, t^0\, M^0\, L^0\, g^0\, \phi_0^0 = \phi \tag{34}$$

$$\Pi_2 = \boldsymbol{X}^{\boldsymbol{S_2}} = \phi^0\, t^1\, M^0\, L^{-1/2}\, g^{1/2}\, \phi_0^0 = t/\sqrt{L/g} \tag{35}$$

$$\Pi_3 = \boldsymbol{X}^{\boldsymbol{S_3}} = \phi^0\, t^0\, M^0\, L^0\, g^0\, \phi_0^1 = \phi_0. \tag{36}$$

The functional relationship among these may be written as $\Pi_1 = F(\Pi_2, \Pi_3)$, or

$$\phi = F(t/\sqrt{L/g}, \phi_0)\,. \tag{37}$$

In analogy with Eqs. (6) and (7) the relationship for the period may be written

$$P/\sqrt{L/g} = F(\phi_0), \tag{38}$$

which is beginning to look familiar. Notice that mass M has an exponent of zero in all of the solution vectors, consistent with the informal analysis of Sec. 3 that showed that there was no way to construct a nondimensional variable from a single parameter having dimensions of mass. Also note that the angles ϕ and ϕ_0 sailed into the null space untouched since they were already nondimensional.

Tension can be analyzed in the same manner; the dimensional matrix is

$$\mathbb{D} = \begin{matrix} \\ m \\ l \\ t \end{matrix} \begin{matrix} \begin{matrix} T & t & M & L & g & \phi_0 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ -2 & 1 & 0 & 0 & -2 & 0 \end{bmatrix} \end{matrix} \tag{39}$$

and the null space basis vectors in matrix form are

$$\mathbb{S} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & -1/2 & 0 \\ -1 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{40}$$

A basis set of nondimensional variables is thus

$$\Pi_1 = \boldsymbol{X}^{\boldsymbol{S}_1} = T^1\, t^0\, M^{-1}\, L^0\, g^{-1}\, \phi_0^0 = T/Mg \tag{41}$$

$$\Pi_2 = \boldsymbol{X}^{\boldsymbol{S}_2} = T^0\, t^1\, M^0\, L^{-1/2}\, g^{1/2}\, \phi_0^0 = t/\sqrt{L/g} \tag{42}$$

$$\Pi_3 = \boldsymbol{X}^{\boldsymbol{S}_3} = T^0\, t^0\, M^0\, L^0\, g^0\, \phi_0^1 = \phi_0. \tag{43}$$

The second and third of these are identical to Π_2 and Π_3 of the angle and the period noted above. The functional relationship for the tension and the maximum tension can then be written

$$\frac{T}{Mg} = F(\frac{t}{\sqrt{L/g}}, \phi_0), \tag{44}$$

and

$$\frac{T_{\max}}{Mg} = F(\phi_0). \tag{45}$$

Notice that the mass, M, has been retained in the nondimensional tension. That the mass must appear is evident when one considers that the tension in the line will equal the weight of the bob, $T = Mg$, in the absence of motion, and can only exceed the weight due to centrifugal acceleration. Notice too that the length, L, has been eliminated from the maximum tension. A little thought will reveal that a length cannot be made nondimensional with T, g, and M in any combination, and thus dimensional analysis reveals that the maximum tension of a simple, inviscid pendulum started from rest must be independent of L. This was suggested by inspection of a few numerical solutions (see Fig. 1b) and now dimensional analysis assures us that this result holds rigorously for all simple pendulums.

It is interesting to pause briefly and consider whether dimensional analysis has provided a satisfactory *explanation* of this observation. Satisfactory is obviously a matter of degree and of opinion, but my opinion is that it does not. On the one hand, an explanation by dimensional analysis is rigorous, in so far as the observed fact has been deduced from a general principle — invariance of a physical law to the choice of units — and a set of specific conditions — the physical model of a simple, inviscid pendulum. But rigorous or not, this and most explanations by dimensional analysis seem oddly shallow and unsatisfying; in this case there is no connection to a larger physical principle, and not the slightest hint of limits, i.e., whether maximum tension would depend upon L if the physical model included a very small viscosity, for example. Dimensional analysis is a marvelously efficient tool that can help us find our way in the most difficult circumstances. But the results of dimensional analysis are seldom complete. In this instance and frequently, we will have to look beyond dimensional analysis when we seek explanations having sufficient depth to confer a useful understanding.[13]

5 The viscous pendulum

Now consider the decay rate (an inverse time scale) defined by

$$\Gamma = \frac{1}{\Phi}\frac{d\Phi}{dt}, \tag{46}$$

where Φ is the amplitude of the motion. We begin with observations of the amplitude Φ made by measuring visually the cord length at intervals of 30 sec to 2 min (the crosses of Fig. 4a). To minimize the measurement noise associated with the rather coarse least count on the measured cord, about 10^{-3} m, it was advantageous to use a longer pendulum, $L = 3.70\,\mathrm{m}$. This pendulum was supported on a needle bearing (a fishhook on a hard metal surface) to minimize interactions with the pivot, and the line was smooth monofilament having a diameter $D_l = 0.40 \times 10^{-3}$ m. The bob was a nearly spherical, more or less smooth lead fishing sinker with a diameter $D_b = 0.0211\,\mathrm{m}$ and mass $M = 0.055\,\mathrm{kg}$. The observed amplitude history, $\Phi(t)$, was quite repeatable and can be roughly characterized as an exponential decay with a time scale of about 10 min.

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[13]Scientific explanation can take various forms, and may not be the aim of all investigations. An interesting, concise discussion of scientific explanation is by Karl Popper, 'The aim of science', Ch. 12 of *Popper Selections* Ed. by D. Miller. (Princeton Univ. Press, 1985). Notice that the maximum tension in Fig. 3b for any ϕ_0 appears to be exactly 5 and occurs at $\phi_0 = \pi$. Can you explain this observation using energy conservation and the radial equation of motion? We have more or less taken it for granted that the period of a simple pendulum is independent of the amplitude of the motion provided that the amplitude is small. Can you explain this? One approach might be to use dimensional analysis to find some limits on this result, i.e., indicate when this will not be true, by considering oscillators having a restoring force that is proportional to some arbitrary power of the displacement.

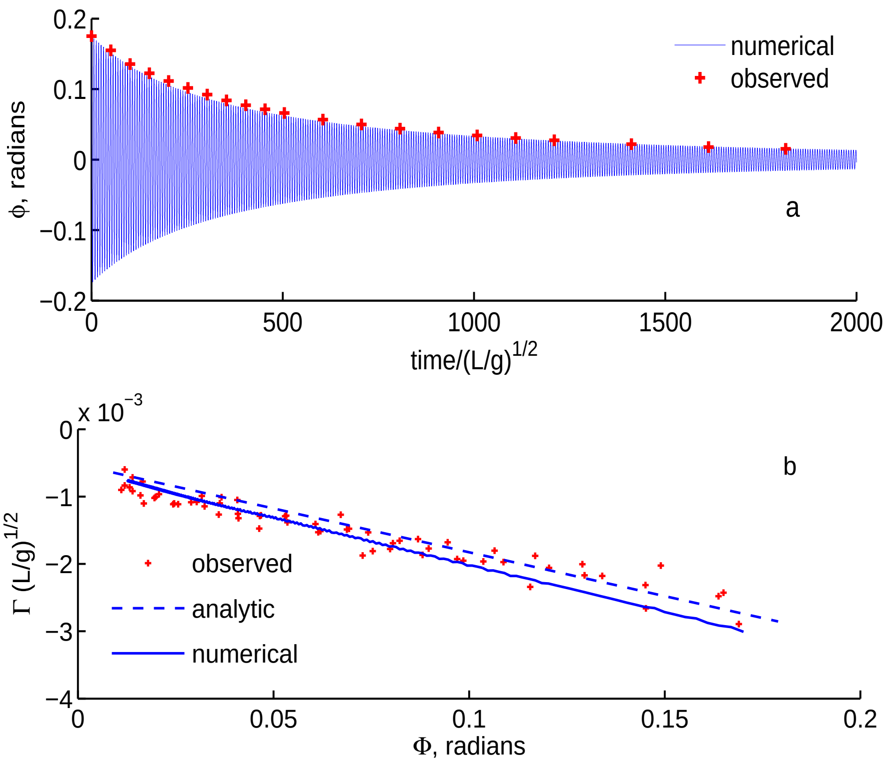

Figure 4: (a) Observations (crosses) and a numerical solution (the thin solid line) of the motion of a simple, viscous pendulum. The crosses are observations of the amplitude at intervals of 30 seconds to 2 minutes. (b) The decay rate computed directly from three repetitions of the experiment (crosses), as estimated from an approximate analytic solution Eq. (61) (dashed line) and as diagnosed from the numerical model solution (solid line). Drag that is linear in the angular velocity produces a constant decay rate and a simple exponential decay in time of the amplitude, and drag that is quadratic in the angular velocity produces a decay rate that increases linearly with the amplitude, Φ.

5.1 A physical model of the viscous pendulum

We presume that hydrodynamic drag with the surrounding air is the primary damping process, [14] and that the diameter of the bob, D_b, and of the line, D_l, are now relevant, as are the density and kinematic viscosity of air, ρ and ν. When we amend the inviscid model of Sec. 2 to include these variables we have

- A physical model for the decay rate of a simple, viscous pendulum:

1. the decay rate, $\Gamma \doteq m^0 l^0 t^{-1}$, the dependent variable;
2. mass of the bob, $M \doteq m^1 l^0 t^0$, a parameter,
3. length of the line, $L \doteq m^0 l^1 t^0$, a parameter,
4. acceleration of gravity, $g \doteq m^0 l^1 t^{-2}$, a parameter,
5. the amplitude of the motion, $\Phi \doteq$ nond, a parameter,
6. diameter of the line, $D_l \doteq m^0 l^1 t^0$, a parameter,
7. diameter of the sphere, $D_b \doteq m^0 l^1 t^0$, a parameter,

[14] Detailed treatment of damping processes are by P. T. Squire, "Pendulum damping," Am. J. Phys. **54**, 984–991 (1986) and R. A. Nelson, and M. G. Olsson, "The pendulum: Rich physics from a simple system," Am. J. Phys. **54**, 112–121 (1985).

8. density of air, $\rho \doteq m^1 l^{-3} t^0$, a parameter (1.2 kg m^{-3}, nominal),
9. kinematic viscosity of air, $\nu \doteq m^0 l^2 t^{-1}$, a parameter (1.5×10^{-5} m^2 s^{-1}, nominal).

(For the purpose of defining the amplitude of the motion we might have used ϕ_o in place of Φ.) Dimensional analysis, from here on omitting all of the intermediate steps, indicates six nondimensional variables;

$$\Gamma\sqrt{L/g} = F(\Phi,\ \frac{D_b}{L},\ \frac{D_l}{L},\ \frac{\rho D_b^3}{M},\ \frac{g^{1/2}L^{3/2}}{\nu}). \qquad (47)$$

The first five nondimensional variables have an obvious interpretation, but the last one involving the viscosity, ν, does not. In any event, we are not ready to make use of such a comprehensive model. We may still be thinking of the nearly conservative pendulum of Sec. 2, but the nine-variable physical model includes all possible pendulums and fluid mediums. Before we can expect a useful result from dimensional analysis we will have to identify the most relevant parameters for the kind of nearly conservative pendulum that we have in mind.

5.2 Drag on a moving sphere

A piecewise approach is tried next. Consider in isolation the hydrodynamic drag on a smooth sphere (the bob) due to a steady motion through an infinite viscous fluid (air) that is otherwise at rest.

- A physical model for drag on a sphere moving through viscous fluid:

1. drag (a force), $H \doteq m^1 l^1 t^{-2}$, the dependent variable,
2. speed of the sphere, $U \doteq m^0 l^1 t^{-1}$, a parameter,
3. diameter of the sphere, $D_b \doteq m^0 l^1 t^0$, a parameter,
4. density of air, $\rho \doteq m^1 l^{-3} t^0$, a parameter,
5. kinematic viscosity of air, $\nu \doteq m^0 l^2 t^{-1}$, a parameter.

Despite the highly idealized configuration of this problem, it is very difficult to compute the drag from first principles in the common case that the flow around the sphere is turbulent. However, dimensional analysis combined with laboratory measurement leads to a useful result. The initial basis set of nondimensional variables for this physical model comes out to be

$$\Pi_1 = \frac{H}{\rho D_b^2 U^2} \quad \text{and} \quad \Pi_2 = \frac{\nu}{U D_b}, \qquad (48)$$

where we recognize that Π_2 is the inverse of an important nondimensional variable called the Reynolds number,

$$\mathrm{Re} = \frac{U D_b}{\nu}, \qquad (49)$$

which we prefer. We know from P2 of the null space (Sec. 3) that Π_1 in Eq. (48) is not uniquely determined by dimensional analysis, and a somewhat general basis set can be written

$${}_n\Pi_1 = \frac{H}{\rho D_b^2 U^2}(\frac{U D_b}{\nu})^n \quad \text{and} \quad \Pi_2 = \frac{U D_b}{\nu}, \qquad (50)$$

where n is any real constant and assuming that H and $\Pi_2 = \text{Re}$ may as well remain to the first power. The functional relation between these nondimensional variables could be written as ${}_n\Pi_1 = F(Re)$, where F depends on n. We will consider next how to choose the value of n that gives the best or most useful form.[15] Regardless of the form finally chosen, an essential result is that the nondimensional drag, ${}_n\Pi_1$, is expected to be a function of Re alone. Laboratory measurements can thus be used to define $F(\text{Re})$ that should hold for all steadily moving spheres, Fig. 5a, just the way that the function $F(\phi_0)$ (Sec. 3.2) sufficed to define the period for all inviscid, simple pendulums. Modern text books[3] and this essay show only the curve that runs through the middle of a tight cloud of data points that have accumulated from many laboratory experiments, see for example, Rouse[3]. What is most important, but not evident from this kind of presentation, is that drag coefficients inferred from experiments made using a very wide range of spheres and cylinders moving at widely differing speeds and through many different viscous fluids (Newtonian fluids) do indeed collapse to a well-defined function of Reynolds number alone, just as dimensional analysis had indicated.

This is a result, characteristic of dimensional analysis generally, that is at once profound and trivial. One might say trivial because, after all, dimensional analysis told us that the drag coefficient must depend upon Re alone. From this perspective, an effective collapse of the experimental data merely verifies that carefully controlled laboratory conditions can indeed approximate the idealized physical model. But it is also profound in that dimensional analysis has shown the way to a useful result (Fig. 5), where there would otherwise have been be an unwieldy mass of highly specific data (as in going from Fig. 1 to Fig. 2).

5.2.1 Zero order solution

The crucial (and in this case the only) choice is that of the dependent nondimensional variable, Π_1. One strategy is to form Π_1 so that it reflects a physically meaningful 'zero order' solution for the dependent variable. This amounts to 'scaling' a dimensional variable in a model equation so that the corresponding nondimensional variable has a maximum size of about one (more about scaling in Section 6, and see Ref. 3, Lin and Segel).

A zero order solution requires some sense of the physics of the problem. Visual observations of the flow around a sphere provide hints that drag can arise from two distinct processes. If the sphere is moving very slowly so that the wake behind the sphere is nearly undisturbed, or laminar, then the drag will be mainly viscous and proprtional to the viscosity of the fluid times the shear of the flow around the sphere, U/D_b, or $\rho\nu U/D_b$. If viscous stress acts over an area proportional to D_b^2, then the zero order solution for viscous drag on the sphere would be $H \propto \rho\nu D_b U$. This corresponds to the basis set $n = 1$ of Eq. (50). If we expected that this laminar, viscous flow was the dominant drag-producing process, then it would be appropriate to nondimensionalize the drag as

$$\frac{H}{\rho\nu D_b U} = F(\text{Re}) = C_v(\text{Re}) \,, \tag{51}$$

because the Re-dependence of C_v, the so-called viscous drag coefficient, would then be minimized.

[15] One criterion for choosing the form of the nondimensional variables is to follow conventions of your field. In this case Π_1 is a drag coefficient, usually defined as $C_d = H/\frac{1}{2}\rho A U^2$, where A is the frontal area of the object. For the purpose of this essay we will consider other possible forms for Π_1.

Even if the fluid were nearly inviscid, there would still be drag because fluid must be accelerated as it is displaced by the moving sphere. If the displaced fluid is carried along in a highly disturbed, turbulent wake, as is more or less observed behind a rapidly moving sphere (we will clarify what is meant by rapidly), then the drag would be roughly proportional to the density of the fluid times the speed squared multiplied by the frontal area, $A = \pi D_b^2/4$. Thus the drag would be estimated as $H \propto \rho A U^2$. If we expected that this turbulent, inertial drag process was dominant, then the initial basis set corresponding to $n = 0$ would be appropriate:

$$\frac{H}{\rho A U^2} = F(\mathrm{Re}) = C_i(\mathrm{Re}) \,, \tag{52}$$

and the Re-dependence of the inertial drag coefficient C_i would show the departures from inertial drag due to viscous effects. Either form of the drag coefficient effectively conveys the laboratory data and in that regard there is nothing to choose between them.

5.2.2 The other nondimensional variables: remarks on the Reynolds number

Once the dependent nondimensional variable, Π_1, has been selected, the remaining nondimensional variables can be formed in ways that most clearly define the geometry of the problem, that reflect a balance of terms in a governing equation, or that follow the established norms of your field. This is necessarily vague because the possibilities are limitless, however the task is often easier than might be expected. In the example of drag on a moving sphere, there is only one remaining nondimensional variable, the Reynolds number or its inverse. There are many other such ratios, often termed nondimensional numbers, that succinctly characterize the balances among terms in mathematical models and thus are the natural terminology of theoretical mechanics. Like any language, these nondimensional numbers are conventions that have to be learned. Here a few that are likely to be encountered in geophysical fluid mechanics:[16]

- Reynolds number, $Re = \frac{UL}{\nu}$, where U is the speed of a current, L is the spatial scale over which U changes by roughly 100%, and ν is the kinematic viscosity of the fluid. The Reynolds number is the ratio of advective to viscous terms in the Navier-Stokes momentum balance and arises very frequently in fluid mechanics. There are many different definitions of Reynolds numbers, differing most often by the length scale, L. For nondimensionalizing the drag on spheres it is conventional to use the diameter of the sphere, an external, geometric parameter. In other definitions of Reynolds number L may be an internal variable that is set by the flow itself, e.g., the thickness of a boundary layer. Selecting or knowing the appropriate length scale may thus be a nontrivial issue.

- Rossby number, $R_o = \frac{U}{Lf}$, where U and L are as above and $f \doteq t^{-1}$ is the Coriolis parameter, proportional to the Earth's rotation rate. The Rossby number is the ratio of advective to Coriolis terms in a momentum equation for fluid observed in a rotating reference frame (the Earth).

- Strouhal number, $S = \frac{\omega L}{U}$, where ω is the frequency or the inverse of the time scale, $\frac{\partial}{\partial t}$, of the current, U. The Strouhal number is the ratio of the local rate of change of U to the advective rate of change. The inverse of S is often called the temporal Rossby number in geophysical fluid dynamics.

[16] An excellent compilation is at http://www.atm.damtp.cam.ac.uk/people/mem/GEFD-SUMMER-dimless-params.pdf

- Froude number, $\frac{U}{\sqrt{gH}}$, where H is the thickness of a fluid layer. The Froude number is the ratio of advection of momentum to the pressure gradient and arises in problems in which the acceleration of gravity is important, for example, in the wave drag on ships.
- Ekman number, $E = \frac{K}{f}$, where K is the drag coefficient that appears in a linear (Rayleigh) drag law. The Ekman number is a measure of the ratio of frictional to Coriolis terms in a momentum equation. There are as many definitions of the Ekman number as there are parameterizations of frictional terms.

Recall that for the purpose of modeling drag, a slowly moving sphere is one that has a nearly undisturbed, laminar wake. Observational evidence shows that laminar flow occurs when Re is small, Re ≤ 1, regardless of speed *per se*; dimensional analysis tells us as much in that the drag coefficient depends only upon Re. The small Re range is that of a very small bug swimming through water, for example. In the small Re range the viscous drag coefficient C_v is O(1),[17] both numerically and in the sense that C_v is nearly independent of Re (Fig. 5a). For creatures and objects anywhere near our size, e.g., birds or bicyclers, Reynolds numbers of $O(10^5)$ and greater are the norm, and inertial drag, often termed form drag, is generally more important than is viscous drag. Notice that for moderately large values of Re, $10^3 \leq \text{Re} \leq 10^5$, the inertial drag coefficient C_i is O(1) in magnitude and very roughly constant within subranges of Re.[18] We can anticipate that our pendulum will have a Reynolds number in an intermediate range in which both viscous and inertial drag are likely to be important.

5.3 A numerical simulation

To model the decay process we will include hydrodynamic drag on the line and bob in the angular momentum balance (1). Drag will be estimated by means of the steady drag laws discussed above, and so it is implicitly assumed that the instantaneous speed of the bob or line gives the same drag as would a steady motion of the same speed. Whether this assumption is appropriate remains to be seen.

The main task is to account for the Re-dependence of the drag coefficients Because the line is quite thin, the Reynolds numbers of the line are rather small, $\text{Re}_l = UD_l/\nu \leq 20$, where $U = rd\phi/dt$, r is the distance from the pivot and an *a priori* estimate of $d\phi/dt$ is $\phi_o/\sqrt{L/g}$. In that small Re range the viscous drag coefficient on a cylinder can be approximated well by $C_v = 3/2 + \text{Re}/3$ (the heavy dotted line of Fig. 5b). The drag per unit length of the line, $\delta = dr$, can then be computed by the drag law corresponding to Eq. (51) as $H = \pi\rho\nu C_v U dr$, and the (dimensional) torque due to drag over the length of the line is then

$$\tau_l = \int_0^L rHdr = \rho\left(\frac{\pi}{2}\nu L^3 + \frac{1}{12}D_l L^4 \mid \frac{d\phi}{dt} \mid\right)\frac{d\phi}{dt}. \quad (53)$$

[17]The symbol O( ) can be used to indicate the rough numerical size of a term, usually to within an ’order’ of magnitude; thus 1/3 and 3 are both O(1), while 0.01 or 10 would not be. In a later section, 7, we will extend this definition.

[18]Even at very large Re it does not follow that viscosity is entirely irrelevant. Significant changes in the drag coefficient occur at around Re $\approx 2 \times 10^5$ due to changes in the viscous boundary layer and the width of the wake behind a moving sphere. This is the Re range of a well-hit golf ball or tennis ball, and is part of the reason that aerodynamic drag on these objects has a surprising sensitivity to surface roughness or spin. For much more detail on these phenomenon see S. Vogel, *Life in Moving Fluids* (Princeton Univ. Press, 1994) and P. Timmerman, and J. P van der Weele, “On the rise and fall of a ball with linear and quadratic drag,” *Am. J. Phys.*, **67**, 538–546 (1999).

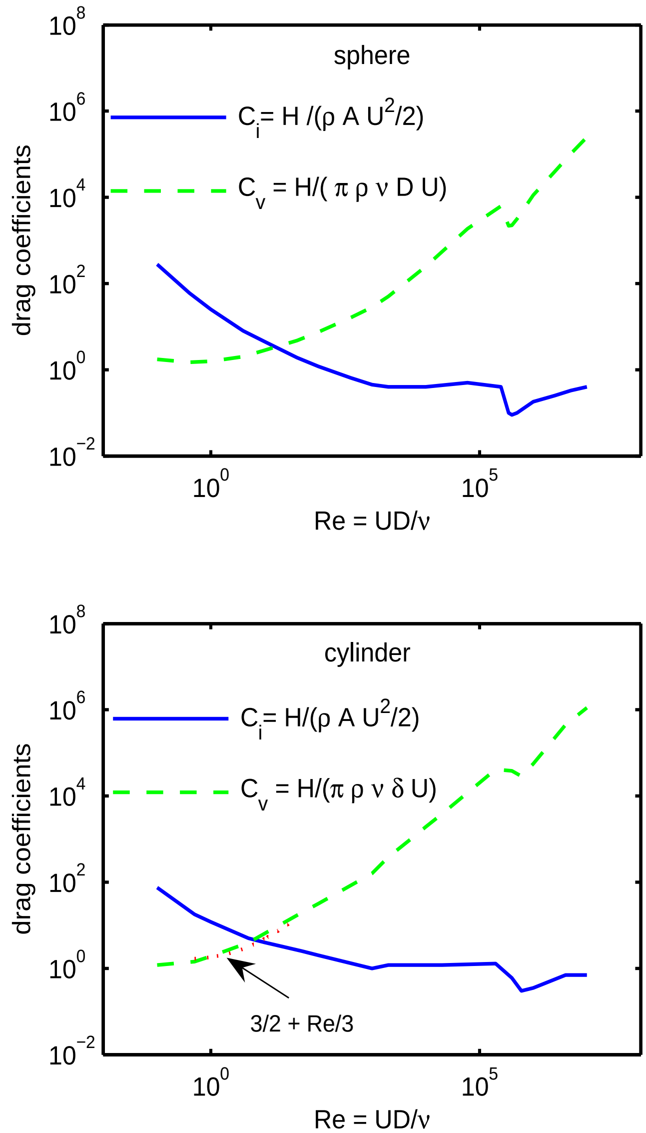

Figure 5: Drag coefficients of a sphere (a) and a cylinder (b) moving at a steady speed U through viscous fluid. Two forms of drag coefficient are shown here, the viscous drag coefficient is denoted by C_v (the dashed line), and the inertial drag coefficient denoted by C_i (the solid line, usually denoted C_d, and by far the most commonly encountered form). Note that C_v is O(1) (numerically and that it is approximately independent of Re) if Re is very small, and that C_i is O(1) if the Reynolds number is very large. The inertial drag coefficients were read from Munson et al., Fig. 7.7 and Rouse, Figs. 125-126.[3,16]

The absolute value operator insures that the drag force always opposes the motion. The bob has a much larger diameter and thus a much larger Reynolds number; $\mathrm{Re}_b = L\frac{d\phi}{dt}D_b/\nu$ is in the range $\mathrm{Re}_b \leq 1000$ where no simple formula for a drag coefficient is highly accurate. Thus we will allow an arbitrary $C_i(\mathrm{Re}_b)$ and compute the drag-induced torque on the bob as

$$\tau_b = \frac{\pi\rho}{8}C_i(Re_b)D_b^2L^3 \mid \frac{d\phi}{dt} \mid \frac{d\phi}{dt} \tag{54}$$

where Re_b and C_i are evaluated at each time step of the numerical integration using the data of Fig. 5a. The amended angular momentum balance (in dimensional variables),

$$\frac{d^2\phi}{dt^2} = -\frac{g}{L}\sin(\phi) - \frac{\tau_l + \tau_b}{L^2M}, \tag{55}$$

together with Eqs. (53) and (54) and the data of Fig 4.1 plus the initial condition Eq. (3) make a complete if rather cumbersome model that can be integrated numerically.

With drag terms included, the period of the oscillation is nearly unchanged, but the amplitude slowly decays (Fig. 4a). The decay simulated by the numerical model looks plausible when compared with the observations, suggesting that the steady drag laws have the gist of it (a more critical appraisal is given below).

5.4 An approximate model of the decay rate

Numerical solutions are not revealing of parameter dependence, but given two modest approximations we can go on to deduce a model of the viscous pendulum that has transparent solutions. First, the angle ϕ is small enough in the case shown in Fig. 4a that $\sin\phi$ of Eq. (55) can be approximated well as ϕ. Second, the drag overall is due mostly, $\approx 85\%$, to the line, and so it should be acceptable to make the severe approximation that the inertial drag coefficient for the bob is a constant, $C_i = 0.7$, an average for the Re_b range of the bob in the present case. With these approximations we obtain a solvable model for the simple, viscous pendulum (now in nondimensional variables)

$$\frac{d^2\phi}{dt^{*2}} = -\phi - a\frac{d\phi}{dt^*} - b \mid \frac{d\phi}{dt^*} \mid \frac{d\phi}{dt^*}, \tag{56}$$

where the coefficient in the linear drag term is

$$a = \frac{\pi}{2}\frac{\rho\nu L^{3/2}}{Mg^{1/2}} \tag{57}$$

and the coefficient in the quadratic term is

$$b = \frac{\rho}{8M}(0.7D_b^2L + \frac{2\pi}{3}D_lL^2). \tag{58}$$

Approximate solutions for small damping are given in Ref. 14; linear drag causes the amplitude to decay at a nondimensional rate

$$\frac{1}{\Phi}\frac{d\Phi}{dt^*} = -\frac{a}{2} \tag{59}$$

and the quadratic term causes decay at a rate

$$\frac{1}{\Phi}\frac{d\Phi}{dt^*} = -\frac{8b}{6\pi}\Phi\,, \tag{60}$$

where again Φ is the slowly varying amplitude. For small damping, these can be added together and evaluated to give an approximate decay rate,

$$\Gamma\sqrt{L/g} = \frac{1}{\Phi}\frac{d\Phi}{dt^*} \approx -5.2\times10^{-4} - 1.6\times10^{-2}\Phi \tag{61}$$

shown as the dashed line of Fig. 4b. This approximate model shows clearly how the decay rate is expected to vary with the parameters that characterize the pendulum and the surrounding fluid, and in fact it gives an excellent account found in numerical simulations even for quite strong damping. All of the pieces of this model were present in our first attempt at dimensional analysis of the viscous pendulum, Eq. (47), though we had no way to recognize them at the time.

The decay rate can be estimated from the observations and from the numerical solution by first differencing (Fig. 4b). A comparison of decay rates makes a much more sensitive test of the drag formulation than does the amplitude itself (cf. Fig. 4a) and reveals that the decay is not simple exponential as it first appears. In fact, there is a significant dependence of the decay rate upon amplitude, which in the approximate model follows from the quadratic drag term, Eq. (58). This shows that drag on the pendulum is due mostly to inertial drag rather than the purely viscous drag of the very small Re range, which is not unexpected.

While the modeled decay rate is fairly accurate, there is at least a hint that the appropriate drag law for this pendulum has a somewhat greater linear drag than is found in the models, and slightly less quadratic drag. This behavior is found over a fairly wide range of parameters, but further study of drag phenomena is outside the scope of this essay.[19]

---

[19]Three problems for you to solve using dimensional analysis:

1) Can you calculate a Reynolds number for the bob and the line from the original six nondimensional variables of Eq. (47)? Which nondimensional variable is present in Eq. (47) but not in Eqs. (57) and (58)? How or why was it omitted? Under what conditions (what parameter range) would you expect to see a significant effect of the time-dependent motion? How could you test (in principle and in practice) that the steady drag formulations really are appropriate for modeling the damping of a simple pendulum? You might, for example, consider that the fluid medium was water in place of air (the approximate density and kinematic viscosity of water are $\rho = 1.0\times10^3$ kg m^{-3} and $\nu = 1.8\times10^{-6}$ m^2 s^{-1} at a temperature $=$ 0°C, and $\rho = 1.0\times10^3$ kg m^{-3} and $\nu = 0.7\times10^{-6}$ m^2 s^{-1} at a temperature $=$ 40°C). Can you think of a name more apt than 'viscous' pendulum?

2) Find an expression for the frequency, ω, and the phase speed of surface gravity waves assuming that the frequency depends upon the wavelength, λ, and the acceleration of gravity, g. Measurements made in a case where the water depth was very large compared to the wavelength gave the following results: λ = [2 5 10 20 40 75] m, and ω = [5.54 3.50 2.48 1.75 1.24 0.90] rad s^{-1}. What is the function relating the frequency to wavenumber, $k = 2\pi/\lambda$, for these deep water waves?

3) The flow rate, V, through a smooth-walled pipe is presumed to depend upon the density and kinematic viscosity of the fluid, ρ and $\nu = [l^2\,t^{-1}]$ the diameter of the pipe, d, and the pressure gradient, $P_x = [m\,l^{-2}\,t^{-2}]$ (a force per unit volume). Find a nondimensional relationship among these variables, treating P_x as the dependent variable, and look for a Reynolds number on the right hand side. The following data were taken from a pipe with d = 0.01 m, carrying water having $\rho = 1.0$ kg m^{-3} and $\nu = 1\times10^{-6}$m^2 s^{-1}; $Px =$ 1.e3*[0.0026 0.0087 0.026 0.124 0.809 3.59] kg m^{-2} s^{-2}, and V = [0.12 0.24 0.45 1.11 3.22 7.40] m s^{-1}. Show that these data are reasonably consistent with the empirical Blasius formula for turbulent through a smooth-walled pipe, $\frac{dP_x}{\rho V^2} = 1.58(\frac{dV}{\nu})^{-0.25}$.

6 A similarity solution for diffusion in one dimension

Sometimes the use of dimensional analysis can lead to solutions that might otherwise have been missed. A good example is afforded by the study of Stokes First Problem, in which a fluid column is driven from rest by an imposed surface speed, V_o (a standard problem, e.g., Kundu and Cohen[3]). A key physical assumption is that the momentum supplied at the upper boundary is assumed to diffuse downward into the fluid at a rate set by a kinematic viscosity, ν, that is presumed to be a given constant and not dependent upon the flow. In that case the governing equation for the current, U, is the elementary one-dimensional diffusion equation,

$$\frac{\partial U}{\partial t} = \nu \frac{\partial^2 U}{\partial z^2}. \tag{62}$$

The initial condition is presumed to be a state of rest,

$$U(z, t = 0) = 0 \tag{63}$$

and the boundary conditions are that the fluid sticks to an upper surface that is moving at speed Vo, and to a lower surface at $z = -L$ that is at rest,

$$U(z = 0, t \geq 0) = Vo, \quad \text{and} \quad U(z = -L, t) = 0. \tag{64}$$

It is easy to generate solutions to this linear model; Fourier transform leads to an infinite trigonometric series that can be summed to very high accuracy, and numerical solution is almost deceptively easy. What role can dimensional analysis have in this case? There is yet another avenue, known as a similarity solution, that may yield more insight than we are likely to derive from an infinite series or from numerical data. To arrive at a similarity solution we will begin with a dimensional analysis of the elementary diffusion model. As usual, we will start by making a straightforward list of the important variables.

- A physical model of one-dimensional, elementary diffusion:

1. current speed, $U \doteq l^1\ t^{-1}$, the dependent variable
2. time, $t \doteq t^1$, an independent variable
3. depth, $z \doteq l^1$, a second independent variable
4. surface boundary value, $Vo \doteq l^1\ t^{-1}$, a parameter
5. the kinematic viscosity, $\nu \doteq l^2\ t^{-1}$ a parameter
6. the depth of the fluid column, $L \doteq l^1$, a parameter.

Given this physical model, the nondimensional functional relationship might then read

$$\frac{U}{Vo} = F(\frac{zVo}{\nu},\ \frac{tVo^2}{\nu},\ \frac{z}{L}). \tag{65}$$

6.1 Honing the physical model

Solutions to Eqs. (62 - 64) show that the current at a given depth and time is directly proportional to the boundary value, Vo, as might have been inferred also from inspection of the mathematical model. This important property has not been built into the physical model nor is it reflected in the initial basis set of nondimensional variables Eq (65). Instead, the physical model covers a much more general problem in which Vo appears in the nondimensional variables combined with z or t as if Vo effected the diffusion process. On physical grounds this can be expected to happen when the diffusion process results from turbulence generated by the boundary forcing rather than molecular diffusion. Whether the flow is turbulent or laminar depends upon the distance from the boundary, the current speed at that point, and the fluid viscosity, i.e., a Reynolds number! If we insist that the diffusion process be represented by a constant viscosity, which is what is meant by the elementary diffusion model, then we are implicitly limiting the analysis to small Reynolds number flows, or to something like heat diffusion in a solid. To assert this important physical property in the physical model we will make a small but highly significant change — we will replace the dependent variable U by U/Vo and then remove Vo from the list of parameters. The basis set for this revised physical model is then

$$\frac{U}{Vo} = F(\frac{z}{\sqrt{t\nu}}, \frac{z}{L}), \tag{66}$$

which notice has one fewer variables than before. Now we are going to consider the limit that L is very large compared to the depth that diffusion has reached (we will elaborate on this point below). In that case the nondimensional current will depend upon only the single independent variable

$$\frac{U}{Vo} = F(\eta), \tag{67}$$

where

$$\eta = \frac{z}{\sqrt{t\nu}}, \tag{68}$$

and not upon z and t separately as the first set of nondimensional variables indicated. The current profiles at various times thus have a similar shape, being more less stretched out depending upon $\sqrt{t}$, Fig. 6. The variable η is said to be a 'similarity' variable and the function $F(\eta)$ a similarity function, as noted in connection with the drag coefficients of Section 5.

6.2 A similarity solution

The analysis above suggests that the governing equation (a partial differential equation in z and t) might be transformable into an ordinary differential equation in the single independent variable η. To see if this holds we will substitute $U(\eta(z,t))$ from Eq. (67) into the governing equation; the partial time derivative becomes

$$\frac{\partial U}{\partial t} = Vo\frac{\partial F}{\partial \eta}\frac{\partial \eta}{\partial t} = -VoF'\frac{\eta}{2t}, \tag{69}$$

where $F' = \frac{dF}{d\eta}$, and the second derivative with respect to z becomes

$$\frac{\partial^2 U}{\partial z^2} = VoF''\frac{1}{t\nu}. \tag{70}$$

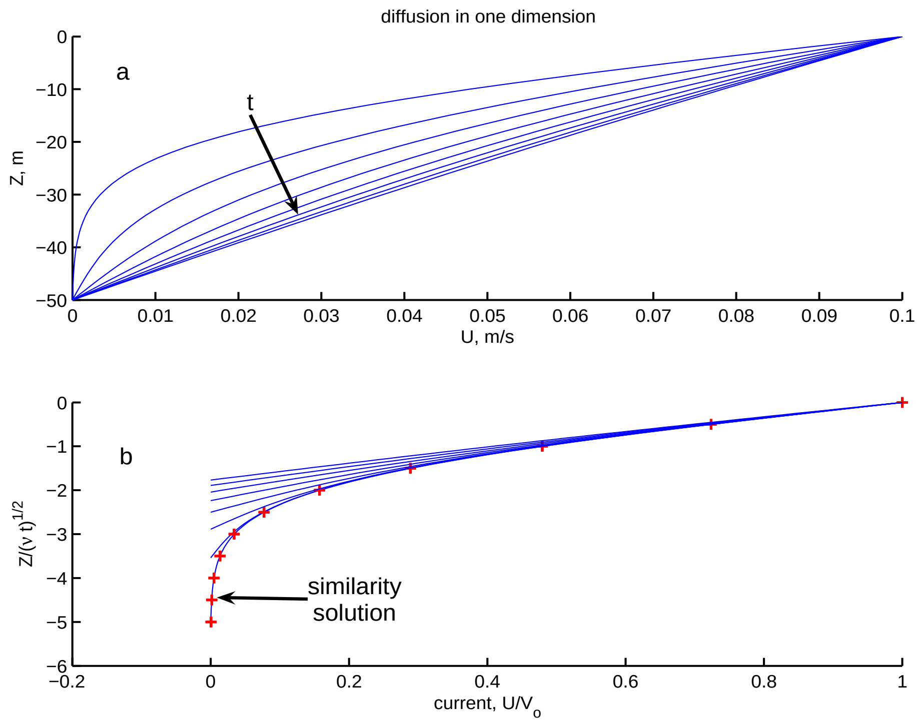

Figure 6: Solutions $U(z,t)$ from an elementary diffusion model, Eqs. (62) - (64) solved numerically. (a) The dimensional current plotted at intervals of 10^4 s. The dimensional values are roughly those of an upper ocean ($\nu = 100$ Munks $= 10^{-2}\mathrm{m}^2\mathrm{s}^{-1}$). (b) The numerical solutions nondimensionalized and plotted along with the similarity solution, Eq. (73) (the crosses). At small time, the numerical solutions lie exactly on top of the similarity solution. At longer times, the effect of the lower no-slip boundary condition becomes appreciable and the numerical solution departs more and more from the similarity form.

Substitution into the governing equation and noting that z and t appear only in the combination $z/\sqrt{t}$ shows that the second order partial differential equation is indeed transformed into the second order ordinary differential equation

$$F'' + \frac{1}{2}\eta F' = 0. \tag{71}$$

Substitution into the upper and lower boundary conditions gives

$$F(\eta = 0) = 1, \text{ and } F(\eta = -\infty) = 0. \tag{72}$$

So far as the current is concerned, small time and large z are now equivalent, and the initial condition is identical to the lower boundary condition. The model Eqs. (71) and (72) are a canonical form with solution,

$$U/Vo = 1 - erf(\eta/2), \tag{73}$$

where

$$erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x exp(-y^2)dy \tag{74}$$

is the error function. The error function is tabulated and this similarity solution can be considered exact and closed. This solution is valuable in at least two ways. Because it is exact, it can serve as a precise test of

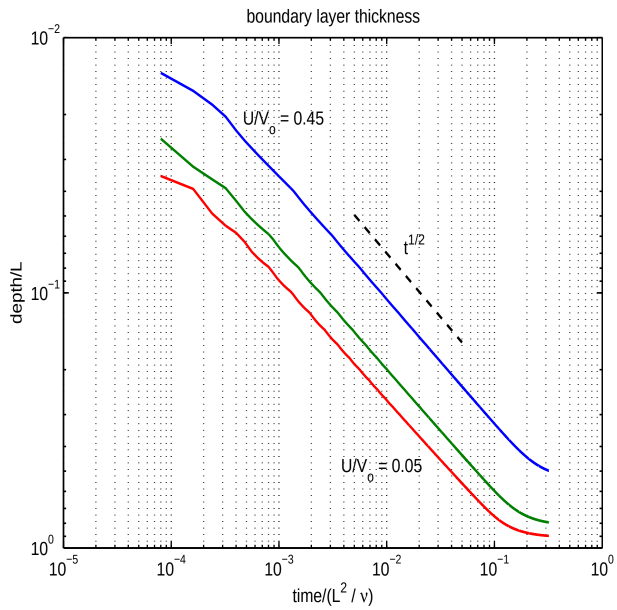

Figure 7: Boundary layer thickness, defined as the depth at which the current was 0.45, 0.15 or 0.05 of the surface value (the three separate curves) diagnosed from the numerical solution of Fig. 6. Note that there was an intermediate (nondimensional) time from about 5×10^{-3} to 10^{-1} during which the boundary layer thickness grew like $t^{1/2}$. The similarity solution Eq. (73) holds during this intermediate time and otherwise it does not.

numerical or numerically evaluated solutions whose accuracy might be hard to evaluate *a priori*. Secondly, because of its simplicity we can see qualitative features in a similarity solution that might have been missed in a mass of numerical data. In particular, note that a given value of η, say $\eta = C$, and thus a given value of $U/Vo = F(C)$, moves downward over time as

$$z = C\sqrt{\nu t}. \tag{75}$$

Thus the thickness of the layer directly effected by the boundary condition, i.e., the boundary layer, grows like the square root of time, Fig. (7), which is characteristic of 1-dimensional elementary diffusion and random processes alike. This result can be turned around; the time needed to diffuse a distance L is roughly L^2/ν, which forms a natural time scale for a one-dimensional diffusion problem.[20]

This elegant and precise solution has the significant built-in limitation that it holds only for an intermediate time interval after the boundary conditions is imposed. The time has to be long enough that the growing boundary layer does not retain the detailed imprint of the startup, which could never be a pure step function as is assumed, or, in a numerical model, the boundary layer must be thick enough that the structure is not highly dependent upon the vertical resolution. Judging from the curves of Fig. (6), this requires about $t/(L^2/\nu) \geq 5 \times 10^{-3}$, in this particular numerical solution. The time also has to be short enough that diffusion has not caused an appreciable current near the lower boundary at $z = -L$. Once the current becomes appreciable near the lower boundary, for $t/(L^2/\nu) \leq 10^{-1}$ judging from Fig. (6), we can expect that similarity will no longer hold accurately, and the more general form Eq. (66) will be relevant from then on. This second limit, unlike the previous one, is independent of the details of the solution method.

[20] When we say that diffusion reaches a certain distance we mean that an appreciable (or given) fraction of the boundary value amplitude will be found at that distance from the boundary, say. The continuous diffusion equation has the property that any given point is effected by the entire domain instantaneously. However, if the point is far away from the boundary in the sense that η is large, then the boundary effect will be correspondingly small, though never literally zero.

One way to think about this diffusion problem in the large is that the governing equation (62) determines the structure of the solution for short and intermediate times, but the boundary condition wins out at long times, i.e, at steady state. This shows in an especially clear way how important boundary conditions are in fluid mechanics; fluid motion is often forced or initiated by or through a boundary, and fluid flows often have a global dependence upon a domain defined by some kind of boundary.[21]

7 Scaling analysis

Dimensional analysis can be taken as the starting point of another general and important procedure called scaling analysis, that we consider here, briefly. The specific goal of a scaling analysis is often to identify small terms in a model equation so that an approximate solution can be sought. A scaling analysis can be done without going through the three step procedure of a dimensional analysis touted here. However, one way to think about the method of a scaling analysis is that it seeks a nondimensional basis set that reflects a meaningful 'zero order' solution of just the sort that we have already had occasion to discuss in Sec. 5.2. A dimensional analysis can be performed with no thought given to scaling *per se*, and we have done that, too. However, the results of a dimensional analysis will often be much more useful if aspects of a scaling analysis are considered either from the outset, or at the third, interpretive step of a dimensional analysis, the main point that we hope to make here.

7.1 A nonlinear projectile problem

To illustrate the purpose and the elementary methods of a scaling analysis we follow the nonlinear projectile problem that was treated in detail by Lin and Segel.[3] The problem is to calculate the motion of a projectile of mass m that is launched upwards with speed V from the surface of a planet having a radius R and mass M. The only external force on the projectile is presumed to be gravitational mass attraction to the planet. To make the problem interesting for our purpose, the variation of the gravitational acceleration with height above the surface, z, is considered (but the very important effect of drag with the air will be ignored). The equation of motion for the projectile would then be

$$m\frac{d^2z}{dt^2} = \frac{-mMG}{(R+z)^2},$$

[21]Two questions for you. 1) Where is the steady state evident in Fig. (6)? Can you interpret the steady solution, i.e., describe the stress profile and the momentum balance? Suppose that the lower boundary was one of free-slip, which in practice means no gradient normal to the surface, what then? Suppose that the upper boundary condition was an imposed stress, which implies that the gradient normal to the boundary is imposed rather than the speed as here, what then? 2) The thickness of a classical layer grows like $t^{1/2}$ until the boundary layer hits the bottom. On the other hand, we know that wind has been exerting as a stress on the ocean surface for a very long time, and yet the surface boundary layer of the ocean, often called the Ekman layer, has a finite depth, very roughly, 20 m. The ocean surface layer is evidently not a classical boundary layer of the type considered here. One important reason for the difference is that the ocean (outside of equatorial regions) is rotating along with the Earth, at a rate (midlatitude) $f = 10^{-4}$ sec^{-1}, noted already in Section 5.2.2. The rotation rate f is thus an additional variable that should appear in a physical model of the ocean surface boundary layer. Under the assumption that the fluid thickness L is not relevant, can you use dimensional analysis to show that a steady state boundary layer thickness could be possible (dimensionally)? Assuming that the diffusivity is *roughly* $\nu = 10^{-2}$ m^2 sec^{-1}, can you make a correspondingly rough estimate of the boundary layer thickness?

where G is the universal gravitational constant. The acceleration of gravity on the planet's surface can be defined as $g = MG/R^2$, a parameter, and the equation of motion rewritten

$$\frac{d^2z}{dt^2} = \frac{-g}{(1 + z/R)^2}. \tag{76}$$

Suitable initial conditions are

$$z(t = 0) = 0 \quad \text{and} \quad \frac{dz}{dt}(t = 0) = V. \tag{77}$$

As the height of the projectile becomes comparable to R, the local gravitational acceleration thus decreases. For a sufficiently large z the projectile could escape the gravitational tug of the planet altogether and continue into deep space. Clearly this is going to have something to do with the parameters g, R and V. To see just how, we will analyze a nondimensional form of the model equations (77) and (78), beginning with

- A physical model of projectile motion in variable gravity:

1. height above the planet surface, $z \doteq l^1$, the dependent variable,
2. time, $t \doteq t^1$, an independent variable,
3. the acceleration of gravity on the planet surface, $g \doteq l^1\, t^{-2}$, a parameter,
4. radius of the planet, $R \doteq l^1$, a parameter,
5. initial (vertical) speed, $V \doteq l^1\, t^{-1}$ a parameter.

For this arbitrary ordering of variables in the physical model, the initial basis set of nondimensional variables comes out to be

$$\Pi_1 = \frac{z}{R}, \quad \Pi_2 = \frac{t}{R/V} \quad \text{and} \quad \Pi_3 = \frac{V^2}{gR}, \tag{78}$$

and the relation between these three nondimensional variables could be written

$$\frac{z}{R} = F(\frac{t}{R/V}, \frac{V^2}{gR}). \tag{79}$$

It will be helpful to denote these particular nondimensional variables with a $(\ )'$, i.e.,

$$z' = \frac{z}{R}, \quad \text{and} \quad t' = \frac{t}{R/V}$$

and the combination

$$\epsilon = \frac{V^2}{gR}.$$

With this notation Eq.(79) is

$$z' = z'(t',\ \epsilon).$$

The maximum height that the projectile will reach, say Z, is of special interest, and from the above we can see that the nondimensional form corresponding to this initial basis set will be

$$\frac{Z}{R} = F(\frac{V^2}{gR}), \quad \text{or} \tag{80}$$

$$Z' = Z'(\epsilon). \tag{81}$$

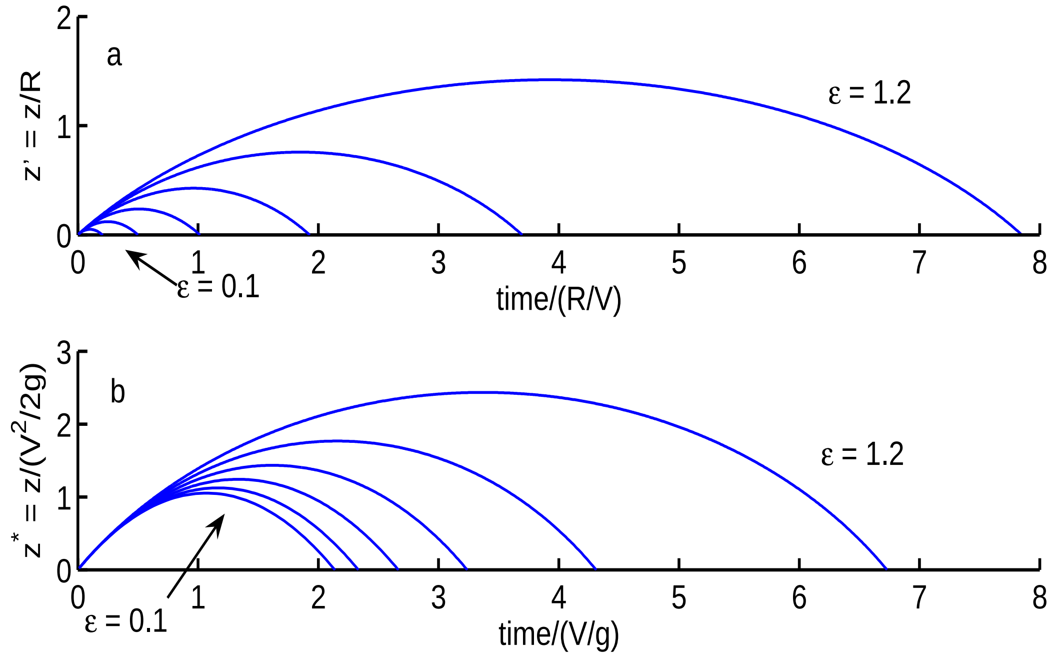

Figure 8: Projectile height computed numerically for several ϵ varying from 0.09 to 1.19. The solutions have been nondimensionalized in one of two ways; in (a) using the initial basis set, Eqs. (79) and in (b) using a second basis set built around a zero order solution described in Section 7.3. Notice that for small values of ϵ the set of curves in (b) appear to collapse toward one curve, while the set of curves in (a) do not.

Notice that $\epsilon = \frac{V^2}{gR}$ is the only nondimensional parameter in the relation for maximum height, i.e., g, R and V will appear only in this combination. The parameter ϵ is the ratio of twice the kinetic energy of the projectile to the depth of the potential energy well of the planet evaluated on the planet's surface. For larger ϵ we would expect a larger maximum height, and when $\epsilon \geq 2$, the kinetic energy exceeds the work required to climb out of the potential energy well. V is then said to be the escape velocity, approx. 12 km s^{-1} for Earth (recall that air drag was ignored). Thus, in the usual, partial way of dimensional analysis, we have already learned something useful about this problem.

Numerical solutions of Eqs. (76) and (77) nondimensionalized by this basis set look entirely reasonable (Figs. 8 and 9); Z' is a well-defined function of the nondimensional parameter ϵ. [22] This first basis set thus serves one of the main goals of dimensional analysis - to make a compact and useful presentation of what would otherwise be an unwieldy mass of data (imagine plotting the maximum height as a function of the three relevant dimensional variables).

[22] This is a mathematical certainty since these 'data' are solutions of a numerical model whose parameters we know exactly and hence the physical model is exactly consistent. Suppose, though, that these data did *not* collapse to a single, well-defined curve — what could you infer? In contrast to these numerical data, the drag coefficient estimates that were compiled to make Fig. 5 came from observations of physical systems. The collapse of those data to a well-defined function of the Reynolds number is a result of considerable physical (and practical) significance.

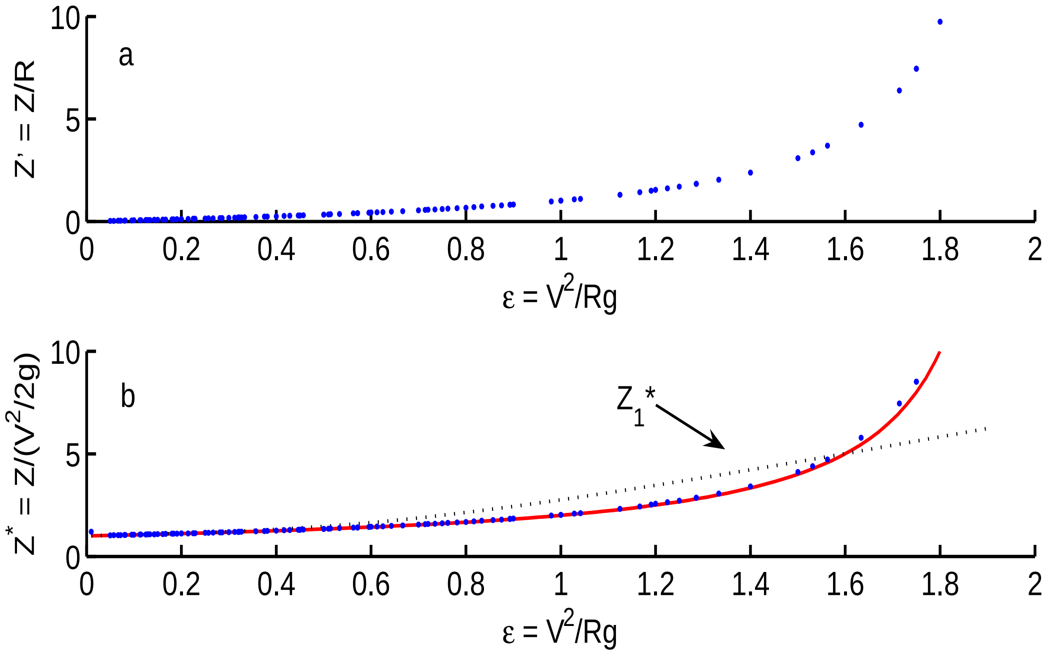

Figure 9: The maximum height of a projectile launched upwards at various speeds V from planets with various R and g. Each dot is the maximum height diagnosed from a numerical solution. In (a), the maximum height has been nondimensionalized by R, consistent with the initial basis set of nondimensional variables, Eq. (79). In (b), the basis set is one deduced from a scaling analysis (Section 7.3). Both of these indicate a clear-cut dependence of maximum height upon the single nondimensional parameter $\epsilon = V^2/gR$. Notice that as ϵ approaches 2 the nondimensional height goes to infinity, indicating that the projectile has escaped the tug of gravity. The dotted line in (b) is the maximum height of the first order solution, Z_1^*, and the solid line that runs nearly through the data points is an approximate solution in which the effective gravitational acceleration is taken to be $g(1-\epsilon/2)$; both of these approximate solutions are developed in Section 7.4.

7.2 Small parameter → small term?

An aspect of nondimensionalization that goes beyond our previous discussions is the question of how or whether a nondimensional model equation can be used to develop an approximate solution. The basis of approximation considered here is that one or more of the most difficult terms of an equation might be dropped to yield a solvable problem. Once an approximate solution is at hand, then the equation can be iterated to arrive at successively better solutions that take account of the term dropped on the first pass.

Our initial basis set amounts to nondimensionalizing the projectile height by the radius of the planet, R, and nondimensionalizing time by the time interval required to move the distance R at a rate V. For the purpose of writing the model equation in nondimensional we will use that $z' = z/R$ and $t' = t/(R/V)$ and hence the nondimensional velocity of the projectile is

$$\frac{dz'}{dt'} = \frac{dz/R}{dt/(R/V)} = \frac{1}{V}\frac{dz}{dt}, \tag{82}$$

and the acceleration is

$$\frac{d^2z'}{dt'^2} = \frac{d^2z/R}{d(t/(R/V))^2} = \frac{R}{V^2}\frac{d^2z}{dt^2}. \tag{83}$$

If we rewrite the equation of motion, Eq. (76), using these nondimensional variables the result is

$$\epsilon \frac{d^2 z'}{dt'^2} = -\frac{1}{(1+z')^2} \tag{84}$$

and the ICs are just

$$z'(t=0) = 0 \quad \text{and} \quad \frac{dz'}{dt}(t=0) = 1. \tag{85}$$

The nondimensional equation of motion contains the single parameter, ϵ. For Earth-like values of R and g, and for V = 2000 m s^{-1} or less, say, ϵ is a small parameter, roughly 10^{-2}.

In this problem, the difficult, nonlinear, term is the z-dependent, gravitational acceleration term of Eq. (78). It is plausible that the z-dependence could be ignored if $z \ll R$, and by extension, if $\epsilon \ll 1$. Thus, we should be able to solve the problem in the limit that $\epsilon \to 0$ and then go on to find a better solution by iteration. On that basis we might guess that a first approximation to Eq. (84) can be obtained by dropping the term multiplied by the small parameter ϵ, which happens to be the acceleration term. However, the solution to the reduced equation, $0 = 1/(1+z')^2$, is $z' = \infty$, which is contrary to our assumption of small z', and is nonsensical, generally. Either the idea that we could find a useful approximation by starting from small ϵ is wrong, or, we erred in dropping the acceleration term. In fact, it was the latter step that failed; there was no reason to conclude that the acceleration term could be dropped simply because it is multiplied by the small parameter ϵ because we have no idea how big the nondimensional acceleration $\frac{d^2z'}{dt'^2}$ is compared with the terms kept, i.e., compared with 1. It turns out that $\frac{d^2z'}{dt'^2} \gg 1$ for $\epsilon \to 0$. As a result, if we drop the acceleration term we will not be able to proceed toward an improved solution. It is important to understand that the nondimensional Eq. (84) is not at fault here; the terms of Eq. (84) still have the ratio one to another of the dimensional Eq. (76) since all that has been done is to divide by parameters. It is only the inferences that might be drawn from Eq. (84) that could be at fault.

If we intend to estimate the relative size of terms that are multiplied by nondimensional parameters, then we have to take care that the nondimensional variables, in Eq.(84) z' and $\frac{d^2z'}{dt'^2}$, will be O(1) in the limit that the small parameter, ϵ, goes to zero (the relevant limit here).[23] In the first nondimensional basis set considered for this problem we used R as the length scale for the height of the projectile. Though R is certainly an important length scale in this problem, it nevertheless has no direct relation to the maximum size of z and clearly does not have the small ϵ dependence of Z that we now seek. The imposition of this additional requirement on the choice of the scales is often said to comprise a 'scaling' analysis, implying a purposeful and thoughtful choice of the basis set of nondimensional variables.

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[23]In Section 5 we used the 'big O' notation to indicate the order of magnitude of a numerical value, e.g., O(1) or O(10^2), say. Here the meaning is extended to indicate the asymptotic behavior of a function. When we say that a function $f(\epsilon)$ is O(1) we have in mind a limit, $\epsilon \to 0$, and that the function $f(\epsilon) \to C$ in that limit. The constant C need not be 1, though we have taken care that it will be here. If $f(\epsilon)/\epsilon^n \to C$, then $f(\epsilon)$ is said to be O(ϵ^n) in that limit. There are other possible gauges against which to measure asymptotic behavior besides the simple power law dependence that will suffice here. For more detail on the big O notation see http://encyclopedia.thefreedictionary.com/Big%20O%20notation A few questions: Based upon what you can see in Figs. 9a and 9b, how would you characterize the order of the functions $Z'(\epsilon)$ and $Z^*(\epsilon)$, i.e., are they order ϵ^{-1}, ϵ^0 or ϵ? What is the order of d^2z'/dt'^2, and how does this impact the inferences we might draw from Eq. (84)?

7.3 Scaling the dependent variable

To insure that a nondimensional dependent variable is O(1) as $\epsilon \rightarrow 0$ we have to select a scale that is consistent with a physically motivated, even if highly simplified, model solution in that limit. We can form such a 'zero order' model for the dimensional height z by ignoring the height dependence of the gravitational acceleration in Eq. (76),

$$\frac{d^2 z_0}{dt^2} = -g, \tag{86}$$

where $(\,)_0$ refers to the order, which we will formalize shortly. The ICs are exactly as before

$$z_0(t=0) = 0 \text{ and } \frac{dz_0}{dt}(t=0) = V, \tag{87}$$

and the solution is

$$z_0 = Vt - gt^2/2. \tag{88}$$

The maximum height is $V^2/2g$, which is the appropriate length scale for the height, z, if we want to insure that the maximum of the nondimensional height ≈ 1 in the range of small ϵ. An appropriate time scale is the time it takes the acceleration g to produce (or erase) the initial velocity V, and is V/g. This new choice of scales amounts to a new basis set of nondimensional variables that is derivable from Eq. (81) when Π_1 and Π_2 are multiplied by Π_3^{-1}, or, reusing the Πs,

$$\Pi_1 = \frac{z}{V^2/2g}, \quad \Pi_2 = \frac{t}{V/g} \text{ and } \Pi_3 = \frac{V^2}{gR}. \tag{89}$$

The relation between these three nondimensional variables can be written

$$\frac{z}{V^2/2g} = F(\frac{t}{V/g}, \frac{V^2}{gR}), \tag{90}$$

or using a $(\,)^*$ to denote these nondimensional variables,

$$z^* = F(t^*, \epsilon)$$

and the nondimensional maximum height is then

$$\frac{Z}{V^2/2g} = F(\frac{V^2}{gR}), \text{ or} \tag{91}$$

$$Z^* = Z^*(\epsilon), \tag{92}$$

where $\epsilon = V^2/gR$, as before in Eq. (81). This second basis sets leads to a second interpretation, that ϵ is proportional to the ratio of the zero order maximum height to the radius of the planet.

The result of using this new basis set is another clearly defined functional dependence of maximum height upon the single parameter ϵ (Fig. 9b). This $Z^*(\epsilon)$ happens to look something like the initial form, $Z'(\epsilon)$, though with one significant difference. At small values of ϵ the new $Z^*(\epsilon)$ goes to a constant, 1, where the initial $Z'(\epsilon)$ decreased to zero as $\epsilon \rightarrow 0$. Thus the new scaling, i.e., the new basis set, is consistent with an underlying zero order solution in the limit of vanishing ϵ, where the initial basis set was evidently not. This new basis set of nondimensional variables seems an obvious choice now that we have it in front of us, but then the first basis set seemed quite sensible as well.

7.4 Approximate and iterated solutions

We have now chosen length and time scales that are appropriate specifically to the vertical motion of a free projectile in a constant gravitational field, as opposed to just any old length and time scales that may happen to be present in the physical model. Using this new basis set, and that

$$z^* = \frac{z}{V^2/2g}, \quad \text{and} \quad t' = \frac{t}{V/g}$$

are the nondimensional height and time, the nondimensional velocity of the projectile is

$$\frac{dz*}{dt*} = \frac{dz/(V^2/2g)}{dt/(V/g)} = \frac{2}{V}\frac{dz}{dt}, \tag{93}$$

and the acceleration is

$$\frac{d^2z^*}{dt^{*2}} = \frac{d^2z/(V^2/2g)}{d(t/(V/g))^2} = \frac{2}{g}\frac{d^2z}{dt^2}. \tag{94}$$

When ϵ is small, the dimensional acceleration is approximately equal to g and $\frac{d^2z^*}{dt^{*2}}$ is O(1). The same is true for the nondimensional velocity; despite that the velocity goes to zero at the top of the trajectory we still say that dz^*/dt^* is O(1), because our concern is with the largest magnitude of a term, rather than it's average or smallest value. If we rewrite the equation of motion Eq. (84) using these nondimensional variables, the result is

$$\frac{d^2z^*}{dt^{*2}} = \frac{-2}{(1+\epsilon z^*/2)^2} \tag{95}$$

and the initial condition is

$$z^*(t=0) = 0 \quad \text{and} \quad \frac{dz^*}{dt}(t=0) = 2. \tag{96}$$

It is very helpful to expand the righthand side of Eq. (95) using a binomial expansion,

$$\frac{d^2z^*}{dt^{*2}} = -2 + 2\epsilon z^* - \frac{3}{2}\epsilon^2 z^{*2} + \epsilon^3 z^{*3} + HOT, \tag{97}$$

that will converge for $\epsilon z^*/2 \leq 1$. Because we have taken care to nondimensionalize the height and time with appropriate scales, the size of each of the terms can be told by the exponent on the parameter ϵ. The acceleration term and the first term on the righthand side are independent of ϵ or O(ϵ^0), which is to say O(1) (a factor 2 notwithstanding). The second term on the righthand side is O(ϵ), the third term is O(ϵ^2), and so on, and the HOT is the sum of all the terms that are higher order in ϵ. When we refer to the order of a model, we mean the highest exponent of ϵ in the terms retained in the expansion.

When we drop the terms multiplied by ϵ we recover a useful first approximation to the projectile problem,

$$\frac{d^2z_0^*}{dt^{*2}} = -2,$$

which is the 'zero order' model Eq. (88) written in nondimensional form. The ICs are exactly as Eq. (96) and the solution is

$$z_0^* = 2t^* - t^{*2}. \tag{98}$$

It seems that we have gone in a circle, but with Eq. (97) in hand we know how to proceed toward an improved solution. We can use this zero order solution as an estimate of z^* in the first order model (in which all terms higher order than ϵ^2 are omitted),

$$\frac{d^2 z_1^*}{dt^{*2}} = -2 + 2\epsilon z^* \quad \approx \quad -2 + 2\epsilon z_0^* \tag{99}$$

$$= \quad -2 + 2\epsilon(2t^* - t^{*2}), \tag{100}$$

which is easily integrable. The ICs do not involve ϵ, and the ICs of the first order model are Eq. (96) (the initial velocity is satisfied by the zero order solution alone). The first order solution is easily found to be

$$z_1^* = 2t^* - t^{*2} + \epsilon(\frac{2}{3}t^{*3} - \frac{1}{6}t^{*4}). \tag{101}$$

Compared to the zero order solution, which appears here as the first and second terms on the right hand side, the new term is O(ϵ) and may thus be regarded as a correction to the zero order solution. The consequence of this new term is not particularly transparent, but for small ϵ so that t^* is O(1), we can tell that it will indicate an increase in height compared to the zero order solution. The maximum nondimensional height evaluated from Eq. (101), Z_1^*, shown as the dotted line in Fig. 9b, compares well with Z^* diagnosed from numerical solutions in the range $0 \leq \epsilon \leq 1/2$, and then begins to diverge from the numerical solution for larger ϵ, first above then below. Thus the first order solution represents approximately the effect of decreasing gravitational attraction with height above the planet surface.[24]

An improved solution can be inferred by noting that the average value of z_0^* is $\approx 1/2$, and from Eq. (95) we might guess that the height-dependent gravitational term in Eq. (100) could be estimated as the nominal gravity reduced by the factor $1 - \epsilon\frac{1}{2}$. This has the correct asymptotic behavior, i.e., it goes to 1 as ϵ vanishes, and it has the right qualitative behavior at $\epsilon = 2$ when the projectile will escape into deep space. The solution for the nondimensional maximum height is then $Z^\dagger = 1/(1 - \epsilon/2)$, the solid line of Fig. 9b, which compares well to Z^* diagnosed from the numerical solutions from the full model up to $\epsilon = 2$.[25]

Here and frequently, it happens that the first order model gives valuable insight into the parameter dependence and the physics of a phenomenon in a way that numerical solutions, no matter how extensive and

[24]This iteration procedure is a simple and intuitive kind of perturbation analysis known as 'successive approximations' combined with an expansion in the small variable ϵ. Perturbation methods are powerful and very important techniques that are far from the main theme here. Excellent references are Ch. 7 of Lin and Segel,[3] and a very clear and concise text by J. G. Simmonds and J. E. Mann, 'A First Look at Perturbation Theory' (Dover Pub., 1986).

[25]Note, though, that if we did not have the numerical solution to bolster our confidence in the first order solution and the informal (but effective) guess at the solution that we just made, then we would probably want to proceed at least one further step in the iteration to verify that the iterated solution does indeed converge. Two problems for you to consider: 1) Add the zero order solution for maximum height on to Fig. 9b so that we will have Z_0^* and Z_1^* to compare with Z^*. How does the evident convergence of the iterated solution for maximum height compare qualitatively with the convergence of terms in a binomial expansion? How does the guessed solution, $Z^\dagger$, behave when $\epsilon \geq 2$, which is an important and realizable case. 2) The nonlinear pendulum, Eq. (2), is amenable to this kind of analysis; solve the first order model for initial displacements 0.1, 1 and 2 radians and estimate the period of the nonlinear pendulum (by eye is sufficient). How does your result compare with the numerical and experimental data of Figs. 2a and 3a? Can you explain the change in the pendulum's period? If your method is an expansion of the sort shown here, your first order solution will give a good account of the change in the period, but will be in error by showing that the oscillation amplitude increases linearly in time. This spurious secular dependence can be cured by a more sophisticated perturbation analysis that seeks consistency of the solution at long times, see Simmonds and Mann[24] for an example.

precise, may not. Some of this we can credit to the mathematical formalism, especially dimensional analysis and scaling analysis, but insight comes mainly from careful, critical thinking about the phenomenon and it's mathematical representation.

8 Summary and closing remarks

The goal of this essay has been to motivate the use of dimensional analysis and to present a method of dimensional analysis that is systematic, objective and quick to implement. The first, physical step of a dimensional analysis is to define a model for a specific dependent variable. For the purpose of a dimensional analysis, the only thing required is a list of the variables and parameters that define the model; if a parameter would appear in a mathematical model of the dependent variable, then it has to appear in this list, which is then called the physical model. The second, mathematical step of a dimensional analysis is the calculation of a null space basis of the corresponding dimensional matrix. This step is readily automated.[4] Nondimensional variables correspond one-for-one with vectors of the null space basis, and in most cases their detailed form is not determined by dimensional analysis alone. The third, interpretive step is to choose an optimal form for the basis set. One very useful strategy is to construct a basis set so that the dependent variable is nondimensionalized by the solution of a simplified, physically motivated model.

The mathematical step of a dimensional analysis is certain and quick, and the physical model is a finite list of variables. The ease with which a dimensional analysis can be done might engender confidence that the procedure is without risk of error. When dimensional analysis is applied to a mathematical or numerical model, that may well be true. But when dimensional analysis is meant to describe a real, physical system, that is not the case. Though the mathematical analysis is certain, it remains that the definition of an appropriate physical model is seldom as straightforward as the examples here might suggest. The absolute requirement that the physical model be complete is always at odds with the practical need to keep the physical model concise. The success of a dimensional analysis depends upon finding a satisfactory compromise; this requires judgment that comes with experience and from continual reference to relevant observations and numerical integrations.

A claim was made in Section 1 that dimensional analysis was applicable to virtually any data set or model and that it was sometimes quite powerful. With some experience we can see that dimensional analysis is most useful, indeed, almost indispensable, in cases wherein a mathematical model is either not known in detail or cannot be solved usefully. In the case that there are only three or four nondimensional variables in a problem, dimensional analysis can lead most of the way to a solution, e.g., the inviscid pendulum of Section 3, or an efficient way to correlate a large data set, e.g., the drag coefficient on a moving sphere of Section 5.2. When there is a larger number of dimensional variables, as commonly occurs, dimensional analysis is still quite valuable as an adjunct to other analysis methods.

The benefits of dimensional analysis can be characterized as data compression and model clarification. By means of a dimensional analysis we can distill a large mass of data down to something that is more manageable and useful. Even if we never expected to generate a mass of new data, we will almost certainly have occasion to use historical data compilations, e.g., the drag coefficient data of Section 5, and to understand the scope and limitations it is necessary to understand the assumptions behind the analysis. By writing a model equation in nondimensional variables we cast the equation in its simplest form that might then be recognized as a canonical form, e.g., the diffusion equation of Section 6.2. If a dimensional analysis is combined with a scale analysis, as in Section 7, we can identify reliably the comparative size of terms in a model equation, and from there go on to find useful, approximate solutions. An equation written in nondimensional variables, for example Eq. (17), is more efficient of parameters than its dimensional counterpart, Eq. (5), but it is also more abstract. An equation written in nondimensional variables must be accompanied by a definition of the nondimensional variables and where possible, a brief explanation of just why a particular basis set of nondimensional variables was chosen. The thoughtful use of dimensional analysis is a hallmark of insightful analysis, but the cavalier use of nondimensional variables can easily obscure what might otherwise have been a valuable message.

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Resource: Topics in Fluid Dynamics
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