CHAPTER 5

Integrals

5.1 The Idea of the Integral

This chapter is about the ***idea*** of integration, and also about the ***technique*** of integration. We explain how it is done *in principle*, and then how it is done *in practice*. Integration is a problem of adding up infinitely many things, each of which is infinitesimally small. Doing the addition is not recommended. The whole point of calculus is to offer a better way.

The problem of integration is to find a limit of sums. The key is to work backward from a limit of differences (which is the derivative). ***We can integrate*** $v(x)$ ***if it turns up as the derivative of another function*** $f(x)$. The integral of $v = \cos x$ is $f = \sin x$. The integral of $v = x$ is $f = \frac{1}{2}x^2$. Basically, $f(x)$ is an "***anti****derivative*". The list of f's will grow much longer (Section 5.4 is crucial). A selection is inside the cover of this book. If we don't find a suitable $f(x)$, numerical integration can still give an excellent answer.

I could go directly to the formulas for integrals, which allow you to compute areas under the most amazing curves. (Area is the clearest example of adding up infinitely many infinitely thin rectangles, so it always comes first. It is certainly not the only problem that integral calculus can solve.) But I am really unwilling just to write down formulas, and skip over all the ideas. Newton and Leibniz had an absolutely brilliant intuition, and there is no reason why we can't share it.

They started with something simple. We will do the same.

SUMS AND DIFFERENCES

Integrals and derivatives can be mostly explained by working (very briefly) with sums and differences. Instead of functions, we have n ordinary numbers. The key idea is nothing more than a basic fact of algebra. In the limit as $n \to \infty$, it becomes the basic fact of calculus. The step of "going to the limit" is the essential difference between algebra and calculus! It has to be taken, in order to add up infinitely many infinitesimals—but we start out this side of it.

To see what happens before the limiting step, we need ***two sets of*** n ***numbers***. The first set will be $v_1, v_2, \ldots, v_n$, where v suggests velocity. The second set of numbers will be $f_1, f_2, \ldots, f_n$, where f recalls the idea of distance. You might think d would be a better symbol for distance, but that is needed for the dx and dy of calculus.

A first example has $n = 4$:

$$v_1, v_2, v_3, v_4 = 1, 2, 3, 4 \qquad f_1, f_2, f_3, f_4 = 1, 3, 6, 10.$$

The relation between the v's and f's is seen in that example. When you are given $1, 3, 6, 10$, how do you produce $1, 2, 3, 4$? ***By taking differences***. The difference between 10 and 6 is 4. Subtracting $6 - 3$ is 3. The difference $f_2 - f_1 = 3 - 1$ is $v_2 = 2$. Each v is the difference between two f's:

$$v_j \textit{ is the difference } f_j - f_{j-1}.$$

This is the discrete form of the derivative. I admit to a small difficulty at $j = 1$, from the fact that there is no f_0. The first v should be $f_1 - f_0$, and the natural idea is to agree that f_0 is zero. This need for a starting point will come back to haunt us (or help us) in calculus.

Now look again at those same numbers—but start with v. From $v = 1, 2, 3, 4$ how do you produce $f = 1, 3, 6, 10$? ***By taking sums***. The first two v's add to 3, which is f_2. The first three v's add to $f_3 = 6$. The sum of all four v's is $1 + 2 + 3 + 4 = 10$. ***Taking sums is the opposite of taking differences***.

That idea from algebra is the key to calculus. The sum f_j involves all the numbers $v_1 + v_2 + \cdots + v_j$. The difference v_j involves only the *two* numbers $f_j - f_{j-1}$. The fact that one reverses the other is the "Fundamental Theorem." Calculus will change sums to integrals and differences to derivatives—but why not let the key idea come through now?

5A ***Fundamental Theorem of Calculus*** (before limits):

$$\textit{If each } v_j = f_j - f_{j-1}, \textit{ then } v_1 + v_2 + \cdots + v_n = f_n - f_0.$$

The differences of the f's add up to $f_n - f_0$. All f's in between are canceled, leaving only the last f_n and the starting f_0. *The sum "telescopes"*:

$$v_1 + v_2 + v_3 + \cdots + v_n = (f_1 - f_0) + (f_2 - f_1) + (f_3 - f_2) + \cdots + (f_n - f_{n-1}).$$

The number f_1 is canceled by $-f_1$. Similarly $-f_2$ cancels f_2 and $-f_3$ cancels f_3. Eventually f_n and $-f_0$ are left. When f_0 is zero, the sum is the final f_n.

That completes the algebra. ***We add the*** v's ***by finding the*** f's.

Question How do you add the odd numbers $1 + 3 + 5 + \cdots + 99$ (the v's)?
Answer They are the differences between $0, 1, 4, 9, \ldots$. These f's are squares. By the Fundamental Theorem, the sum of 50 odd numbers is $(50)^2$.

The tricky part is to discover the right f's! Their differences must produce the v's. In calculus, the tricky part is to find the right $f(x)$. Its derivative must produce $v(x)$. It is remarkable how often f can be found—more often for integrals than for sums. Our next step is to understand how ***the integral is a limit of sums***.

SUMS APPROACH INTEGRALS

Suppose you start a successful company. The rate of income is increasing. After x years, the income per year is $\sqrt{x}$ million dollars. In the first four years you reach $\sqrt{1}, \sqrt{2}, \sqrt{3}$, and $\sqrt{4}$ million dollars. Those numbers are displayed in a bar graph (Figure 5.1a, for investors). I realize that most start-up companies make losses, but your company is an exception. If the example is too good to be true, please keep reading.

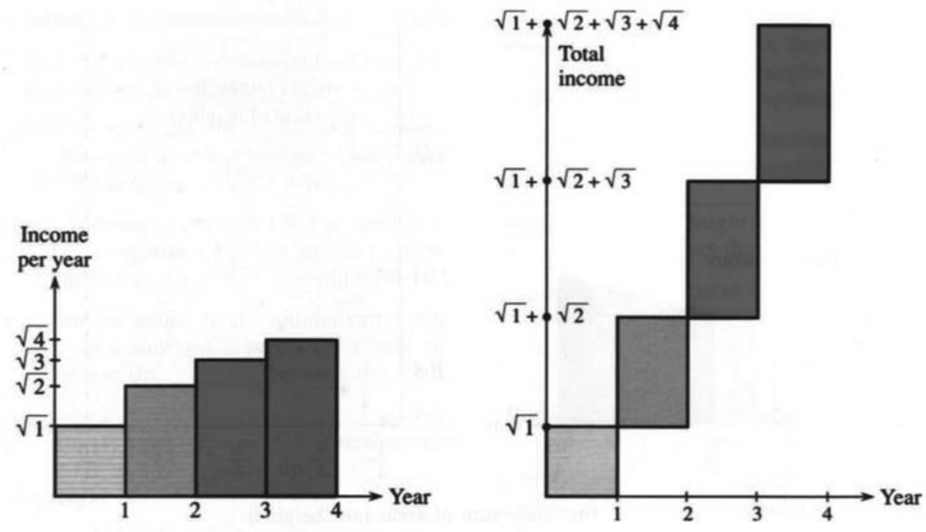


Fig. 5.1 Total income = total area of rectangles = 6.15.

The graph shows four rectangles, of heights $\sqrt{1}, \sqrt{2}, \sqrt{3}, \sqrt{4}$. Since the base of each rectangle is one year, those numbers are also the ***areas*** of the rectangles. One investor, possibly weak in arithmetic, asks a simple question: *What is the total income for all four years*? There are two ways to answer, and I will give both.

The first answer is $\sqrt{1}+\sqrt{2}+\sqrt{3}+\sqrt{4}$. Addition gives 6.15 million dollars. Figure 5.1b shows this total—which is reached at year 4. This is exactly like velocities and distances, but now v is the ***income per year*** and f is the ***total income***. Algebraically, f_j is still $v_1 + \cdots + v_j$.

The second answer comes from geometry. *The* ***total income*** *is the* ***total area*** *of the rectangles*. We are emphasizing the correspondence between ***addition*** and ***area***. That point may seem obvious, but it becomes important when a second investor (smarter than the first) asks a harder question.

Here is the problem. ***The incomes as stated are false***. The company did not make a million dollars the first year. After three months, when x was $1/4$, the rate of income was only $\sqrt{x} = 1/2$. The bar graph showed $\sqrt{1} = 1$ for the whole year, but that was an overstatement. The income in three months was not more than $1/2$ times $1/4$, the rate multiplied by the time.

All other quarters and years were also overstated. Figure 5.2a is closer to reality, with 4 years divided into 16 quarters. It gives a new estimate for total income.

Again there are two ways to find the total. We add $\sqrt{1/4}+\sqrt{2/4}+\cdots+\sqrt{16/4}$, remembering to multiply them all by $1/4$ (because each rate applies to $1/4$ year). This is also the area of the 16 rectangles. The area approach is better because the $1/4$ is automatic. Each rectangle has base $1/4$, so that factor enters each area. The total area is now 5.56 million dollars, closer to the truth.

You see what is coming. The next step divides time into weeks. After one week the rate $\sqrt{x}$ only $\sqrt{1/52}$. That is the height of the first rectangle—its base is $\Delta x = 1/52$. There is a rectangle for every week. Then a hard-working investor divides time into days, and the base of each rectangle is $\Delta x = 1/365$. At that point there are $4 \times 365 = 1460$ rectangles, or 1461 because of leap year, with a total area below $5\frac{1}{2}$

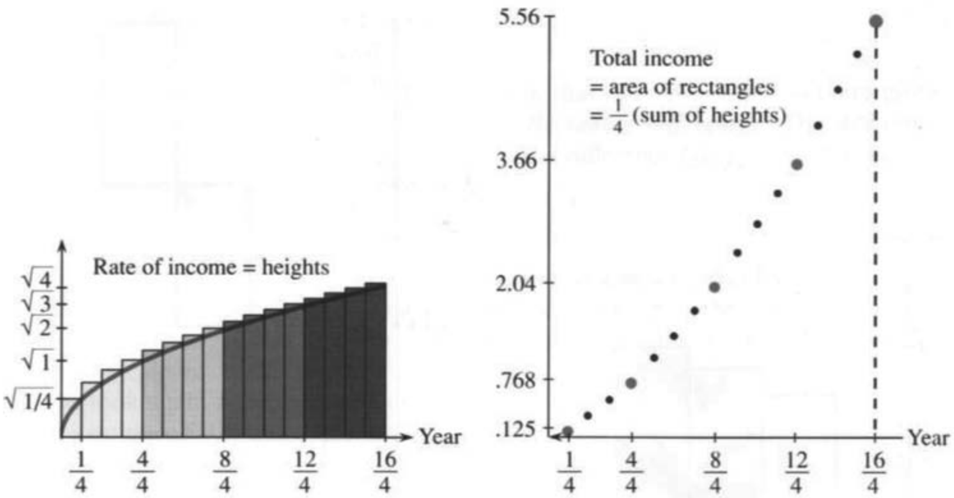


Fig. 5.2 Income = sum of areas (not heights)

$$= \frac{1}{4}\left(\sqrt{\frac{1}{4}}+\sqrt{\frac{2}{4}}+\cdots+\sqrt{\frac{16}{4}}\right)$$

million dollars. The calculation is elementary but depressing—adding up thousands of square roots, each multiplied by Δx from the base. There has to be a better way.

The better way, in fact the best way, is calculus. The whole idea is to allow for ***continuous change***. ***The geometry problem is to find the area under the square root curve***. That question cannot be answered by arithmetic, because it involves a *limit*. The rectangles have base Δx and heights $\sqrt{\Delta x}, \sqrt{\Delta 2x}, \ldots, \sqrt{4}$. There are $4/\Delta x$ rectangles—more and more terms from thinner and thinner rectangles. ***The area is the limit of the sum as*** $\Delta x \to 0$.

This limiting area is the "integral." We are looking for a number below $5\frac{1}{2}$.

Algebra (*area of n rectangles*): Compute $v_1+\cdots+v_n$ by finding f's.
Key idea: If $v_j = f_j - f_{j-1}$, then the sum is $f_n - f_0$.
Calculus (*area under curve*): *Compute the limit of* $\Delta x[v(\Delta x)+v(2\Delta x)+\cdots]$.
Key idea: ***If*** $v(x) = df/dx$ ***then area = integral to be explained next***.

5.1 EXERCISES

Read-through questions

The problem of summation is to add $v_1+\cdots+v_n$. It is solved if we find f's such that $v_j =$ ___a___. Then $v_1+\cdots+v_n$ equals ___b___. The cancellation in $(f_1-f_0)+(f_2-f_1)+\cdots+(f_n-f_{n-1})$ leaves only ___c___. Taking sums is the ___d___ of taking differences.

The differences between $0,1,4,9$ are $v_1, v_2, v_3 =$ ___e___. For $f_j = j^2$ the difference between f_{10}, and f_9 is $v_{10} =$ ___f___. From this pattern $1+3+5+\cdots+19$ equals ___g___.

For functions, finding the integral is the reverse of ___h___. If the derivative of $f(x)$ is $v(x)$, then the ___i___ of $v(x)$ is $f(x)$. If $v(x) = 10x$ then $f(x) =$ ___j___. This is the ___k___ of a triangle with base x and height $10x$.

Integrals begin with sums. The triangle under $v = 10x$ out to $x = 4$ has area ___l___. It is approximated by four rectangles of heights $10, 20, 30, 40$ and area ___m___. It is better approximated by eight rectangles of heights ___n___ and area ___o___. For n rectangles covering the triangle the area is the sum of ___p___. As $n \to \infty$

this sum should approach the number __q__. *That is the integral of* $v = 10x$ *from* 0 *to* 4.

Problems 1–6 are about sums f_j and differences v_j.

1 With $v = 1,2,4,8$, the formula for v_j ______ is (not 2^j). Find f_1, f_2, f_3, f_4, starting from $f_0 = 0$. What is f_7?

2 The same $v = 1,2,4,8,\dots$ are the differences between $f = 1,2,4,8,16,\dots$. Now $f_0 = 1$ and $f_j = 2^j$. (a) Check that $2^5 - 2^4$ equal v_5. (b) What is $1+2+4+8+16$?

3 The differences between $f = 1, 1/2, 1/4, 1/8$ are $v = -1/2, -1/4, -1/8$. These negative v's do not add up to these positive f's. Verify that $v_1 + v_2 + v_3 + v_4 = f_4 - f_0$ is still true.

4 Any constant C can be added to the antiderivative $f(x)$ because the ______ of a constant is zero. Any C can be added to $f_0, f_1, \dots$ because the ______ between the f's is not changed.

5 Show that $f_j = r^j/(r-1)$ has $f_j - f_{j-1} = r^{j-1}$. Therefore the geometric series $1 + r + \cdots + r^{j-1}$ adds up to ______ (remember to subtract f_0).

6 The sums $f_j = (r^j - 1)/(r-1)$ also have $f_j - f_{j-1} = r^{j-1}$. Now $f_0 =$ ______. Therefore $1 + r + \cdots + r^{j-1}$ adds up to f_j. The sum $1 + r + \cdots + r^n$ equals ______.

7 Suppose $v(x) = 3$ for $x < 1$ and $v(x) = 7$ for $x > 1$. Find the area $f(x)$ from 0 to x, under the graph of $v(x)$. (Two pieces.)

8 If $v = 1, -2, 3, -4, \dots$, write down the f's starting from $f_0 = 0$. Find formulas for v_j and f_j when j is odd and j is even.

Problems 9–16 are about the company earning $\sqrt{x}$ per year.

9 When time is divided into weeks there are $4 \times 52 = 208$ rectangles. Write down the first area, the 208th area, and the jth area.

10 How do you know that the sum over 208 weeks is smaller than the sum over 16 quarters?

11 A pessimist would use $\sqrt{x}$ at the *beginning* of each time period as the income rate for that period. Redraw Figure 5.1 (both parts) using heights $\sqrt{0}, \sqrt{1}, \sqrt{2}, \sqrt{3}$. How much lower is the estimate of total income?

12 The same pessimist would redraw Figure 5.2 with heights $0, \sqrt{1/4}, \dots$. What is the height of the last rectangle? How much does this change reduce the total rectangular area 5.56?

13 At every step from years to weeks to days to hours, the pessimist's area goes ______ and the optimist's area goes ______. The difference between them is the area of the last ______.

14 The optimist and pessimist arrive at the same limit as years are divided into weeks, days, hours, seconds. Draw the $\sqrt{x}$ curve between the rectangles to show why the pessimist is always too low and the optimist is too high.

15 (Important) Let $f(x)$ be the area under the $\sqrt{x}$ curve, above the interval from 0 to x. The area to $x + \Delta x$ is $f(x + \Delta x)$. The extra area is $\Delta f =$ ______. This is almost a rectangle with base ______ and height $\sqrt{x}$. So $\Delta f/\Delta x$ is close to ______. As $\Delta x \to 0$ we suspect that $df/dx =$ ______.

16 Draw the $\sqrt{x}$ curve from $x = 0$ to 4 and put triangles below to prove that the area under it is more than 5. Look left and right from the point where $\sqrt{1} = 1$.

Problems 17–22 are about a company whose expense rate $v(x) = 6 - x$ is decreasing.

17 The expenses drop to zero at $x =$ ______. The total expense during those years equals ______. This is the area of ______.

18 The rectangles of heights $6,5,4,3,2,1$ give a total estimated expense of ______. Draw them enclosing the triangle to show why this total is too high.

19 How many rectangles (enclosing the triangle) would you need before their areas are within 1 of the correct triangular area?

20 The accountant uses 2-year intervals and computes $v = 5,3,1$ at the midpoints (the odd-numbered years). What is her estimate, how accurate is it, and why?

21 What is the area $f(x)$ under the line $v(x) = 6 - x$ above the interval from 2 to x? What is the derivative of this $f(x)$?

22 What is the area $f(x)$ under the line $v(x) = 6 - x$ above the interval from x to 6? What is the derivative of this $f(x)$?

23 With $\Delta x = 1/3$, find the area of the three rectangles that enclose the graph of $v(x) = x^2$.

24 Draw graphs of $v = \sqrt{x}$ and $v = x^2$ from 0 to 1. Which areas add to 1? The same is true for $v = x^3$ and $v =$ ______.

25 From x to $x + \Delta x$, the area under $v = x^2$ is Δf. This is almost a rectangle with base Δx and height ______. So $\Delta f/\Delta x$ is close to ______. In the limit we find $df/dx = x^2$ and $f(x) =$ ______.

26 Compute the area of 208 rectangles under $v(x) = \sqrt{x}$ from $x = 0$ to $x = 4$.

5.2 Antiderivatives

The symbol $\int$ was invented by Leibniz to represent the integral. It is a stretched-out **S**, from the Latin word for sum. This symbol is a powerful reminder of the whole construction: ***Sum approaches integral***, **S** ***approaches*** $\int$***, and rectangular area approaches curved area***:

$$\textit{\textbf{curved area}} = \int v(x)\,dx = \int \sqrt{x}\,dx. \tag{1}$$

The rectangles of base Δx lead to this limit—the integral of $\sqrt{x}$. The "dx" indicates that Δx approaches zero. The heights v_j of the rectangles are the heights $v(x)$ of the curve. The sum of v_j times Δx approaches "*the integral of* v *of* $x\,dx$." You can imagine an infinitely thin rectangle above every point, instead of ordinary rectangles above special points.

We now find the area under the square root curve. The "***limits of integration***" are 0 and 4. The lower limit is $x=0$, where the area begins. (*The start could be any point* $x=a$.) The upper limit is $x=4$, since we stop after four years. (*The finish could be any point* $x=b$.) The area of the rectangles is a sum of base Δx times heights $\sqrt{x}$. The curved area is the limit of this sum. ***That limit is the integral of*** $\sqrt{x}$ ***from*** 0 ***to*** 4:

$$\lim_{\Delta x\to 0}\left[(\sqrt{\Delta x})(\Delta x)+(\sqrt{2\Delta x})(\Delta x)+\cdots+(\sqrt{4})(\Delta x)\right]=\int_{x=0}^{x=4}\sqrt{x}\,dx. \tag{2}$$

The outstanding problem of integral calculus is still to be solved. *What is this limiting area*? We have a symbol for the answer, involving $\int$ and $\sqrt{x}$ and dx—but we don't have a number.

THE ANTIDERIVATIVE

I wish I knew who discovered the area under the graph of $\sqrt{x}$. It may have been Newton. The answer was available earlier, but the key idea was shared by Newton and Leibniz. They understood the parallels between sums and integrals, and between differences and derivatives. I can give the answer, by following that analogy. I can't give the proof (yet)—it is the Fundamental Theorem of Calculus.

In algebra the difference $f_j - f_{j-1}$ is v_j. When we add, the sum of the v's is $f_n - f_0$. ***In calculus the derivative of*** $f(x)$ ***is*** $v(x)$. When we integrate, ***the area under the*** $v(x)$ ***curve is*** $f(x)$ ***minus*** $f(0)$. Our problem asks for the area out to $x=4$:

> **5B** (Discrete vs. continuous, rectangles vs. curved areas, addition vs. integration)
> ***The integral of*** $v(x)$ ***is the difference in*** $f(x)$:
>
> $$\textit{\textbf{If}}\quad df/dx=\sqrt{x}\quad \textit{\textbf{then area}} = \int_{x=0}^{x=4}\sqrt{x}\,dx = f(4)-f(0). \tag{3}$$

What is $f(x)$? Instead of the derivative of $\sqrt{x}$, we need its "***antiderivative***." We have to find a function $f(x)$ whose derivative is $\sqrt{x}$. It is the opposite of Chapters $2-4$, and requires us to ***work backwards***. The derivative of x^n is nx^{n-1}—now we need the antiderivative. The quick formula is $f(x)=x^{n+1}/(n+1)$—we aim to understand it.

Solution Since the derivative lowers the exponent, the antiderivative *raises* it. We go from $x^{1/2}$ to $x^{3/2}$. But then the derivative is $(3/2)x^{1/2}$. It contains an unwanted factor $3/2$. *To cancel that factor, put* $2/3$ *into the antiderivative*:

$$f(x) = \tfrac{2}{3}x^{3/2} \textit{ has the required derivative } v(x) = x^{1/2} = \sqrt{x}.$$

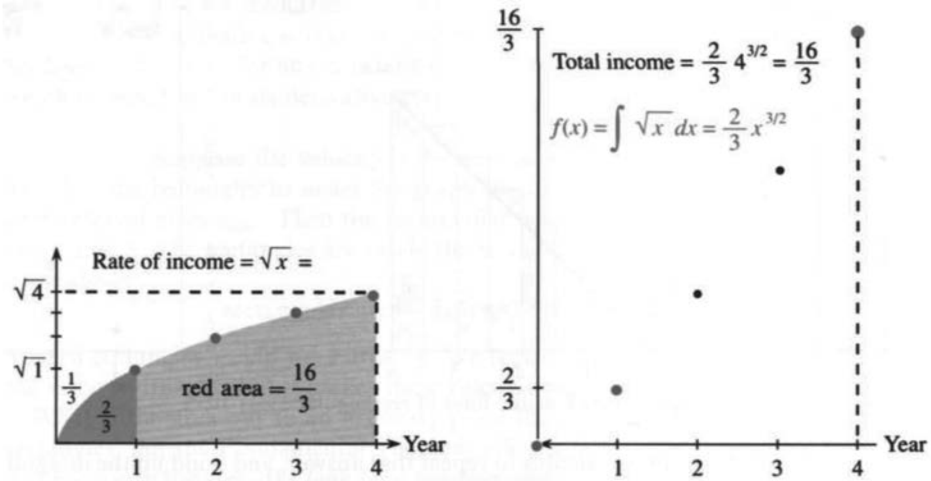


Fig. 5.3 The integral of $v(x) = \sqrt{x}$ is the exact area $16/3$ under the curve.

There you see the key to integrals: Work backward from derivatives (and adjust).

Now comes a number—the exact area. At $x = 4$ we find $x^{3/2} = 8$. Multiply by $2/3$ to get $16/3$. Then subtract $f(0) = 0$:

$$\int_{x=0}^{x=4} \sqrt{x}\,dx = \frac{2}{3}(4)^{3/2} - \frac{2}{3}(0)^{3/2} = \frac{2}{3}(8) = \frac{16}{3} \tag{4}$$

The total income over four years is $16/3 = 5\frac{1}{3}$ ***million dollars***. This is $f(4) - f(0)$. The sum from thousands of rectangles was slowly approaching this exact area $5\frac{1}{3}$.

Other areas The income in the first year, at $x = 1$, is $\frac{2}{3}(1)^{3/2} = \frac{2}{3}$ million dollars. (The false income was 1 million dollars.) The total income after x years is $\frac{2}{3}(x)^{3/2}$, which is the antiderivative $f(x)$. *The square root curve covers* $2/3$ *of the overall rectangle it sits in*. The rectangle goes out to x and up to $\sqrt{x}$, with area $x^{3/2}$, and $2/3$ of that rectangle is below the curve. ($1/3$ is above.)

Other antiderivatives The derivative of x^5 is $5x^4$. Therefore the antiderivative of x^4 is $x^5/5$. Divide by 5 (or $n+1$) to cancel the 5 (or $n+1$) from the derivative. And don't allow $n + 1 = 0$:

$$\textit{The derivative } v(x) = x^n \textit{ has the antiderivative } f(x) = x^{n+1}/(n+1).$$

EXAMPLE 1 The antiderivative of x^2 is $\frac{1}{3}x^3$. This is the area under the parabola $v(x) = x^2$. The area out to $x = 1$ is $\frac{1}{3}(1)^3 - \frac{1}{3}(0)^3$, or $1/3$.

Remark on $\sqrt{x}$ *and* x^2 The $2/3$ from $\sqrt{x}$ and the $1/3$ from x^2 add to 1. Those are the areas below and above the $\sqrt{x}$ curve, in the corner of Figure 5.3. If you turn the curve by $90°$, it becomes the parabola. The functions $y = \sqrt{x}$ and $x = y^2$ are inverses! The areas for these inverse functions add to a square of area 1.

AREA UNDER A STRAIGHT LINE

You already know the area of a triangle. The region is below the diagonal line $v = x$ in Figure 5.4. The base is 4, the height is 4, and the area is $\frac{1}{2}(4)(4) = 8$. Integration is

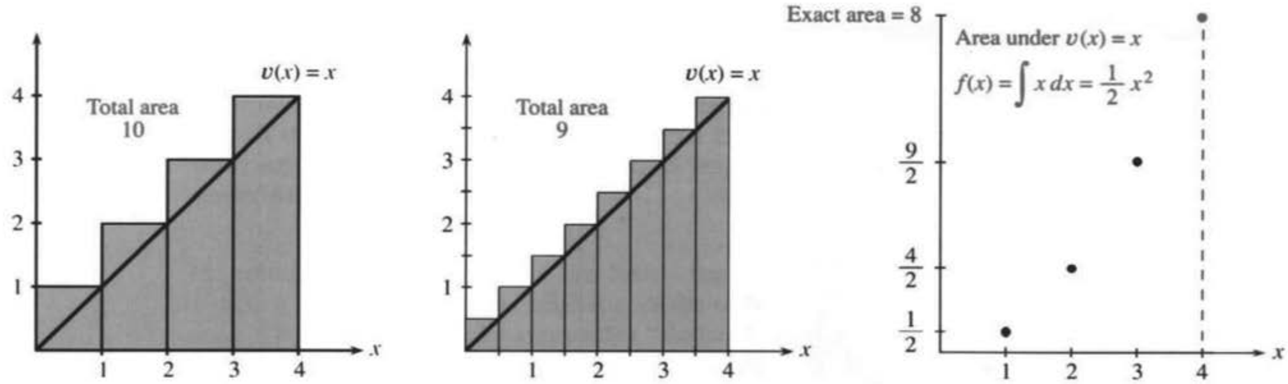


Fig. 5.4 Triangular area 8 as the limit of rectangular areas $10, 9, 8\frac{1}{2}, \ldots$.

not required! But if you allow calculus to repeat that answer, and build up the integral $f(x) = \frac{1}{2}x^2$ as the limiting area of many rectangles, you will have the beginning of something important.

The four rectangles have area $1+2+3+4 = 10$. That is greater than 8, because the triangle is inside. 10 is a first approximation to the triangular area 8, and to improve it we need more rectangles.

The next rectangles will be thinner, of width $\Delta x = 1/2$ instead of the original $\Delta x = 1$. There will be eight rectangles instead of four. They extend above the line, so the answer is still too high. The new heights are $1/2$, 1, $3/2$, 2, $5/2$, 3, $7/2$, 4. The total area in Figure 5.4b is the sum of the base $\Delta x = 1/2$ times those heights:

$$\text{area } = \tfrac{1}{2}\left(\tfrac{1}{2} + 1 + \tfrac{3}{2} + 2 + \cdots + 4\right) = 9 \text{ (which is closer to 8).}$$

Question What is the area of 16 rectangles? Their heights are $\frac{1}{4}, \frac{1}{2}, \ldots, 4$.
Answer With base $\Delta x = \frac{1}{4}$ the area is $\frac{1}{4}\left(\frac{1}{4} + \frac{1}{2} + \cdots + 4\right) = 8\frac{1}{2}$.

The effort of doing the addition is increasing. A formula for the sums is needed, and will be established soon. (The next answer would be $8\frac{1}{4}$.) But more important than the formula is the idea. ***We are carrying out a Iimiting process, one step at a time***. The area of the rectangles is approaching the area of the triangle, as Δx decreases. The same limiting process will apply to other areas, in which the region is much more complicated. Therefore we pause to comment on what is important.

Area Under a Curve

What requirements are imposed on those thinner and thinner rectangles? It is not essential that they all have the same width. And it is not required that they cover the triangle completely. The rectangles could lie *below* the curve. The limiting answer will still be 8, even if the widths Δx are unequal and the rectangles fit inside the triangle or across it. We only impose two rules:

1. The largest width $\Delta x_{\max}$ must approach zero.
2. The top of each rectangle must touch or cross the curve.

The area under the graph is defined to be the limit of these rectangular areas, if that limit exists. For the straight line, the limit does exist and equals 8. That limit is independent of the particular widths and heights—as we absolutely insist it should be.

Section 5.5 allows any continuous $v(x)$. The question will be the same—***Does the limit exist***? The answer will be the same—***Yes***. That limit will be the *integral of* $v(x)$, and it will be the area under the curve. It will be $f(x)$.

EXAMPLE 2 The triangular area from 0 to x is $\frac{1}{2}$(base)(height) $= \frac{1}{2}(x)(x)$. That is $f(x) = \frac{1}{2}x^2$. Its derivative is $v(x) = x$. But notice that $\frac{1}{2}x^2 + 1$ has the *same derivative*. So does $f = \frac{1}{2}x^2 + C$, for any constant C. There is a "***constant of integration***" in $f(x)$, which is wiped out in its derivative $v(x)$.

EXAMPLE 3 Suppose the velocity is decreasing: $v(x) = 4 - x$. If we sample v at $x = 1, 2, 3, 4$, the rectangles lie *under* the graph. Because v is decreasing, the right end of each interval gives $v_{\min}$. Then the rectangular area $3 + 2 + 1 + 0 = 6$ is less than the exact area 8. The rectangles are *inside* the triangle, and eight rectangles with base $\frac{1}{2}$ come closer:

$$\text{rectangular area } = \tfrac{1}{2}(3\tfrac{1}{2} + 3 + \cdots + \tfrac{1}{2} + 0) = 7.$$

Sixteen rectangles would have area $7\frac{1}{2}$. We repeat that the rectangles need not have the same widths Δx, but it makes these calculations easier.

What is the area out to an arbitrary point (like $x = 3$ or $x = 1$)? We could insert rectangles, but the Fundamental Theorem offers a faster way. Any antiderivative of $4 - x$ will give the area. ***We look for a function whose derivative is*** $4 - x$. The derivative of $4x$ is 4, the derivative of $\frac{1}{2}x^2$ is x, so work backward:

$$\text{to achieve } df/dx = 4 - x \text{ choose } f(x) = 4x - \tfrac{1}{2}x^2.$$

Calculus skips past the rectangles and computes $f(3) = 7\frac{1}{2}$. ***The area between*** $x = 1$ ***and*** $x = 3$ ***is the difference*** $7\frac{1}{2} - 3\frac{1}{2} = 4$. In Figure 5.5, this is the area of the trapezoid.

The f-curve flattens out when the v-curve touches zero. No new area is being added.

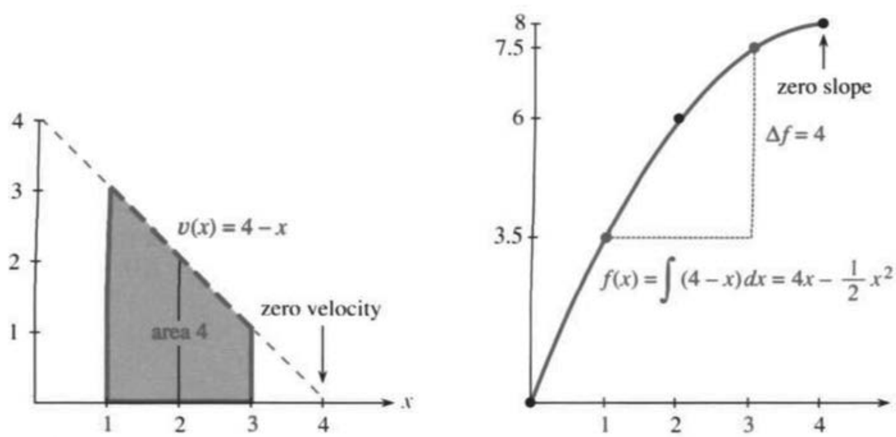


Fig. 5.5 The area is $\Delta f = 7\frac{1}{2} - 3\frac{1}{2} = 4$. Since $v(x)$ decreases, $f(x)$ bends down.

INDEFINITE INTEGRALS AND DEFINITE INTEGRALS

We have to distinguish two different kinds of integrals. They both use the antiderivative $f(x)$. The definite one involves the limits 0 and 4, the indefinite one doesn't:

The ***indefinite integral*** is a ***function*** $f(x)=4x-\frac{1}{2}x^2$.

The ***definite integral*** from $x=0$ to $x=4$ is the ***number*** $f(4)-f(0)$.

The definite integral is definitely 8. But the indefinite integral is not necessarily $4x-\frac{1}{2}x^2$. ***We can change $f(x)$ by a constant without changing its derivative*** (since the derivative of a constant is zero). The following functions are also antiderivatives:

$$f(x)=4x-\tfrac{1}{2}x^2+1, \qquad f(x)=4x-\tfrac{1}{2}x^2-9, \qquad f(x)=4x-\tfrac{1}{2}x^2+C.$$

The first two are particular examples. The last is the general case. The constant C can be anything (including zero), to give all functions with the required derivative. The theory of calculus will show that there are no others. The indefinite integral is the most general antiderivative (with no limits):

$$\textbf{\textit{indefinite integral}}\quad f(x)=\int v(x)\,dx=4x-\tfrac{1}{2}x^2+C. \tag{5}$$

By contrast, the definite integral is a number. It contains no arbitrary constant C. More that that, it contains no variable x. The definite integral is determined by the function $v(x)$ and the limits of integration (also known as the ***endpoints***). It is the area under the graph between those endpoints.

To see the relation of indefinite to definite, answer this question: *What is the definite integral between $x=1$ and $x=3$?* The indefinite integral gives $f(3)=7\frac{1}{2}+C$ and $f(1)=3\frac{1}{2}+C$. To find the area between the limits, ***subtract f at one limit from f at the other limit***:

$$\int_{x=1}^{3} v(x)\,dx=f(3)-f(1)=(7\tfrac{1}{2}+C)-(3\tfrac{1}{2}+C)=4. \tag{6}$$

The constant cancels itself! The definite integral is the *difference* between the values of the indefinite integral. C disappears in the subtraction.

The difference $f(3)-f(1)$ is like f_n-f_0. The sum of v_j from 1 to n has become "***the integral of*** $v(x)$ ***from*** 1 ***to*** 3." Section 5.3 computes other areas from sums, and 5.4 computes many more from antiderivatives. Then we come back to the definite integral and the Fundamental Theorem:

$$\int_a^b v(x)\,dx=\int_a^b \frac{df}{dx}\,dx=f(b)-f(a). \tag{7}$$

5.2 EXERCISES

Read-through questions

Integration yields the ___a___ under a curve $y=v(x)$. It starts from rectangles with base ___b___ and heights $v(x)$ and areas ___c___. As $\Delta x\to 0$ the area $v_1\Delta x+\cdots+v_n\Delta x$ becomes the ___d___ of $v(x)$. The symbol for the indefinite integral of $v(x)$ is ___e___.

The problem of integration is solved if we find $f(x)$ such that ___f___. Then f is the ___g___ of v, and $\int_2^6 v(x)\,dx$ equals ___h___ minus ___i___. The limits of integration are ___j___. This is a ___k___ integral, which is a ___l___ and not a function $f(x)$.

The example $v(x)=x$ has $f(x)=$ ___m___. It also has $f(x)=$ ___n___. The area under $v(x)$ from 2 to 6 is ___o___. The constant is canceled in computing the difference ___p___ minus ___q___. If $v(x)=x^8$ then $f(x)=$ ___r___.

The sum $v_1 + \cdots + v_n = f_n - f_0$ leads to the Fundamental Theorem $\int_a^b v(x)\,dx =$ ___s___. The ___t___ integral is $f(x)$ and the ___u___ integral is $f(b) - f(a)$. Finding the ___v___ under the v-graph is the opposite of finding the ___w___ of the f-graph.

Find an antiderivative $f(x)$ for $v(x)$ in 1–14. Then compute the definite integral $\int_0^1 v(x)\,dx = f(1) - f(0)$.

1 $5x^4 + 4x^5$	**2** $x + 12x^2$
3 $1/\sqrt{x}$ (or $x^{-1/2}$)	**4** $(\sqrt{x})^3$ (or $x^{3/2}$)
5 $x^{1/3} + (2x)^{1/3}$	**6** $x^{1/3}/x^{2/3}$
7 $2 \sin x + \sin 2x$	**8** $\sec^2 x + 1$
9 $x \cos x$ (by experiment)	**10** $x \sin x$ (by experiment)
11 $\sin x \cos x$	**12** $\sin^2 x \cos x$
13 0 (find all f)	**14** -1 (find all f)

15 If $df/dx = v(x)$ then the definite integral of $v(x)$ from a to b is ______. If $f_j - f_{j-1} = v_j$ then the definite sum of $v_3 + \cdots + v_7$ is ______.

16 The areas include a factor Δx, the base of each rectangle. So the sum of v's is multiplied by ______ to approach the integral. The difference of f's is divided by ______ to approach the derivative.

17 The areas of 4, 8, and 16 rectangles were 10, 9, and $8\frac{1}{2}$, containing the triangle out to $x = 4$. Find a formula for the area A_N of N rectangles and test it for $N = 3$ and $N = 6$.

18 Draw four rectangles with base 1 *below* the $y = x$ line, and find the total area. What is the area with N rectangles?

19 Draw $y = \sin x$ from 0 to π. Three rectangles (base $\pi/3$) and six rectangles (base $\pi/6$) contain an arch of the sine function. Find the areas and guess the limit.

20 Draw an example where three lower rectangles under a curve (heights m_1, m_2, m_3) have less area than two rectangles.

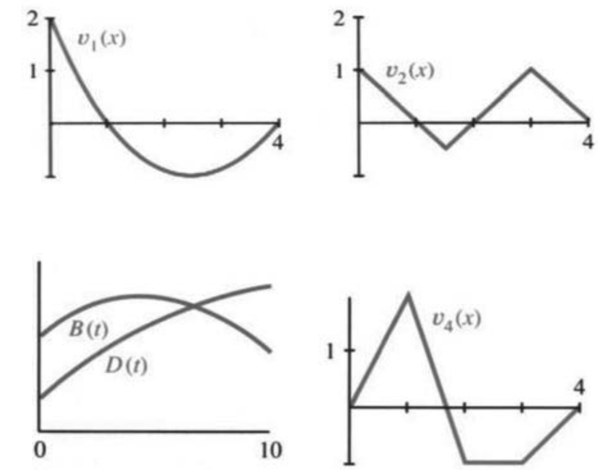
21 Draw $y = 1/x^2$ for $0 < x < 1$ with two rectangles under it (base $1/2$). What is their area, and what is the area for four rectangles? Guess the limit.

22 Repeat Problem 21 for $y = 1/x$.

23 (with calculator) For $v(x) = 1/\sqrt{x}$ take enough rectangles over $0 \leqslant x \leqslant 1$ to convince any reasonable professor that the area is 2. Find $f(x)$ and verify that $f(1) - f(0) = 2$.

24 Find the area under the parabola $v = x^2$ from $x = 0$ to $x = 4$. Relate it to the area $16/3$ below $\sqrt{x}$.

25 For v_1 and v_2 in the figure estimate the areas $f(2)$ and $f(4)$. Start with $f(0) = 0$.



26 Draw $y = v(x)$ so that the area $f(x)$ increases until $x = 1$, stays constant to $x = 2$, and decreases to $f(3) = 1$.

27 Describe the indefinite integrals of v_1 and v_2. Do the areas increase? Increase then decrease? ...

28 For $v_4(x)$ find the area $f(4) - f(1)$. Draw $f_4(x)$.

29 The graph of $B(t)$ shows the birth rate: births per unit time at time t. $D(t)$ is the death rate. In what way do these numbers appear on the graph?

1. The change in population from $t = 0$ to $t = 10$.
2. The time T when the population was largest.
3. The time t^* when the population increased fastest.

30 Draw the graph of a function $y_4(x)$ whose area function is $v_4(x)$.

31 If $v_2(x)$ is an antiderivative of $y_2(x)$, draw $y_2(x)$.

32 Suppose $v(x)$ increases from $v(0) = 0$ to $v(3) = 4$. The area under $y = v(x)$ plus the area on the left side of $x = v^{-1}(y)$ equals ______.

33 ***True or false***, when $f(x)$ is an antiderivative of $v(x)$.

(a) $2f(x)$ is an antiderivative of $2v(x)$ (*try examples*)
(b) $f(2x)$ is an antiderivative of $v(2x)$
(c) $f(x) + 1$ is an antiderivative of $v(x) + 1$
(d) $f(x+1)$ is an antiderivative of $v(x+1)$.
(e) $(f(x))^2$ is an antiderivative of $(v(x))^2$.

5.3 Summation versus Integration

This section does integration the hard way. We find explicit formulas for $f_n = v_1 + \cdots + v_n$. From areas of rectangles, the limits produce the area $f(x)$ under a curve. According to the Fundamental Theorem, df/dx should return us to $v(x)$—and we verify in each case that it does.

May I recall that there is sometimes an easier way? If we can find an $f(x)$ whose derivative is $v(x)$, then the integral of v is f. Sums and limits are not required, when f is spotted directly. The next section, which explains how to look for $f(x)$, will displace this one. (If we can't find an antiderivative we fall back on summation.) Given a successful f, adding any constant produces another f—since the derivative of the constant is zero. The right constant achieves $f(0) = 0$, with no extra effort.

This section constructs $f(x)$ from sums. The next section searches for antiderivatives.

THE SIGMA NOTATION

In a section about sums, there has to be a decent way to express them. Consider $1^2 + 2^2 + 3^2 + 4^2$. The individual terms are $v_j = j^2$. Their sum can be written in ***summation notation***, using the capital Greek letter Σ (pronounced sigma):

$$1^2 + 2^2 + 3^2 + 4^2 \text{ is written } \sum_{j=1}^{4} j^2.$$

Spoken aloud, that becomes "***the sum of*** j^2 ***from*** $j = 1$ ***to*** 4." It equals 30. The limits on j (written below and above Σ) indicate where to start and stop:

$$v_1 + \cdots + v_n = \sum_{j=1}^{n} v_j \quad \text{and} \quad v_3 + \cdots + v_9 = \sum_{k=3}^{9} v_k. \tag{1}$$

The k at the end of (1) makes an additional point. There is nothing special about the letter j. That is a "*dummy variable*," no better and no worse than k (or i). Dummy variables are only on one side (the side with Σ), and they have no effect on the sum. *The upper limit* n *is on both sides*. Here are six sums:

$$\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n \qquad \sum_{j=1}^{4} (-1)^j = -1 + 1 - 1 + 1 = 0$$

$$\sum_{j=1}^{5} (2j - 1) = 1 + 3 + 5 + 7 + 9 = 5^2 \qquad \sum_{i=0}^{0} v_i = v_0 \left[\text{only one term}\right]$$

$$\sum_{i=1}^{4} j^2 = \left[\text{meaningless?}\right] \qquad \sum_{k=0}^{\infty} \frac{1}{2^k} = 1 + \frac{1}{2} + \frac{1}{4} + \cdots = 2 \left[\text{infinite series}\right]$$

The numbers 1 and n or 1 and 4 (or 0 and ∞) are the ***lower limit*** and ***upper limit***. The dummy variable i or j or k is the *index* of summation. I hope it seems reasonable that the infinite series $1 + \frac{1}{2} + \frac{1}{4} + \cdots$ adds to 2. We will come back to it in Chapter 10.†

A sum like $\Sigma_{j=1}^{n} 6$ looks meaningless, but it is actually $6 + 6 + \cdots + 6 = 6n$. It follows the rules. In fact $\Sigma_{i=1}^{4} j^2$ is not meaningless either. Every term is j^2 and by

† Zeno the Greek believed it was impossible to get anywhere, since he would only go halfway and then half again and half again. Infinite series would have changed his whole life.

the same rules, that sum is $4j^2$. However the i was probably intended to be j. Then the sum is $1+4+9+16=30$.

Question What happens to these sums when the upper limits are changed to n?

Answer *The sum depends on the stopping point* n. A formula is required (when possible). Integrals stop at x, sums stop at n, and we now look for special cases when $f(x)$ or f_n can be found.

A SPECIAL SUMMATION FORMULA

How do you add the first 100 whole numbers? The problem is to compute

$$\sum_{j=1}^{100} j = 1+2+3+\cdots+98+99+100 = ?$$

If you were Gauss, you would see the answer at once. (He solved this problem at a ridiculous age, which gave his friends the idea of getting him into another class.) His solution was to combine $1+100$, and $2+99$, and $3+98$, *always adding to* 101. There are fifty of those combinations. Thus the sum is $(50)(101)=5050$.

The sum from 1 to n uses the same idea. The first and last terms add to $n+1$. The next terms $n-1$ and 2 also add to $n+1$. If n is even (as 100 was) then there are $\frac{1}{2}n$ parts. Therefore the sum is $\frac{1}{2}n$ times $n+1$:

$$\sum_{j=1}^{n} j = 1+2+\cdots+(n-1)+n = \frac{1}{2}n(n+1). \tag{2}$$

The important term is $\frac{1}{2}n^2$, but the exact sum is $\frac{1}{2}n^2+\frac{1}{2}n$.

What happens if n is an odd number (like $n=99$)? Formula (2) remains true. The combinations $1+99$ and $2+98$ still add to $n+1=100$. There are $\frac{1}{2}(99)=49\frac{1}{2}$ such pairs, because the middle term (which is 50) has nothing to combine with. Thus $1+2+\cdots+99$ equals $49\frac{1}{2}$ times 100, or 4950.

Remark That sum had to be 4950, because it is 5050 minus 100. The sum up to 99 equals the sum up to 100 with the last term removed. Our key formula $f_n - f_{n-1} = v_n$ has turned up again!

EXAMPLE Find the sum $101+102+\cdots+200$ of the *second* hundred numbers.

First solution This is the sum from 1 to 200 minus the sum from 1 to 100:

$$\sum_{101}^{200} j = \sum_{1}^{200} j - \sum_{1}^{100} j. \tag{3}$$

The middle sum is $\frac{1}{2}(200)(201)$ and the last is $\frac{1}{2}(100)(101)$. Their difference is 15050.

Note! I left out "$j=$" in the limits. It is there, but not written.

Second solution The answer 15050 is exactly the sum of the first hundred numbers (which was 5050) plus an additional 10000. Believing that a number like 10000 can never turn up by accident, we look for a reason. It is found through ***changing the limits of summation***:

$$\sum_{j=101}^{200} j \text{ is the same sum as } \sum_{k=1}^{100}(k+100). \tag{4}$$

This is important, to be able to shift limits around. Often the lower limit is moved to zero or one, for convenience. Both sums have 100 terms (that doesn't change). The dummy variable j is replaced by another dummy variable k. They are related by $j = k + 100$ or equivalently by $k = j - 100$.

The variable must change everywhere—in the lower limit and the upper limit as well as inside the sum. If j starts at 101, then $k = j - 100$ starts at 1. If j ends at 200, k ends at 100. If j appears in the sum, it is replaced by $k+100$ (and if j^2 appeared it would become $(k+100)^2$).

From equation (4) you see why the answer is 15050. The sum $1 + 2 + \cdots + 100$ is 5050 as before. 100 ***is added to each of those*** 100 ***terms***. That gives 10000.

EXAMPLES OF CHANGING THE VARIABLE (and the limits)

$\sum_{i=0}^{3} 2^i$ equals $\sum_{j=1}^{4} 2^{j-1}$ (here $i = j-1$). Both sums are $1+2+4+8$

$\sum_{i=3}^{n} v_i$ equals $\sum_{j=0}^{n-3} v_{j+3}$ (here $i = j+3$ and $j = i-3$). Both sums are $v_3 + \cdots + v_n$.

Why change n to $n-3$? Because the upper limit is $i = n$. So $j + 3 = n$ and $j = n - 3$.

A final step is possible, and you will often see it. ***The new variable*** j ***can be changed back to*** i. Dummy variables have no meaning of their own, but at first the result looks surprising:

$$\sum_{i=0}^{5} 2^i \text{ equals } \sum_{j=1}^{6} 2^{j-1} \text{ equals } \sum_{i=1}^{6} 2^{i-1}.$$

With practice you might do that in one step, skipping the temporary letter j. Every i on the left becomes $i-1$ on the right. Then $i = 0, \ldots, 5$ changes to $i = 1, \ldots, 6$. (At first two steps are safer.) This may seem a minor point, but soon we will be changing the limits on *integrals* instead of sums. Integration is parallel to summation, and it is better to see a "change of variable" here first.

Note about $1 + 2 + \cdots + n$. The good thing is that Gauss found the sum $\frac{1}{2}n(n+1)$. The bad thing is that his method looked too much like a trick. I would like to show how this fits the fundamental rule connecting sums and differences:

$$\textbf{\textit{if}} \;\; v_1 + v_2 + \cdots + v_n = f_n \;\; \textbf{\textit{then}} \;\; v_n = f_n - f_{n-1}. \tag{5}$$

Gauss says that f_n is $\frac{1}{2}n(n+1)$. Reducing n by 1, his formula for f_{n-1} is $\frac{1}{2}(n-1)n$. ***The difference*** $f_n - f_{n-1}$ ***should be the last term*** n ***in the sum***:

$$f_n - f_{n-1} = \tfrac{1}{2}n(n+1) - \tfrac{1}{2}(n-1)n = \tfrac{1}{2}(n^2 + n - n^2 + n) = n. \tag{6}$$

This is the one term $v_n = n$ that is included in f_n but not in f_{n-1}.

There is a deeper point here. For any sum f_n, there are two things to check. The f's must *begin* correctly and they must *change* correctly. The underlying idea is ***mathematical induction***: ***Assume the statement is true below*** n. ***Prove it for*** n.

***Goal*: *To prove that* $1+2+\cdots+n=\frac{1}{2}n(n+1)$. *This is the guess* f_n.**

***Proof by induction*: *Check* f_1 (*it equals* 1). *Check* $f_n - f_{n-1}$(*it equals* n).**

For $n=1$ the answer $\frac{1}{2}n(n+1)=\frac{1}{2}\cdot 1\cdot 2$ is correct. For $n=2$ this formula $\frac{1}{2}\cdot 2\cdot 3$ agrees with $1+2$. But that separate test is not necessary! ***If*** f_1 ***is right*, *and if the change*** $f_n - f_{n-1}$ ***is right for every*** n, ***then*** f_n ***must be right***. Equation (6) was the key test, to show that the change in f's agrees with v.

That is the logic behind mathematical induction, but I am not happy with most of the exercises that use it. There is absolutely no excitement. The answer is given by some higher power (like Gauss), and it is proved correct by some lower power (like us). It is much better when we lower powers find the answer for ourselves.† Therefore I will try to do that for the second problem, which is the ***sum of squares***.

THE SUM OF j^2 AND THE INTEGRAL OF x^2

An important calculation comes next. It is the area in Figure 5.6. One region is made up of rectangles, so its area is a sum of n pieces. The other region lies under the parabola $v=x^2$. It cannot be divided into rectangles, and calculus is needed.

The first problem is to find $f_n = 1^2+2^2+3^2+\cdots+n^2$. This is a sum of squares, with $f_1=1$ and $f_2=5$ and $f_3=14$. The goal is to find the pattern in that sequence. By trying to guess f_n we are copying what will soon be done for integrals.

Calculus looks for an $f(x)$ whose derivative is $v(x)$. There f is an *antiderivative*

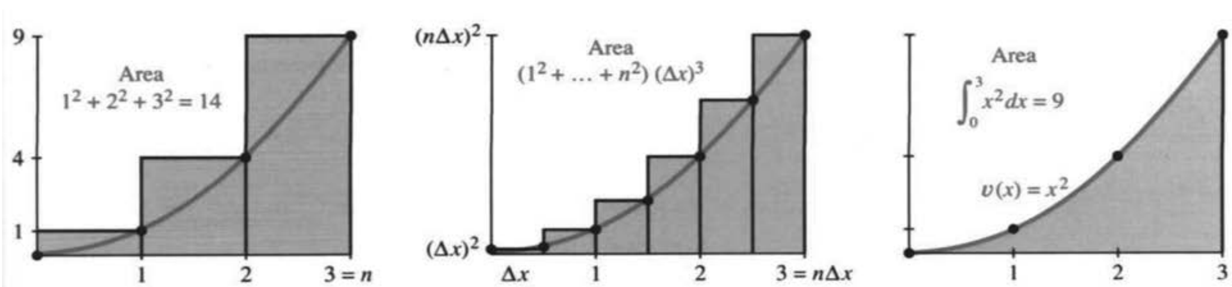


Fig. 5.6 Rectangles enclosing $v=x^2$ have area $\left(\frac{1}{3}n^3+\frac{1}{2}n^2+\frac{1}{6}n\right)(\Delta x)^3 \approx \frac{1}{3}(n\Delta x)^3 = \frac{1}{3}x^3$.

(or an integral). Algebra looks for f_n's whose differences produce v_n. Here f_n could be called an *antidifference* (better to call it a sum).

The best start is a good guess. Copying directly from integrals, we might try $f_n=\frac{1}{3}n^3$. To test if it is right, check whether $f_n - f_{n-1}$ produces on $v_n=n^2$:

$$\frac{1}{3}n^3-\frac{1}{3}(n-1)^3=\frac{1}{3}n^3-\frac{1}{3}(n^3-3n^2+3n-1)=n^2-n+\frac{1}{3}.$$

We see n^2, but also $-n+\frac{1}{3}$. The guess $\frac{1}{3}n^3$ needs ***correction terms***. To cancel $\frac{1}{3}$ in the difference, I subtract $\frac{1}{3}n$ from the sum. To put back n in the difference, I add $1+2+\cdots+n=\frac{1}{2}n(n+1)$ to the sum. The new guess (which should be right) is

$$f_n=\frac{1}{3}n^3+\frac{1}{2}n(n+1)-\frac{1}{3}n=\frac{1}{3}n^3+\frac{1}{2}n^2+\frac{1}{6}n. \tag{7}$$

To check this answer, verify first that $f_1=1$. Also $f_2=5$ and $f_3=14$. To be certain, verify that $f_n - f_{n-1}=n^2$. For calculus the important term is $\frac{1}{3}n^3$:

†The goal of real teaching is for the *student* to find the answer. And also the problem.

$$\textit{\textbf{The sum}} \sum_{j=1}^{n} j^2 \textit{\textbf{ of the first }} n \textit{\textbf{ squares is }} \frac{1}{3}n^3 \textit{\textbf{ plus corrections }} \frac{1}{2}n^2 \textit{\textbf{ and }} \frac{1}{6}n.$$

In practice $\frac{1}{3}n^3$ is an excellent estimate. The sum of the first 100 squares is approximately $\frac{1}{3}(100)^3$, or a third of a million. If we need the exact answer, equation (7) is available: the sum is $338,350$. Many applications (example: the number of steps to solve 100 linear equations) can settle for $\frac{1}{3}n^3$.

What is fascinating is the contrast with calculus. ***Calculus has no correction terms***! They get washed away in the limit of thin rectangles. When the sum is replaced by the integral (the area), we get an absolutely clean answer:

$$\textit{\textbf{The integral of }} v = x^2 \textit{\textbf{ from }} x = 0 \textit{\textbf{ to }} x = n \textit{\textbf{ is exactly }} \tfrac{1}{3}n^3.$$

The area under the parabola, out to the point $x = 100$, is precisely a third of a million. We have to explain why, with many rectangles.

The idea is to approach an infinite number of infinitely thin rectangles. A hundred rectangles gave an area of $338,350$. Now take a thousand rectangles. Their heights are $\left(\frac{1}{10}\right)^2, \left(\frac{2}{10}\right)^2, \ldots$ because the curve is $v = x^2$. The base of every rectangle is $\Delta x = \frac{1}{10}$, and we add heights times base:

$$\text{area of rectangles} = \left(\frac{1}{10}\right)^2\left(\frac{1}{10}\right) + \left(\frac{2}{10}\right)^2\left(\frac{1}{10}\right) + \cdots + \left(\frac{1000}{10}\right)^2\left(\frac{1}{10}\right).$$

Factor out $\left(\frac{1}{10}\right)^3$. What you have left is $1^2 + 2^2 + \cdots + 1000^2$, which fits the sum of squares formula. The exact area of the thousand rectangles is $333,833.5$. I could try to guess ten thousand rectangles but I won't.

Main point: The area is approaching $333,333.333\ldots$. But the calculations are getting worse. It is time for algebra—which means that we keep "Δx" and avoid numbers.

The interval of length 100 is divided into n pieces of length Δx. (Thus $n = 100/\Delta x$.) The jth rectangle meets the curve $v = x^2$, so its height is $(j\Delta x)^2$. Its base is Δx, and we add areas:

$$\textit{\textbf{area}} = (\Delta x)^2(\Delta x) + (2\Delta x)^2(\Delta x) + \cdots + (n\Delta x)^2(\Delta x) = \sum_{j=1}^{n}(j\Delta x)^2(\Delta x). \tag{8}$$

Factor out $(\Delta x)^3$, leaving a sum of n squares. The area is $(\Delta x)^3$ times f_n, and $n = \dfrac{100}{\Delta x}$:

$$(\Delta x)^3\left[\frac{1}{3}\left(\frac{100}{\Delta x}\right)^3 + \frac{1}{2}\left(\frac{100}{\Delta x}\right)^2 + \frac{1}{6}\left(\frac{100}{\Delta x}\right)\right] = \frac{1}{3}100^3 + \frac{1}{2}100^2(\Delta x) + \frac{1}{6}100(\Delta x)^2. \tag{9}$$

This equation shows what is happening. The leading term is a third of a million, as predicted. The other terms are approaching zero! They contain Δx, and as the rectangles get thinner they disappear. They only account for the small corners of rectangles that lie above the curve. The vanishing of those corners will eventually be proved for any continuous functions—*the area from the correction terms goes to zero*—but here in equation (9) you see it explicitly.

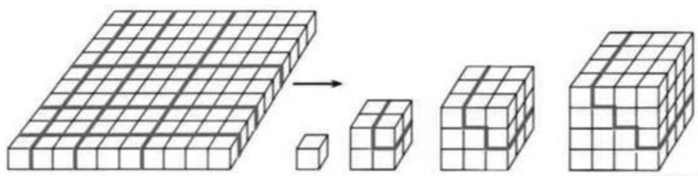
The area under the curve came from the central idea of integration: $100/\Delta x$ rectangles of width Δx approach the limiting area$= \frac{1}{3}(100)^3$. ***The rectangular area is*** $\Sigma v_j \Delta x$. ***The exact area is*** $\int v(x)dx$. ***In the limit*** Σ ***becomes*** $\int$ ***and*** v_j ***becomes*** $v(x)$ ***and*** Δx ***becomes*** dx.

That completes the calculation for a parabola. It used the formula for a sum of squares, which was special. But the underlying idea is much more general. The limit of the sums agrees with the antiderivative: ***The antiderivative of*** $v(x) = x^2$ ***is*** $f(x) = \frac{1}{3}x^3$. According to the Fundamental Theorem, the area under $v(x)$ is $f(x)$:

$$\int_0^{100} v(x)\,dx = f(100) - f(0) = \tfrac{1}{3}(100)^3.$$

That Fundamental Theorem is not yet proved! I mean it is not proved by us. Whether Leibniz or Newton managed to prove it, I am not quite sure. But it can be done. Starting from sums of differences, the difficulty is that we have too many limits at once. The sums of $v_j\,\Delta x$ are approaching the integral. The differences $\Delta f/\Delta x$ approach the derivative. A real proof has to separate those steps, and Section 5.7 will do it.

Proved or not, you are seeing the main point. What was true for the numbers f_j and v_j is true in the limit for $v(x)$ and $f(x)$. Now $v(x)$ can vary continuously, but it is still the slope of $f(x)$. ***The reverse of slope is area***.



$$(\mathbf{1+2+3+4})^2 = \mathbf{1^3+2^3+3^3+4^3}$$

Proof without words by Roger Nelsen (*Mathematics Magazine* 1990).

Finally we review the area under $v = x$. The sum of $1+2+\cdots+n$ is $\frac{1}{2}n^2 + \frac{1}{2}n$. This gives the area of $n = 4/\Delta x$ rectangles, going out to $x = 4$. The heights are $j\Delta x$, the bases are Δx, and we add areas:

$$\sum_{j=1}^{4/\Delta x} (j\Delta x)(\Delta x) = (\Delta x)^2\left[\frac{1}{2}\left(\frac{4}{\Delta x}\right)^2 + \frac{1}{2}\left(\frac{4}{\Delta x}\right)\right] = 8 + 2\Delta x. \qquad (10)$$

With $\Delta x = 1$ the area is $1+2+3+4 = 10$. With eight rectangles and $\Delta x = \frac{1}{2}$, the area was $8 + 2\Delta x = 9$. Sixteen rectangles of width $\frac{1}{4}$ brought the correction $2\Delta x$ down to $\frac{1}{2}$. The exact area is 8. ***The error is proportional to*** Δx.

Important note There you see a question in applied mathematics. If there is an error, what size is it? How does it behave as $\Delta x \to 0$? The Δx term disappears in the limit, and $(\Delta x)^2$ disappears faster. But to get an error of 10^{-6} we need ***eight million rectangles***:

$$2\Delta x = 2 \cdot 4/8{,}000{,}000 = 10^{-6}.$$

That is horrifying! The numbers $10, 9, 8\frac{1}{2}, 8\frac{1}{4}, \ldots$ seem to approach the area 8 in a satisfactory way, but the convergence is ***much too slow***. It takes twice as much work to get one more binary digit in the answer—which is absolutely unacceptable. Somehow the Δx term must be removed. If the correction is $(\Delta x)^2$ instead of Δx, then a thousand rectangles will reach an accuracy of 10^{-6}.

The problem is that the rectangles are unbalanced. Their right sides touch the graph of v, but their left sides are much too high. The best is to cross the graph in the *middle* of the interval—this is the ***midpoint rule***. Then the rectangle sits halfway across the line $v = x$, and the error is zero. Section 5.8 comes back to this rule—and to Simpson's rule that fits parabolas and removes the $(\Delta x)^2$ term and is built into many calculators.

Finally we try the quick way. The area under $v = x$ is $f = \frac{1}{2}x^2$, because df/dx is v. The area out to $x = 4$ is $\frac{1}{2}(4)^2 = 8$. Done.

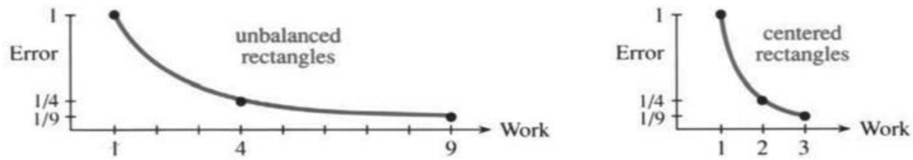


Fig. 5.7 Endpoint rules: error $\sim 1/(\text{work}) \sim 1/n$. Midpoint rule is better: error $\sim 1/(\text{work})^2$.

Optional: pth ***powers*** Our sums are following a pattern. First, $1 + \cdots + n$ is $\frac{1}{2}n^2$ plus $\frac{1}{2}n$. The sum of squares is $\frac{1}{3}n^3$ plus correction terms. *The sum of* pth *powers is*

$$1^p + 2^p + \cdots + n^p = \frac{1}{p+1}n^{p+1} \textbf{\textit{ plus correction terms}}. \tag{11}$$

192 The correction involves lower powers of n, and you know what is coming. ***Those corrections disappear in calculus***. The area under $v = x^p$ from 0 to n is

$$\int_{x=0}^{n} x^p\,dx = \lim_{\Delta x \to 0} \sum_{j=1}^{n/\Delta x} (j\Delta x)^p(\Delta x) = \frac{1}{p+1}n^{p+1}. \tag{12}$$

Calculus doesn't care if the upper limit n is an integer, and it doesn't care if the power p is an integer. We only need $p + 1 > 0$ to be sure n^{p+1} is genuinely the leading term. ***The antiderivative of*** $v = x^p$ ***is*** $f = x^{p+1}/(p+1)$.

We are close to interesting experiments. The correction terms disappear and the sum approaches the integral. Here are actual numbers for $p = 1$, when the sum and integral are easy: $S_n = 1 + \cdots + n$ and $I_n = \int x\,dx = \frac{1}{2}n^2$. The difference is $D_n = \frac{1}{2}n$. The thing to watch is the ***relative error*** $E_n = D_n/I_n$:

n	S_n	I_n	$D_n = S_n - I_n$	$E_n = D_n/I_n$
100	5050	5000	50	.010
200	20100	20000	100	.005

The number 20,100 is $\frac{1}{2}(200)(201)$. Please write down the next line $n = 400$, ***and please find a formula for*** E_n. You can guess E_n from the table, or you can derive it from knowing S_n and I_n. The formula should show that E_n goes to zero. More important, it should show how quick (or slow) that convergence will be.

One more number—a third of a million—was mentioned earlier. It came from integrating x^2 from 0 to 100, which compares to the sum S_{100} of 100 squares:

n	p	S_n	$I_n = \frac{1}{3}n^3$	$D = S - I$	$E = D/I$
100	2	338350	$333333\frac{1}{3}$	$5016\frac{2}{3}$	.01505
200	2	2686700	$2666666\frac{2}{3}$	$20033\frac{1}{3}$	.0075125

These numbers suggest a new idea, ***to keep*** n ***fixed and change*** p. The computer can find sums without a formula! With its help we go to fourth powers and square roots:

n	p	$S = 1^p + \cdots + n^p$	$I = n^{p+1}/(p+1)$	$D = S - I$	$E_{n,p} = D/I$
100	4	2050333330	$\frac{1}{5}(100)^5$	50333330	0.0252
100	$\frac{1}{2}$	671.4629	$\frac{2}{3}(100)^{3/2}$	4.7963	0.0072

In this and future tables we don't expect exact values. The last entries are rounded off, and the goal is to see the pattern. The errors $E_{n,p}$ are sure to obey a systematic rule—they are proportional to $1/n$ and to an unknown number $C(p)$ that depends on p. I hope you can push the experiments far enough to discover $C(p)$. This is not an exercise with an answer in the back of the book—it is mathematics.

5.3 EXERCISES

Read-through questions

The Greek letter __a__ indicates summation. In $\Sigma_1^n v_j$ the dummy variable is __b__. The limits are __c__, so the first term is __d__ and the last term is __e__. When $v_j = j$ this sum equals __f__. For $n = 100$ the leading term is __g__. The correction term is __h__. The leading term equals the integral of $v = x$ from 0 to 100, which is written __i__. The sum is the total __j__ of 100 rectangles. The correction term is the area between the __k__ and the __l__.

The sum $\Sigma_{i=3}^6 i^2$ is the same as $\Sigma_{j=1}^4$ __m__ and equals __n__. The sum $\Sigma_{i=4}^5 v_i$ is the same as __o__ v_{i+4} and equals __p__. For $f_n = \Sigma_{j=1}^n v_j$ the difference $f_n - f_{n-1}$ equals __q__.

The formula for $1^2 + 2^2 + \cdots + n^2$ is $f_n =$ __r__. To prove it by mathematical induction, check $f_1 =$ __s__ and check $f_n - f_{n-1} =$ __t__. The area under the parabola $v = x^2$ from $x = 0$ to $x = 9$ is __u__. This is close to the area of __v__ rectangles of base Δx. The correction terms approach zero very __w__.

1 Compute the numbers $\sum_{n=1}^{4} 1/n$ and $\sum_{i=2}^{5} (2i-3)$.

2 Compute $\sum_{j=0}^{3} (j^2 - j)$ and $\sum_{j=1}^{n} 1/2^j$.

3 Evaluate the sum $\sum_{i=0}^{6} 2^i$ and $\sum_{i=0}^{n} 2^i$.

4 Evaluate $\sum_{i=1}^{6} (-1)^i i$ and $\sum_{j=1}^{n} (-1)^j j$.

5 Write these sums in sigma notation and compute them:

$$2+4+6+\cdots+100 \qquad 1+3+5+\cdots+199 \qquad 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}$$

6 Express these sums in sigma notation:

$$v_1 - v_2 + v_3 - v_4 \qquad v_1w_1 + v_2w_2 + \cdots + v_nw_n \qquad v_1 + v_3 + v_5$$

7 Convert these sums to sigma notation:

$$a_0 + a_1x + \cdots + a_nx^n \qquad \sin\frac{2\pi}{n} + \sin\frac{4\pi}{n} + \cdots + \sin 2\pi$$

8 The binomial formula uses coefficients $\binom{n}{j} = \dfrac{n!}{j!(n-j)!}$:

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \cdots + \binom{n}{n}b^n = \sum_{j=0}^{n} \_\_\_\_\_ \, b^j.$$

9 With electronic help compute $\sum_{1}^{100} 1/j$ and $\sum_{1}^{1000} 1/j$.

10 On a computer find $\sum_{0}^{10} (-1)^j/j!$ times $\sum_{0}^{10} 1/j!$

11 Simplify $\sum_{i=1}^{n} (a_i + b_i)^2 + \sum_{i=1}^{n} (a_i - b_i)^2$ to $\sum_{i=1}^{n}$ _____.

12 Show that $\left(\sum_{i=1}^{n} a_i\right)^2 \neq \sum_{i=1}^{n} a_i^2$ and $\sum_{i=1}^{n} a_ib_i \neq \sum_{j=1}^{n} a_j \sum_{k=1}^{n} b_k$.

13 *"Telescope" the sums* $\sum_{k=1}^{n} (2^k - 2^{k-1})$ and $\sum_{j=1}^{10} \left(\frac{1}{j+1} - \frac{1}{j}\right)$. All but two terms cancel.

14 Simplify the sums $\sum_{j=1}^{n} (f_j - f_{j-1})$ and $\sum_{j=3}^{12} (f_{j+1} - f_j)$.

15 ***True or false***: (a) $\sum_{j=4}^{8} v_j = \sum_{i=2}^{6} v_{i-2}$ (b) $\sum_{i=1}^{9} v_i = \sum_{i=3}^{11} v_{i-2}$

16 $\sum_{i=1}^{n} v_i = \sum_{j=0}^{n-1}$ _____ and $\sum_{i=0}^{6} i^2 = \sum_{i=2}^{8}$ _____.

17 The antiderivative of d^2f/dx^2 is df/dx. What is the sum $(f_2 - 2f_1 + f_0) + (f_3 - 2f_2 + f_1) + \cdots + (f_9 - 2f_8 + f_7)$?

18 ***Induction***: Verify that $1^2 + 2^2 + \cdots + n^2$ is $f_n = n(n+1)(2n+1)/6$ by checking that f_1 is correct and $f_n - f_{n-1} = n^2$.

19 Prove by induction: $1 + 3 + \cdots + (2n-1) = n^2$.

20 Verify that $1^3 + 2^3 + \cdots + n^3$ is $f_n = \frac{1}{4}n^2(n+1)^2$ by checking f_1 and $f_n - f_{n-1}$. The text has a *proof without words*.

21 Suppose f_n has the form $an + bn^2 + cn^3$. If you know $f_1 = 1$, $f_2 = 5$, $f_3 = 14$, turn those into three equations for a, b, c. The solutions $a = \frac{1}{6}$, $b = \frac{1}{2}$, $c = \frac{1}{3}$ give what formula?

22 Find q in the formula $1^8 + \cdots + n^8 = qn^9 +$ correction.

23 Add $n = 400$ to the table for $S_n = 1 + \cdots + n$ and find the relative error E_n. Guess and prove a formula for E_n.

24 Add $n = 50$ to the table for $S_n = 1^2 + \cdots + n^2$ and compute E_{50}. Find an approximate formula for E_n.

25 Add $p = \frac{1}{3}$ and $p = 3$ to the table for $S_{100,p} = 1^p + \cdots + 100^p$. Guess an approximate formula for $E_{100,p}$.

26 Guess $C(p)$ in the formula $E_{n,p} \approx C(p)/n$.

27 Show that $|1-5| < |1| + |-5|$. Always $|v_1 + v_2| < |v_1| + |v_2|$ unless ______.

28 Let S be the sum $1 + x + x^2 + \cdots$ of the (infinite) geometric series. Then $xS = x + x^2 + x^3 + \cdots$ is the same as S minus ______. Therefore $S =$ ______. None of this makes sense if $x = 2$ because ______.

29 The ***double sum*** $\sum_{i=1}^{2} \left[\sum_{j=1}^{3} (i+j) \right]$ is $v_1 = \sum_{i=1}^{3} (1+j)$ plus $v_2 = \sum_{j=1}^{3} (2+j)$. Compute v_1 and v_2 and the double sum.

30 The double sum $\sum_{i=1}^{2} \left(\sum_{j=1}^{3} w_{i,j} \right)$ is $(w_{1,1} + w_{1,2} + w_{1,3}) +$ ______. The double sum $\sum_{j=1}^{3} \left(\sum_{i=1}^{2} w_{i,j} \right)$ is $(w_{1,1} + w_{2,1}) + (w_{1,2} + w_{2,2}) +$ ______. Compare.

31 Find the flaw in the proof that $2^n = 1$ for every $n = 0, 1, 2, \ldots$. For $n = 0$ we have $2^0 = 1$. If $2^n = 1$ for every $n < N$, then $2^N = 2^{N-1} \cdot 2^{N-1} / 2^{N-2} = 1 \cdot 1/1 = 1$.

32 Write out all terms to see why the following are true:

$$\sum_{1}^{3} 4v_j = 4 \sum_{1}^{3} v_j \qquad \sum_{i=1}^{2} \left(\sum_{j=1}^{3} u_i v_j \right) = \left(\sum_{1}^{2} u_i \right) \left(\sum_{1}^{3} v_j \right)$$

33 The average of $6, 11, 4$ is $\bar{v} = \frac{1}{3}(6 + 11 + 4)$. Then $(6 - \bar{v}) + (11 - \bar{v}) + (4 - \bar{v}) =$ ______. The average of $v_1, \ldots, v_n$ is $\bar{v} =$ ______. Prove that $\Sigma(v_i - \bar{v}) = 0$.

34 The *Schwarz inequality* is $\left(\sum_{1}^{n} a_i b_i \right)^2 \leqslant \left(\sum_{1}^{n} a_i^2 \right) \left(\sum_{1}^{n} b_i^2 \right)$. Compute both sides if $a_1 = 2$, $a_2 = 3$, $b_1 = 1$, $b_2 = 4$. Then compute both sides for any a_1, a_2, b_1, b_2. The proof in Section 11.1 uses vectors.

35 Suppose n rectangles with base Δx touch the graph of $v(x)$ at the points $x = \Delta x, 2\Delta x, \ldots, n\Delta x$. Express the total rectangular area in sigma notation.

36 If $1/\Delta x$ rectangles with base Δx touch the graph of $v(x)$ at the *left* end of each interval (thus at $x = 0, \Delta x, 2\Delta x, \ldots$) express the total area in sigma notation.

37 The sum $\Delta x \sum_{j=1}^{1/\Delta x} \dfrac{f(j\Delta x) - f((j-1)\Delta x)}{\Delta x}$ equals ______. In the limit this becomes $\int_0^1$ ______ $dx =$ ______.

5.4 Indefinite Integrals and Substitutions

This section integrates the easy way, by looking for antiderivatives. We leave aside sums of rectangular areas, and their limits as $\Delta x \to 0$. Instead we search for an $f(x)$ with the required derivative $v(x)$. In practice, this approach is more or less independent of the approach through sums—but it gives the same answer. And also, *the search for an antiderivative may not succeed*. We may not find f. In that case we go back to rectangles, or on to something better in Section 5.8.

A computer is ready to integrate v, but not by discovering f. It integrates between specified limits, to obtain a ***number*** (the definite integral). Here we hope to find a ***function*** (the indefinite integral). That requires a symbolic integration code like MACSYMA or *Mathematica* or MAPLE, or a reasonably nice $v(x)$, or both. An expression for $f(x)$ can have tremendous advantages over a list of numbers.

Thus our goal is to find antiderivatives and use them. The techniques will be further developed in Chapter 7—this section is short but good. First we write down what we know. *On each line, $f(x)$ is an antiderivative of $v(x)$ because $df/dx = v(x)$.*

Known pairs	***Function*** $v(x)$	***Antiderivative*** $f(x)$
Powers of x	x^n	$x^{n+1}/(n+1)+C$

$n=-1$ is not included, because $n+1$ would be zero. $v = x^{-1}$ will lead us to $f = \ln x$.

Trigonometric functions	$\cos x$	$\sin x + C$
	$\sin x$	$-\cos x + C$
	$\sec^2 x$	$\tan x + C$
	$\csc^2 x$	$-\cot x + C$
	$\sec x \tan x$	$\sec x + C$
	$\csc x \cot x$	$-\csc x + C$
Inverse functions	$1/\sqrt{1-x^2}$	$\sin^{-1} x + C$
	$1/(1+x^2)$	$\tan^{-1} x + C$
	$1/\lvert x\rvert\sqrt{x^2-1}$	$\sec^{-1} x + C$

You recognize that each integration formula came directly from a differentiation formula. The integral of the cosine is the sine, because the derivative of the sine is the cosine. For emphasis we list three derivatives above three integrals:

$$\frac{d}{dx}(\text{constant}) = 0 \qquad \frac{d}{dx}(x) = 1 \qquad \frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n$$

$$\int 0\,dx = C \qquad \int 1\,dx = x + C \qquad \int x^n\,dx = \frac{x^{n+1}}{n+1} + C$$

There are two ways to make this list longer. One is to find the derivative of a new $f(x)$. Then f goes in one column and $v = df/dx$ goes in the other co1umn.† The other possibility is to use rules for derivatives to find rules for integrals. That is the way to extend the list, enormously and easily.

†We will soon meet e^x, which goes in *both columns*. It is $f(x)$ and also $v(x)$.

RULES FOR INTEGRALS

Among the rules for derivatives, three were of supreme importance. They were ***linearity***, the ***product rule***, and the ***chain rule***. Everything flowed from those three. In the reverse direction (from v to f) this is still true. The three basic methods of differential calculus also dominate integral calculus:

linearity of derivatives → linearity of integrals

product rule for derivatives → integration by parts

chain rule for derivatives → integrals by substitution

The easiest is linearity, which comes first. Integration by parts will be left for Section 7.1. This section starts on substitutions, reversing the chain rule to make an integral simpler.

LINEARITY OF INTEGRALS

What is the integral of $v(x)+w(x)$? Add the two separate integrals. The graph of $v+w$ has two regions below it, the area under v and the area from v to $v+w$. Adding areas gives the sum rule. Suppose f and g are antiderivatives of v and w:

sum rule: $f+g$ is an antiderivative of $v+w$

constant rule: cf is an antiderivative of cv

linearity: $af+bg$ is an antiderivative of $av+bw$

This is a case of overkill. The first two rules are special cases of the third, so logically the last rule is enough. However it is so important to deal quickly with constants— just "*factor them out*"—that the rule $cv \leftrightarrow cf$ is stated separately. The proofs come from the linearity of derivatives: $(af+bg)'$ *equals* $af'+bg'$ *which equals* $av+bw$. The rules can be restated with integral signs:

sum rule: $$\int \left[v(x)+w(x)\right]dx = \int v(x)\,dx + \int w(x)\,dx$$

constant rule: $$\int cv(x)\,dx = c\int v(x)\,dx$$

linearity: $$\int \left[av(x)+bw(x)\right]dx = a\int v(x)\,dx + b\int w(x)\,dx$$

Note about the constant in $f(x)+C$. All antiderivatives allow the addition of a constant. For a combination like $av(x)+bw(x)$, the antiderivative is $af(x)+bg(x)+C$. *The constants for each part combine into a single constant.* To give all possible antiderivatives of a function, just remember to write "$+C$" after one of them. The real problem is to find that one antiderivative.

EXAMPLE 1 The antiderivative of $v=x^2+x^{-2}$ is $f=x^3/3+(x^{-1})/(-1)+C$.

EXAMPLE 2 The antiderivative of $6\cos t+7\sin t$ is $6\sin t-7\cos t+C$.

EXAMPLE 3 Rewrite $\dfrac{1}{1-\sin x}$ as $\dfrac{1-\sin x}{1-\sin^2 x}=\dfrac{1-\sin x}{\cos^2 x}=\sec^2 x-\sec x\tan x$.

The antiderivative is $\tan x-\sec x+C$. That rewriting is done by a symbolic algebra code (or by you). Differentiation is often simple, so most people check that $df/dx=v(x)$.

Question How to integrate $\tan^2 x$?
Method Write it as $\sec^2 x-1$. Answer $\tan x-x+C$.

INTEGRALS BY SUBSTITUTION

We now present the most valuable technique in this section—***substitution***. To see the idea, you have to remember the chain rule:

$$\begin{array}{rll} f(g(x)) & \text{has derivative} & f'(g(x))(dg/dx) \\ \sin x^2 & \text{has derivative} & (\cos x^2)(2x) \\ (x^3+1)^5 & \text{has derivative} & 5(x^3+1)^4(3x^2) \end{array}$$

If the function on the right is given, the function on the left is its antiderivative! There are two points to emphasize right away:

1. ***Constants are no problem—they can always be fixed.*** Divide by 2 or 15:

$$\int x\cos(x)^2 dx = \frac{1}{2}\sin(x^2) + C \qquad \int x^2(x^3+1)^4 dx = \frac{1}{15}(x^3+1)^5 + C$$

Notice the 2 from x^2, the 5 from the fifth power, and the 3 from x^3.

2. ***Choosing the inside function*** g (or u) ***commits us to its derivative***:

$$\begin{array}{ll} \text{the integral of } 2x\cos x^2 \text{ is } \sin x^2 + C & (g = x^2, dg/dx = 2x) \\ \text{the integral of } \cos x^2 \text{ is } (\textit{failure}) & (\text{no } dg/dx) \\ \text{the integral of } x^2\cos x^2 \text{ is } (\textit{failure}) & (\text{wrong } dg/dx) \end{array}$$

To substitute g for x^2, we need its derivative. The trick is to spot an inside function whose derivative is present. We can fix constants like 2 or 15, but otherwise dg/dx has to be there. *Very often the inside function g is written u.* We use that letter to state the ***substitution rule***, when f is the integral of v:

$$\int v(u(x))\frac{du}{dx}\,dx = f(u(x)) + C. \tag{1}$$

EXAMPLE 4 $\int \sin x\cos x\,dx = \frac{1}{2}(\sin x)^2 + C$ $\quad u = \sin x$ (compare Example 6)

EXAMPLE 5 $\int \sin^2 x\,\cos x\,dx = \frac{1}{3}(\sin x)^3 + C$ $\quad u = \sin x$

EXAMPLE 6 $\int \cos x\,\sin x\,dx = -\frac{1}{2}(\cos x)^2 + C$ $\quad u = \cos x$ (compare Example 4)

EXAMPLE 7 $\int \tan^4 x\sec^2 x\,dx = \frac{1}{5}(\tan x)^5 + C$ $\quad u = \tan x$

The next example has $u = x^2 - 1$ and $du/dx = 2x$. The key step is choosing u:

EXAMPLE 8 $\int x\,dx/\sqrt{x^2-1} = \sqrt{x^2-1} + C$ $\quad \int x\sqrt{x^2-1}\,dx = \frac{1}{3}(x^2-1)^{3/2} + C$

A ***shift*** of x (to $x+2$) or a ***multiple*** of x (rescaling to $2x$) is particularly easy:

EXAMPLES 9–10 $\int (x+2)^3 dx = \frac{1}{4}(x+2)^4 + C$ $\quad \int \cos 2x\,dx = \frac{1}{2}\sin 2x + C$

You will soon be able to do those in your sleep. Officially the derivative of $(x+2)^4$ uses the chain rule. But the inside function $u = x+2$ has $du/dx = 1$. The "1" is there automatically, and the graph shifts over—as in Figure 5.8b.

For Example 10 the inside function is $u = 2x$. Its derivative is $du/dx = 2$. This

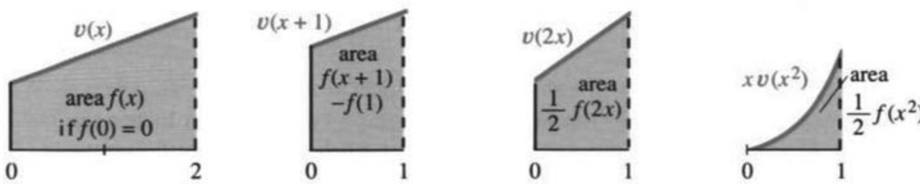


Fig. 5.8 Substituting $u = x+1$ and $u = 2x$ and $u = x^2$. The last graph has half of $du/dx = 2x$.

required factor 2 is missing in $\int \cos 2x\, dx$, but we put it there by multiplying and dividing by 2. Check the derivative of $\frac{1}{2}\sin 2x$: the 2 from the chain rule cancels the $\frac{1}{2}$. The rule for any nonzero constant is similar:

$$\int v(x+c)\,dx = f(x+c) \quad \text{and} \quad \int v(cx)\,dx = \frac{1}{c}f(cx). \tag{2}$$

Squeezing the graph by c divides the area by c. Now $3x+7$ rescales *and* shifts:

EXAMPLE 11 $\int \cos(3x+7)\,dx = \frac{1}{3}\sin(3x+7)+C \quad \int(3x+7)^2 dx = \frac{1}{3}\cdot\frac{1}{3}(3x+7)^3+C$

Remark on writing down the steps When the substitution is complicated, it is a good idea to get du/dx where you need it. Here $3x^2+1$ needs $6x$:

$$\int 7x(3x^2+1)^4 dx = \frac{7}{6}\int (3x^2+1)^4 6x\,dx = \frac{7}{6}\int u^4\frac{du}{dx}\,dx$$

$$\text{Now integrate:} \qquad \frac{7}{6}\frac{u^5}{5}+C = \frac{7}{6}\frac{(3x^2+1)^5}{5}+C. \tag{3}$$

Check the derivative at the end. The exponent 5 cancels 5 in the denominator, $6x$ from the chain rule cancels 6, and $7x$ is what we started with.

Remark on differentials In place of $(du/dx)dx$, many people just write du:

$$\int (3x^2+1)^4 6x\,dx = \int u^4 du = \tfrac{1}{5}u^5+C. \tag{4}$$

This really shows how substitution works. *We switch from x to u*, ***and we also switch from dx to du***. The most common mistake is to confuse dx with du. The factor du/dx from the chain rule is absolutely needed, to reach du. The change of variables (dummy variables anyway!) leaves an easy integral, and then u turns back into $3x^2+1$. Here are the four steps to substitute u for x:

1. Choose $u(x)$ and compute du/dx
2. Locate $v(u)$ times du/dx times dx, or $v(u)$ times du
3. Integrate $\int v(u)\,du$ to find $f(u)+C$
4. Substitute $u(x)$ back into this antiderivative f.

EXAMPLE 12 $\int(\cos\sqrt{x})\,dx/2\sqrt{x} = \int \cos u\,du = \sin u + C = \sin\sqrt{x}+C$

(***put in*** u) (***integrate***) (***put back*** x)

The choice of u must be right, to change everything from x to u. With ingenuity, some remarkable integrals are possible. But most will remain impossible forever. The functions $\cos x^2$ and $1/\sqrt{4-\sin^2 x}$ have no "elementary" antiderivative. Those integrals are well defined and they come up in applications—the latter gives the distance around an ellipse. That can be computed to tremendous accuracy, but not to perfect accuracy.

The exercises concentrate on substitutions, which need and deserve practice. We give a ***nonexample***—$\int (x^2+1)^2 dx$ *does not equal* $\frac{1}{3}(x^2+1)^3$—to emphasize the need for du/dx. Since $2x$ is missing, $u = x^2+1$ does not work. But we can fix up π:

$$\int \sin \pi x \, dx = \int \sin u \frac{du}{\pi} = -\frac{1}{\pi}\cos u + C = -\frac{1}{\pi}\cos \pi x + C.$$

5.4 EXERCISES

Read-through questions

Finding integrals by substitution is the reverse of the __a__ rule. The derivative of $(\sin x)^3$ is __b__. Therefore the antiderivative of __c__ is __d__. To compute $\int (1+\sin x)^2 \cos x \, dx$, substitute $u =$ __e__. Then $du/dx =$ __f__ so substitute $du =$ __g__. In terms of u the integral is $\int$ __h__ = __i__. Returning to x gives the final answer.

The best substitutions for $\int \tan(x+3)\sec^2(x+3)\,dx$ and $\int (x^2+1)^{10} x\,dx$ are $u =$ __j__ and $u =$ __k__. Then $du =$ __l__ and __m__. The answers are __n__ and __o__. The antiderivative of $v\,dv/dx$ is __p__. $\int 2x\,dx/(1+x^2)$ leads to $\int$ __q__, which we don't yet know. The integral $\int dx/(1+x^2)$ is known immediately as __r__.

Find the indefinite integrals in 1–20.

1 $\int \sqrt{2+x}\,dx$ (add $+C$)

2 $\int \sqrt{3-x}\,dx$ (always $+C$)

3 $\int (x+1)^n\,dx$

4 $\int (x+1)^{-n}\,dx$

5 $\int (x^2+1)^5 x\,dx$

6 $\int \sqrt{1-3x}\,dx$

7 $\int \cos^3 x \sin x\,dx$

8 $\int \cos x\,dx/\sin^3 x$

9 $\int \cos^3 2x \sin 2x\,dx$

10 $\int \cos^3 x \sin 2x\,dx$

11 $\int dt/\sqrt{1-t^2}$

12 $\int t\sqrt{1-t^2}\,dt$

13 $\int t^3 dt/\sqrt{1+t^2}$

14 $\int t^3\sqrt{1-t^2}\,dt$

15 $\int (1+\sqrt{x})\,dx/\sqrt{x}$

16 $\int (1+x^{3/2})\sqrt{x}\,dx$

17 $\int \sec x \tan x\,dx$

18 $\int \sec^2 x \tan^2 x\,dx$

19 $\int \cos x \tan x\,dx$

20 $\int \sin^3 x\,dx$

In 21–32 find a function $y(x)$ that solves the differential equation.

21 $dy/dx = x^2 + \sqrt{x}$

22 $dy/dx = y^2$ (try $y = cx^n$)

23 $dy/dx = \sqrt{1-2x}$

24 $dy/dx = 1/\sqrt{1-2x}$

25 $dy/dx = 1/y$

26 $dy/dx = x/y$

27 $d^2y/dx^2 = 1$

28 $d^5y/dx^5 = 1$

29 $d^2y/dx^2 = -y$

30 $dy/dx = \sqrt{xy}$

31 $d^2y/dx^2 = \sqrt{x}$

32 $(dy/dx)^2 = \sqrt{x}$

33 ***True or false***, when f is an antiderivative of v:

(a) $\int v(u(x))\,dx = f(u(x)) + C$

(b) $\int v^2(x)\,dx = \frac{1}{3}f^3(x) + C$

(c) $\int v(x)(du/dx)\,dx = f(u(x)) + C$

(d) $\int v(x)(dv/dx)\,dx = \frac{1}{2}f^2(x) + C$

34 ***True or false***, when f is an antiderivative of v:

(a) $\int f(x)(dv/dx)\,dx = \frac{1}{2}f^2(x) + C$

(b) $\int v(v(x))(dv/dx)\,dx = f(v(x)) + C$

(c) Integral is inverse to derivative so $f(v(x)) = x$

(d) Integral is inverse to derivative so $\int (df/dx)\,dx = f(x)$

35 If $df/dx = v(x)$ then $\int v(x-1)\,dx =$ ______ and $\int v(x/2)\,dx =$ ______.

36 If $df/dx = v(x)$ then $\int v(2x-1)\,dx =$ ______ and $\int v(x^2)x\,dx =$ ______.

37 $\dfrac{x^2}{1+x^2} = 1 - \dfrac{1}{1+x^2}$ so $\displaystyle\int \frac{x^2dx}{1+x^2} =$ ______.

38 $\int (x^2+1)^2 dx$ is not $\frac{1}{3}(x^2+1)^3$ but ______.

39 $\int 2x\,dx/(x^2+1)$ is $\int$ ______ du which will soon be $\ln u$.

40 Show that $\int 2x^3 dx/(1+x^2)^3 = \int (u-1)du/u^3 =$ ______.

41 The acceleration $d^2f/dt^2 = 9.8$ gives $f(t) =$ ______ (*two* integration constants).

42 The solution to $d^4y/dx^4 = 0$ is ______ (*four* constants).

43 If $f(t)$ is an antiderivative of $v(t)$, find antiderivatives of

(a) $v(t+3)$ (b) $v(t)+3$ (c) $3v(t)$ (d) $v(3t)$.

5.5 The Definite Integral

The integral of $v(x)$ is an antiderivative $f(x)$ plus a constant C. This section takes two steps. First, we choose C. Second, we construct $f(x)$. The object is ***to define the integral***—in the most frequent case when a suitable $f(x)$ is not directly known.

The indefinite integral contains "$+C$." The constant is not settled because $f(x)+C$ has the same slope for every C. When we care only about the derivative, C makes no difference. When the goal is a number—a ***definite integral***—C can be assigned a definite value at the starting point.

For mileage traveled, ***we subtract the reading at the start***. This section does the same for area. Distance is $f(t)$ and area is $f(x)$—while the definite integral is $f(b)-f(a)$. Don't pay attention to t or x, pay attention to the great formula of integral calculus:

$$\int_a^b v(t)\,dt=\int_a^b v(x)\,dx=f(b)-f(a). \tag{1}$$

Viewpoint 1 : When f is known, the equation gives the area from a to b.
Viewpoint 2 : When f is *not* known, the equation defines f from the area.

For a typical $v(x)$, we can't find $f(x)$ by guessing or substitution. But still $v(x)$ has an "area" under its graph—and this yields the desired integral $f(x)$.

Most of this section is theoretical, leading to the definition of the integral. You may think we should have defined integrals before computing them—which is logically true. But the idea of area (and the use of rectangles) was already pretty clear in our first examples. Now we go much further. ***Every continuous function*** $v(x)$ ***has an integral*** (also some discontinuous functions). Then the Fundamental Theorem completes the circle: The integral leads back to $df/dx=v(x)$. The area up to x is the antiderivative that we couldn't otherwise discover.

THE CONSTANT OF INTEGRATION

Our goal is to turn $f(x)+C$ into a definite integral— the area between a and b. The first requirement is to have *area* $=$ *zero* at the start:

$$f(a)+C=\text{ starting area }=0 \quad \text{so} \quad C=-f(a). \tag{2}$$

For the area up to x (moving endpoint, indefinite integral), use t as the dummy variable:

the area from a ***to*** x ***is*** $\int_a^x v(t)\,dt=f(x)-f(a)$ (***indefinite integral***)
the area from a ***to*** b ***is*** $\int_a^b v(x)\,dx=f(b)-f(a)$ (***definite integral***)

EXAMPLE 1 The area under the graph of $5(x+1)^4$ from a to b has $f(x)=(x+1)^5$:

$$\int_a^b 5(x+1)^4dx=(x+1)^5\Big]_a^b=(b+1)^5-(a+1)^5.$$

The calculation has two separate steps—first find $f(x)$, then substitute b and a. After the first step, check that df/dx is v. The upper limit in the second step gives *plus*

$f(b)$, the lower limit gives *minus* $f(a)$. Notice the brackets (or the vertical bar):

$$f(x)\Big]_a^b = f(b) - f(a) \quad x^3\big|_1^2 = 8 - 1 \quad \Big[\cos x\Big]_0^{2t} = \cos 2t - 1.$$

Changing the example to $f(x) = (x+1)^5 - 1$ gives an equally good antiderivative—and now $f(0) = 0$. But $f(b) - f(a)$ stays the same, because the -1 disappears:

$$\Big[(x+1)^5 - 1\Big]_a^b = ((b+1)^5 - 1) - ((a+1)^5 - 1) = (b+1)^5 - (a+1)^5.$$

EXAMPLE 2 When $v = 2x \sin x^2$ we recognize $f = -\cos x^2$. The area from 0 to 3 is

$$\int_0^3 2x \sin x^2\, dx = -\cos x^2\Big]_0^3 = -\cos 9 + \cos 0.$$

The upper limit copies the minus sign. The lower limit gives $-(-\cos 0)$, which is $+\cos 0$. ***That example shows the right form for solving exercises on definite integrals.***

Example 2 jumped directly to $f(x) = -\cos x^2$. But most problems involving the chain rule go more slowly—by *substitution*. Set $u = x^2$, with $du/dx = 2x$:

$$\int_0^3 2x \sin x^2\, dx = \int_0^3 \sin u \frac{du}{dx}\, dx = \int_?^? \sin u\, du. \qquad (3)$$

We need new limits when u ***replaces*** x^2. Those limits on u are a^2 and b^2. (In this case $a^2 = 0^2$ and $b^2 = 3^2 = 9$.) ***If*** x ***goes from*** a ***to*** b***, then*** u ***goes from*** $u(a)$ ***to*** $u(b)$.

$$\int_a^b v(u(x)) \frac{du}{dx}\, dx = \int_{u(a)}^{u(b)} v(u)\, du = f(u(b)) - f(u(a)). \qquad (4)$$

EXAMPLE 3 $\displaystyle\int_{x=0}^1 (x^2+5)^3 x\, dx = \int_{u=5}^6 u^3 \frac{du}{2} = \frac{u^4}{8}\Big]_5^6 = \frac{6^4}{8} - \frac{5^4}{8}.$

In this case $u = x^2 + 5$. Therefore $du/dx = 2x$ (or $du = 2x\, dx$ for differentials). We have to account for the missing 2. The integral is $\frac{1}{8}u^4$. The limits on $u = x^2 + 5$ are $u(0) = 0^2 + 5$ and $u(1) = 1^2 + 5$. That is why the u-integral goes from 5 to 6. The alternative is to find $f(x) = \frac{1}{8}(x^2+5)^4$ in one jump (*and check it*).

EXAMPLE 4 $\int_0^1 \sin x^2\, dx = ??$ (***no elementary function gives this integral***).

If we try $\cos x^2$, the chain rule produces an extra $2x$—no adjustment will work. Does $\sin x^2$ still have an antiderivative? *Yes*! Every continuous $v(x)$ has an $f(x)$. Whether $f(x)$ has an algebraic formula or not, we can write it as $\int v(x) dx$. To define that integral, we now take the limit of rectangular areas.

INTEGRALS AS LIMITS OF "RIEMANN SUMS"

We have come to the ***definition of the integral***. The chapter started with the integrals of x and x^2, from formulas for $1 + \cdots + n$ and $1^2 + \cdots + n^2$. We will not go back to those formulas. But for other functions, too irregular to find exact sums, the rectangular areas also approach a limit.

That limit is the integral. This definition is a major step in the theory of calculus. It can be studied in detail, or understood in principle. The truth is that the definition is not so painful—you virtually know it already.

Problem Integrate the continuous function $v(x)$ ***over the interval*** $[a, b]$.

Step 1 Split $[a, b]$ into n subintervals $[a, x_1], [x_1, x_2], \ldots, [x_{n-1}, b]$.

The "meshpoints" $x_1, x_2, \ldots$ divide up the interval from a to b. The endpoints are $x_0 = a$ and $x_n = b$. The length of subinterval k is $\Delta x_k = x_k - x_{k-1}$. In that smaller interval, the minimum of $v(x)$ is m_k. The maximum is M_k.

Now construct rectangles. The "*lower rectangle*" over interval k has height m_k. The "*upper rectangle*" reaches to M_k. Since v is continuous, there are points $x_{\min}$ and $x_{\max}$ where $v = m_k$ and $v = M_k$ (extreme value theorem). ***The graph of*** $v(x)$ ***is in between***.

Important: The area under $v(x)$ contains the area "s" of the lower rectangles:

$$\int_a^b v(x)\,dx \geqslant m_1\,\Delta x_1 + m_2\,\Delta x_2 + \cdots + m_n\,\Delta x_n = s. \tag{5}$$

The area under $v(x)$ is contained in the area "S" of the upper rectangles:

$$\int_a^b v(x)\,dx \leqslant M_1\,\Delta x_1 + M_2\,\Delta x_2 + \cdots + M_n\,\Delta x_n = S. \tag{6}$$

The ***lower sum*** s and the ***upper sum*** S were computed earlier in special cases—when v was x or x^2 and the spacings Δx were equal. Figure 5.9a shows why $s \leqslant$ *area* $\leqslant S$.

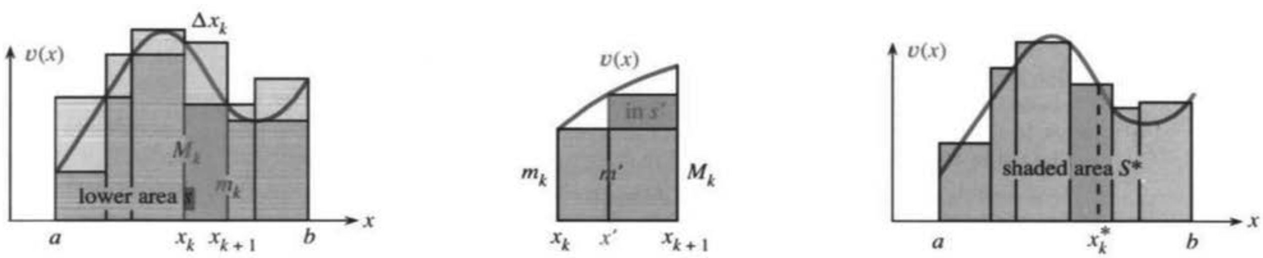


Fig. 5.9 Area of lower rectangles $= s$. Upper sum S includes top pieces. Riemann sum S^* is in between.

Notice an important fact. When a new dividing point x' is added, *the lower sum increases*. The minimum in one piece can be greater (see second figure) than the original m_k. Similarly *the upper sum decreases*. The maximum in one piece can be below the overall maximum. ***As new points are added,*** s ***goes up and*** S ***comes down***. So the sums come closer together:

$$s \leqslant s' \leqslant \qquad \leqslant S' \leqslant S. \tag{7}$$

I have left space in between for the curved area—the integral of $v(x)$.

Now add more and more meshpoints in such a way that $\Delta x_{\max} \to 0$. The lower sums increase and the upper sums decrease. They never pass each other. ***If*** $v(x)$ ***is continuous, those sums close in on a single number*** A. That number is the definite integral—the area under the graph.

DEFINITION The area A is the common limit of the lower and upper sums:

$$s \to A \text{ and } S \to A \text{ as } \Delta x_{\max} \to 0. \tag{8}$$

This limit A exists for all continuous $v(x)$, and also for some discontinuous functions. When it exists, A is the "***Riemann integral***" of $v(x)$ from a to b.

REMARKS ON THE INTEGRAL

As for derivatives, so for integrals: The definition involves a limit. Calculus is built on limits, and we always add "if the limit exists." That is the delicate point. I hope the next five remarks (increasingly technical) will help to distinguish functions that are *Riemann integrable* from functions that are not.

Remark 1 The sums s and S may fail to approach the same limit. A standard example has $V(x) = 1$ at all fractions $x = p/q$, and $V(x) = 0$ at all other points. Every interval contains rational points (fractions) and irrational points (nonrepeating decimals). *Therefore* $m_k = 0$ *and* $M_k = 1$. The lower sum is always $s = 0$. The upper sum is always $S = b - a$ (the sum of 1's times Δx's). ***The gap in equation*** (7) ***stays open***. This function $V(x)$ is not Riemann integrable. The area under its graph is not defined (at least by Riemann—see Remark 5).

Remark 2 The ***step function*** $U(x)$ is discontinuous but still integrable. On every interval the minimum m_k equals the maximum M_k—except on the interval containing the jump. That jump interval has $m_k = 0$ and $M_k = 1$. But when we multiply by Δx_k, and require $\Delta x_{\max} \to 0$, the difference between s and S goes to zero. The area under a step function is clear—the rectangles fit exactly.

Remark 3 With patience another key step could be proved: ***If*** $s \to A$ ***and*** $S \to A$ ***for one sequence of meshpoints, then this limit*** A ***is approached by every choice of meshpoints with*** $\Delta x_{\max} \to 0$. The integral is the lower bound of all upper sums S, and it is the upper bound of all possible s—provided those bounds are equal. The gap must close, to define the integral.

The same limit A is approached by "in-between rectangles." The height $v(x_k^*)$ can be computed at any point x_k^* in subinterval k. See Figures 5.9c and 5.10. Then the total rectangular area is a "***Riemann sum***" between s and S:

$$S^* = v(x_1^*)\,\Delta x_1 + v(x_2^*)\,\Delta x_2 + \cdots + v(x_n^*)\,\Delta x_n. \tag{9}$$

We cannot tell whether the true area is above or below S^*. Very often A is closer to S^* than to s or S. The *midpoint rule* takes x^* in the middle of its interval (Figure 5.10), and Section 5.8 will establish its extra accuracy. The extreme sums s and S are used in the definition while S^* is used in computation.

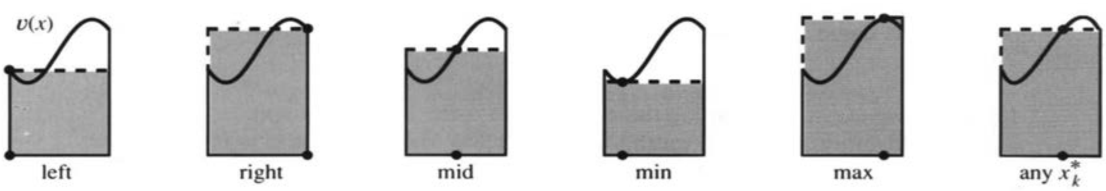


Fig. 5.10 Various positions for x_k^* in the base. The rectangles have height $v(x_k^*)$.

Remark 4 ***Every continuous function is Riemann integrable***. The proof is optional (in my class), but it belongs here for reference. It starts with continuity at x^*: "*For any* ε *there is a* δ ...." When the rectangles sit between $x^* - \delta$ and $x^* + \delta$, the bounds M_k and m_k differ by less than 2ε. Multiplying by the base Δx_k, the areas differ by less than $2\varepsilon(\Delta x_k)$. Combining all rectangles, the upper and lower sums differ by less than $2\varepsilon(\Delta x_1 + \Delta x_2 + \cdots + \Delta x_n) = 2\varepsilon(b - a)$.

As $\varepsilon \to 0$ we conclude that S comes arbitrarily close to s. They squeeze in on a single number A. The Riemann sums approach the Riemann integral, *if v is continuous*.

Two problems are hidden by that reasoning. One is at the end, where S and s come together. We have to know that the line of real numbers has no "holes," so there is a number A to which these sequences converge. That is true.

Any increasing sequence, if it is bounded above, approaches a limit.

The decreasing sequence S, bounded below, converges to the same limit. So A exists.

The other problem is about continuity. We assumed without saying so that the width 2δ is the same around every point x^*. We did not allow for the possibility that δ might approach zero where $v(x)$ is rapidly changing—in which case an infinite number of rectangles could be needed. Our reasoning requires that

*$v(x)$ is **uniformly continuous**: δ depends on ε **but not on the position of** x^*.*

For each ε there is a δ that works at all points in the interval. *A continuous function on a closed interval is **uniformly** continuous*. This fact (proof omitted) makes the reasoning correct, and $v(x)$ is integrable.

On an infinite interval, even $v = x^2$ is not uniformly continuous. It changes across a subinterval by $(x^*+\delta)^2 - (x^*-\delta)^2 = 4x^*\delta$. As x^* gets larger, δ must get smaller—to keep $4x^*\delta$ below ε. No single δ succeeds at all x^*. But on a finite interval $[0,b]$, the choice $\delta = \varepsilon/4b$ works everywhere—so $v = x^2$ is uniformly continuous.

Remark 5 If those four remarks were fairly optional, this one is totally at your discretion. Modern mathematics needs to integrate the zero-one function $V(x)$ in the first remark. Somehow V has more 0's than 1's. The fractions (where $V(x) = 1$) can be put in a list, but the irrational numbers (where $V(x) = 0$) are "uncountable." The integral ought to be zero, but Riemann's upper sums all involve $M_k = 1$.

Lebesgue discovered a major improvement. He allowed *infinitely many subintervals* (smaller and smaller). Then all fractions can be covered with intervals of total width ε. (Amazing, when the fractions are packed so densely.) The idea is to cover $1/q, 2/q, \ldots, q/q$ by narrow intervals of total width $\varepsilon/2^q$. Combining all $q = 1,2,3,\ldots,$ the total width to cover all fractions is no more than $\varepsilon\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots\right) = \varepsilon$. Since $V(x) = 0$ everywhere else, the upper sum S is only ε. And since ε was arbitrary, the "*Lebesgue integral*" is zero as desired.

That completes a fair amount of theory, possibly more than you want or need—but it is satisfying to get things straight. The definition of the integral is still being studied by experts (and so is the derivative, again to allow more functions). By contrast, the *properties* of the integral are used by everybody. Therefore the next section turns from definition to properties, collecting the rules that are needed in applications. They are very straightforward.

5.5 EXERCISES

Read-through questions

In $\int_a^x v(t)\,dt = f(x)+C$, the constant C equals ___a___. Then at $x=a$ the integral is ___b___. At $x=b$ the integral becomes ___c___. The notation $f(x)]_a^b$ means ___d___. Thus $\cos x]_0^\pi$ equals ___e___. Also $[\cos x+3]_0^\pi$ equals ___f___, which shows why the antiderivative includes an arbitrary ___g___. Substituting $u=2x-1$ changes $\int_1^3 \sqrt{2x-1}\,dx$ into ___h___ (with limits on u).

The integral $\int_a^b v(x)dx$ can be defined for any ___i___ function $v(x)$, even if we can't find a simple ___j___. First the meshpoints $x_1, x_2, \ldots$ divide $[a,b]$ into subintervals of length $\Delta x_k =$ ___k___. The upper rectangle with base Δx_k has height $M_k =$ ___l___.The upper sum S is equal to ___m___. The lower sum s is ___n___. The ___o___ is between s and S. As more meshpoints are added, S ___p___ and s ___q___. If S and s approach the same ___r___, that defines the integral. The intermediate sums S^*, named after ___s___, use rectangles of height $v(x_k^*)$. Here x_k^* is any point between ___t___, and $S^* =$ ___u___ approaches the area.

If $v(x) = df/dx$, what constants C make 1–10 true?

1 $\int_2^b v(x)\,dx = f(b)+C$

2 $\int_1^4 v(x)\,dx = f(4)+C$

3 $\int_x^3 v(t)\,dt = -f(x)+C$

4 $\int_{\pi/2}^b v(\sin x)\cos x\,dx = f(\sin b)+C$

5 $\int_1^x v(t)\,dt = f(t)+C$ (careful)

6 $df/dx = v(x)+C$

7 $\int_0^1 (x^2-1)^3 2x\,dx = \int_{-1}^C u^3\,du$.

8 $\int_0^{x^2} v(t)\,dt = f(x^2)+C$

9 $\int_a^b v(-x)\,dx = C$ (change $-x$ to t; also dx and limits)

10 $\int_0^2 v(x)\,dx = C\int_0^1 v(2t)\,dt$.

Choose $u(x)$ in 11–18 and *change limits*. Compute the integral in 11-16.

11 $\int_0^1 (x^2+1)^{10} x\,dx$

12 $\int_0^{\pi/2} \sin^8 x\cos x\,dx$

13 $\int_0^{\pi/4} \tan x\sec^2 x\,dx$

14 $\int_0^2 x^{2n+1}\,dx$ (take $u=x^2$)

15 $\int_0^{\pi/4} \sec^2 x\tan x\,dx$

16 $\int_0^1 x\,dx/\sqrt{1-x^2}$

17 $\int_1^2 dx/x$ (take $u=1/x$)

18 $\int_0^1 x^3(1-x)^3\,dx$ $(u=1-x)$

With $\Delta x = \frac{1}{2}$ in 19–22, find the maximum M_k and minimum m_k and upper and lower sums S and s.

19 $\int_0^1 (x^2+1)^4\,dx$

20 $\int_0^1 \sin 2\pi x\,dx$

21 $\int_0^2 x^3\,dx$

22 $\int_{-1}^1 x\,dx$.

23 Repeat 19 and 20 with $\Delta x = \frac{1}{4}$ and compare with the correct answer.

24 The difference $S-s$ in 21 is the area $2^3\Delta x$ of the far right rectangle. Find Δx so that $S < 4.001$.

25 If $v(x)$ is *increasing* for $a \leqslant x \leqslant b$, the difference $S-s$ is the area of the _____ rectangle minus the area of the _____ rectangle. Those areas approach zero. *So every increasing function on $[a,b]$ is Riemann integrable*.

26 Find the Riemann sum S^* for $V(x)$ in Remark 1, when $\Delta x = 1/n$ and each x_k^* is the midpoint. This S^* is well-behaved but still $V(x)$ is not Riemann integrable.

27 $W(x)$ equals 1 at $x = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$, and elsewhere $W(x)=0$. For $\Delta x = .01$ find the upper sum S. Is $W(x)$ integrable?

28 Suppose $M(x)$ is a *multistep* function with jumps of $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$, at the points $x = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$. Draw a rough graph with $M(0)=0$ and $M(1)=1$. With $\Delta x = \frac{1}{3}$ find S and s.

29 For $M(x)$ in Problem 28 find the difference $S-s$ (which approaches zero as $\Delta x \to 0$). What is the area under the graph?

30 If $df/dx = -v(x)$ and $f(1)=0$, explain $f(x) = \int_x^1 v(t)\,dt$.

31 (a) If $df/dx = +v(x)$ and $f(0)=3$, find $f(x)$.

(b) If $df/dx = +v(x)$ and $f(3)=0$, find $f(x)$.

32 In your own words define the integral of $v(x)$ from a to b.

33 ***True or false***, with reason or example.

(a) Every continuous $v(x)$ has an antiderivative $f(x)$.

(b) If $v(x)$ is not continuous, S and s approach different limits.

(c) If S and s approach A as $\Delta x \to 0$, then all Riemann sums S^* in equation (9) also approach A.

(d) If $v_1(x)+v_2(x) = v_3(x)$, their upper sums satisfy $S_1+S_2 = S_3$.

(e) If $v_1(x)+v_2(x) = v_3(x)$, their Riemann sums at the midpoints x_k^* satisfy $S_1^*+S_2^* = S_3^*$.

(f) The midpoint sum is the average of S and s.

(g) One x_k^* in Figure 5.10 gives the exact area.

5.6 Properties of the Integral and Average Value

The previous section reached the definition of $\int_a^b v(x)\,dx$. But the subject cannot stop there. The integral was defined in order to be used. Its properties are important, and its applications are even more important. The definition was chosen so that the integral has properties that make the applications possible.

One direct application is to the ***average value*** of $v(x)$. The average of n numbers is clear, and the integral extends that idea—it produces the average of a whole continuum of numbers $v(x)$. This develops from the last rule in the following list (Property 7). We now collect together ***seven basic properties of definite integrals***.

The addition rule for $\int [v(x)+w(x)]\,dx$ will not be repeated—even though this property of linearity is the most fundamental. We start instead with a different kind of addition. There is only one function $v(x)$, but now there are two intervals.

The integral from a ***to*** b ***is added to its neighbor from*** b ***to*** c. ***Their sum is the integral from*** a ***to*** c. That is the first (not surprising) property in the list.

***Property* 1** Areas over neighboring intervals add to the area over the combined interval:

$$\int_a^b v(x)\,dx + \int_b^c v(x)\,dx = \int_a^c v(x)\,dx. \tag{1}$$

This sum of areas is graphically obvious (Figure 5.11a). It also comes from the formal definition of the integral. Rectangular areas obey (1)—with a meshpoint at $x=b$ to make sure. When $\Delta x_{\max}$ approaches zero, their limits also obey (1). *All the normal rules for rectangular areas are obeyed in the limit by integrals.*

Property **1** is worth pursuing. It indicates how to define the integral when $a=b$. The integral "*from b to b*" is the area over a point, which we expect to be zero. It is.

***Property* 2** $$\int_b^b v(x)\,dx = 0.$$

That comes from Property **1** when $c=b$. Equation (1) has two identical integrals, so the one from b to b must be zero. Next we see what happens if $c=a$—which makes the second integral go from b to a.

What happens when ***an integral goes backward***? The "lower limit" is now the larger number b. The "upper limit" a is smaller. Going backward reverses the sign:

***Property* 3** $$\int_b^a v(x)\,dx = -\int_a^b v(x)\,dx = f(a)-f(b).$$

Proof When $c=a$ the right side of (1) is zero. Then the integrals on the left side must cancel, which is Property **3**. *In going from* b *to* a *the steps* Δx *are negative*. That justifies a minus sign on the rectangular areas, and a minus sign on the integral (Figure 5.11b). *Conclusion*: Property **1** holds for any ordering of a,b,c.

EXAMPLES $$\int_x^0 t^2\,dt = -\frac{x^3}{3} \qquad \int_1^0 dt = -1 \qquad \int_2^2 \frac{dt}{t} = 0$$

Property* 4** For odd functions $\int_{-a}^a v(x)\,dx = 0$. "Odd***" means that $v(-x) = -v(x)$. For even functions $\int_{-a}^a v(x)\,dx = 2\int_0^a v(x)\,dx$. "***Even***" means that $v(-x) = +v(x)$.

The functions $x, x^3, x^5, \ldots$ are odd. If x changes sign, these powers change sign. The functions $\sin x$ and $\tan x$ are also odd, together with their inverses. This is an important family of functions, and ***the integral of an odd function from $-a$ to a equals zero***. Areas cancel:

$$\int_{-a}^{a} 6x^5\,dx = x^6\Big]_{-a}^{a} = a^6 - (-a)^6 = 0.$$

If $v(x)$ is odd then $f(x)$ is even! All powers $1, x^2, x^4, \ldots$ are even functions. ***Curious fact***: Odd function times even function is *odd*, but odd number times even number is *even*.

For even functions, areas add: $\int_{-a}^{a} \cos x\,dx = \sin a - \sin(-a) = 2\sin a$.

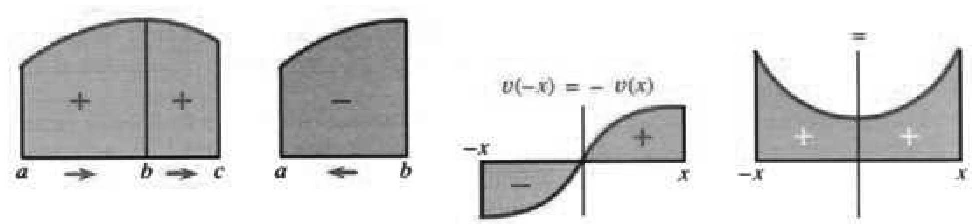


Fig. 5.11 Properties **1–4**: Add areas, change sign to go backward, odd cancels, even adds.

The next properties involve inequalities. If $v(x)$ is positive, the area under its graph is positive (not surprising). Now we have a proof: The lower sums s are positive and they increase toward the area integral. So the integral is positive:

***Property* 5** If $v(x) > 0$ for $a < x < b$ then $\int_a^b v(x)\,dx > 0$.

A positive velocity means a positive distance. A positive v lies above a positive area. A more general statement is true. Suppose $v(x)$ stays between a lower function $l(x)$ and an upper function $u(x)$. Then the rectangles for v stay between the rectangles for l and u. In the limit, the area under v (Figure 5.12) is between the areas under l and u:

***Property* 6** If $l(x) \leqslant v(x) \leqslant u(x)$ for $a \leqslant x \leqslant b$ then

$$\int_a^b l(x)\,dx \leqslant \int_a^b v(x)\,dx \leqslant \int_a^b u(x)\,dx. \qquad (2)$$

EXAMPLE 1 $\cos t \leqslant 1 \quad \Rightarrow \quad \int_0^x \cos t\,dt \leqslant \int_0^x 1\,dt \quad \Rightarrow \quad \sin x \leqslant x$

EXAMPLE 2 $1 \leqslant \sec^2 t \quad \Rightarrow \quad \int_0^x 1\,dt \leqslant \int_0^x \sec^2 t\,dt \quad \Rightarrow \quad x \leqslant \tan x$

EXAMPLE 3 Integrating $\dfrac{1}{1+x^2} \leqslant 1$ leads to $\tan^{-1} x \leqslant x$.

All those examples are for $x > 0$. You may remember that Section 2.4 used geometry to prove $\sin h < h < \tan h$. Examples 1–2 seem to give new and shorter proofs. But I think the reasoning is doubtful. The inequalities were needed to compute the derivatives (therefore the integrals) in the first place.

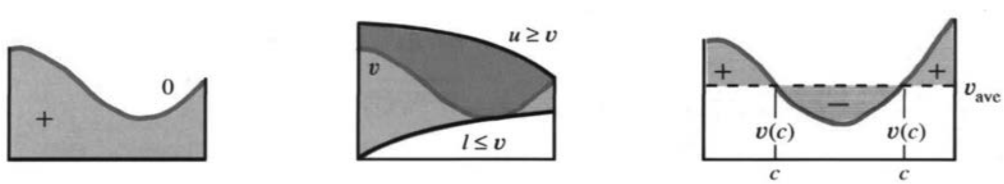


Fig. 5.12 Properties **5–7**: v above zero, v between l and u, average value ($+$ balances $-$).

Property 7 ***(Mean Value Theorem for Integrals)*** If $v(x)$ is continuous, there is a point c between a and b where $v(c)$ equals the average value of $v(x)$:

$$v(c) = \frac{1}{b-a}\int_a^b v(x)\,dx = \text{“}\textbf{\textit{average value of}}\ v(x).\text{”} \qquad (3)$$

This is the same as the ordinary Mean Value Theorem (for the derivative of $f(x)$):

$$f'(c) = \frac{f(b)-f(c)}{b-a} = \text{“average slope of } f.\text{”} \qquad (4)$$

With $f' = v$, (3) and (4) are the same equation. But honesty makes me admit to a flaw in the logic. We need the Fundamental Theorem of Calculus to guarantee that $f(x) = \int_a^x v(t)\,dt$ really gives $f' = v$.

A direct proof of (3) places one rectangle across the interval from a to b. Now raise the top of that rectangle, starting at $v_{\min}$ (the bottom of the curve) and moving up to $v_{\max}$ (the top of the curve). At some height the area will be just right—equal to the area under the curve. Then the rectangular area, which is $(b-a)$ times $v(c)$, equals the curved area $\int_a^b v(x)\,dx$. This is equation (3).

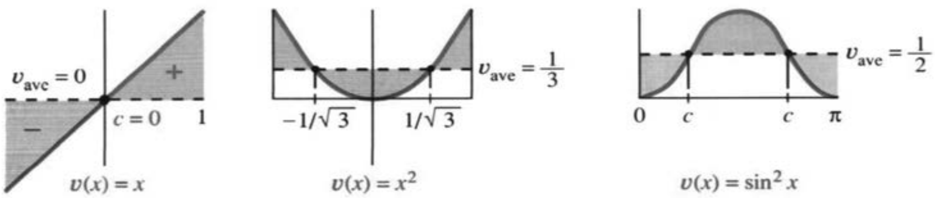


Fig. 5.13 Mean Value Theorem for integrals: area$/(b-a)$ = average height $= v(c)$ at some c.

That direct proof uses the ***intermediate value theorem***: A continuous function $v(x)$ takes on every height between $v_{\min}$ and $v_{\max}$. At some point (at two points in Figure 5.12c) the function $v(x)$ equals its own average value.

Figure 5.13 shows equal areas above and below the average height $v(c) = v_{\text{ave}}$.

EXAMPLE 4 The average value of an odd function is *zero* (between -1 and 1):

$$\frac{1}{2}\int_{-1}^{1} x\,dx = \frac{x^2}{4}\bigg]_{-1}^{1} = \frac{1}{4} - \frac{1}{4} = 0 \qquad \left(\text{note } \frac{1}{b-a} = \frac{1}{2}\right)$$

For once we know c. It is the center point $x = 0$, where $v(c) = v_{\text{ave}} = 0$.

EXAMPLE 5 The average value of x^2 is $\frac{1}{3}$ (between 1 and -1):

$$\frac{1}{2}\int_{-1}^{1} x^2\,dx = \frac{x^3}{6}\bigg]_{-1}^{1} = \frac{1}{6} - \left(-\frac{1}{6}\right) = \frac{1}{3} \qquad \left(\text{note } \frac{1}{b-a} = \frac{1}{2}\right)$$

Where does this function x^2 equal its average value $\frac{1}{3}$? That happens when $c^2 = \frac{1}{3}$, so c can be either of the points $1/\sqrt{3}$ and $-1/\sqrt{3}$ in Figure 5.13b. Those are the ***Gauss points***, which are terrific for numerical integration as Section 5.8 will show.

EXAMPLE 6 The average value of $\sin^2 x$ over a period (zero to π) is $\frac{1}{2}$:

$$\frac{1}{\pi}\int_0^{\pi} \sin x^2\, dx = \frac{x - \sin x\ \cos x}{2\pi}\Big]_0^{\pi} = \frac{1}{2} \qquad \left(\text{note } \frac{1}{b-a} = \frac{1}{\pi}\right)$$

The point c is $\pi/4$ or $3\pi/4$, where $\sin^2 c = \frac{1}{2}$. ***The graph of*** $\sin^2 x$ ***oscillates around its average value*** $\frac{1}{2}$. See the figure or the formula:

$$\sin^2 x = \tfrac{1}{2} - \tfrac{1}{2}\cos 2x. \tag{5}$$

The steady term is $\frac{1}{2}$, the oscillation is $-\frac{1}{2}\cos 2x$. The integral is $f(x) = \frac{1}{2}x - \frac{1}{4}\sin 2x$, which is the same as $\frac{1}{2}x - \frac{1}{2}\sin x \cos x$. *This integral of* $\sin^2 x$ *will be seen again*. Please verify that $df/dx = \sin^2 x$.

THE AVERAGE VALUE AND EXPECTED VALUE

The "average value" from a to b is the integral divided by the length $b-a$. This was computed for x and x^2 and $\sin^2 x$, but not explained. It is a major application of the integral, and it is guided by the ordinary average of n numbers:

$$v_{\text{ave}} = \frac{1}{b-a}\int_a^b v(x)\, dx \qquad \text{comes from} \qquad v_{\text{ave}} = \frac{1}{n}(v_1 + v_2 + \cdots + v_n).$$

Integration is parallel to summation! Sums approach integrals. Discrete averages approach continuous averages. The average of $\frac{1}{3}, \frac{2}{3}, \frac{3}{3}$ is $\frac{2}{3}$. The average of $\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{5}{5}$ is $\frac{3}{5}$. The average of n numbers from $1/n$ to n/n is

$$v_{\text{ave}} = \frac{1}{n}\left(\frac{1}{n} + \frac{2}{n} + \cdots + \frac{n}{n}\right) = \frac{n+1}{2n}. \tag{7}$$

The middle term gives the average, when n is odd. Or we can do the addition. As $n \to \infty$ the sum approaches an integral (do you see the rectangles?). The ordinary average of numbers becomes the continuous average of $v(x) = x$:

$$\frac{n+1}{2n} \to \frac{1}{2} \quad \text{and} \quad \int_0^1 x\, dx = \frac{1}{2} \quad \left(\text{note } \frac{1}{b-a} = 1\right)$$

In ordinary language: "The average value of the numbers between 0 and 1 is $\frac{1}{2}$." Since a whole continuum of numbers lies between 0 and 1, that statement is meaningless until we have integration.

The average value of the *squares* of those numbers is $(x^2)_{\text{ave}} = \int x^2\, dx/(b-a) = \frac{1}{3}$. ***If you pick a number randomly between*** 0 ***and*** 1, ***its expected value is*** $\frac{1}{2}$ ***and its expected square is*** $\frac{1}{3}$.

To me that sentence is a puzzle. First, we don't expect the number to be exactly $\frac{1}{2}$—so we need to define "expected value." Second, if the expected value is $\frac{1}{2}$, why is the expected square equal to $\frac{1}{3}$ instead of $\frac{1}{4}$? The ideas come from probability theory, and calculus is leading us to ***continuous probability***. We introduce it briefly here, and come back to it in Chapter 8.

PREDICTABLE AVERAGES FROM RANDOM EVENTS

Suppose you throw a pair of dice. The outcome is not predictable. Otherwise why throw them? *But the average over more and more throws is totally predictable*. We don't know what will happen, but we know its probability.

For dice, we are adding two numbers between 1 and 6. The outcome is between 2 and 12. The probability of 2 is the chance of two ones: $(1/6)(1/6) = 1/36$. Beside each outcome we can write its probability:

$$2\left(\frac{1}{36}\right) 3\left(\frac{2}{36}\right) 4\left(\frac{3}{36}\right) 5\left(\frac{4}{36}\right) 6\left(\frac{5}{36}\right) 7\left(\frac{6}{36}\right) 8\left(\frac{5}{36}\right) 9\left(\frac{4}{36}\right) 10\left(\frac{3}{36}\right) 11\left(\frac{2}{36}\right) 12\left(\frac{1}{36}\right)$$

To repeat, one roll is unpredictable. Only the probabilities are known, and they add to 1. (Those fractions add to $36/36$; all possibilities are covered.) The total from a million rolls is even more unpredictable—it can be anywhere between $2,000,000$ and $12,000,000$. Nevertheless the *average* of those million outcomes is almost completely predictable. This *expected value* is found by adding the products in that line above:

Expected value: ***multiply*** (*outcome*) ***times*** (*probability of outcome*) ***and add***:

$$\frac{2}{36}+\frac{6}{36}+\frac{12}{36}+\frac{20}{36}+\frac{30}{36}+\frac{42}{36}+\frac{40}{36}+\frac{36}{36}+\frac{30}{36}+\frac{22}{36}+\frac{12}{36}=7.$$

If you throw the dice 1000 times, and the average is not between 6.9 and 7.1, you get an A. Use the random number generator on a computer and round off to integers.

Now comes ***continuous probability***. Suppose all numbers between 2 and 12 are equally probable. This means all numbers—not just integers. What is the probability of hitting the particular number $x = \pi$? *It is zero*! By any reasonable measure, π has *no chance* to occur. In the continuous case, every x has probability zero. But an *interval of* x's has a nonzero probability:

the probability of an outcome between 2 and 3 is $1/10$
the probability of an outcome between x and $x + \Delta x$ is $\Delta x/10$

To find the average, add up each outcome times the probability of that outcome. First divide 2 to 12 into intervals of length $\Delta x = 1$ and probability $p = 1/10$. If we round off x, the average is $6\frac{1}{2}$:

$$2\left(\frac{1}{10}\right)+3\left(\frac{1}{10}\right)+\cdots+11\left(\frac{1}{10}\right)=6.5.$$

Here all outcomes are integers (as with dice). It is more accurate to use 20 intervals of length $1/2$ and probability $1/20$. The average is $6\frac{3}{4}$, and you see what is coming. These are rectangular areas (Riemann sums). As $\Delta x \to 0$ they approach an integral. The probability of an outcome between x and $x + dx$ is $p(x)\,dx$, and this problem has $p(x) = 1/10$. ***The average outcome in the continuous case is not a sum but an integral***:

$$\textbf{\textit{expected value}}\ E(x)=\int_2^{12} x\,p(x)\,dx=\int_2^{12} x\frac{dx}{10}=\frac{x^2}{20}\bigg]_2^{12}=7.$$

That is a big jump. From the point of view of integration, it is a limit of sums. From the point of view of probability, the chance of each outcome is zero but the ***probability***

density at x is $p(x) = 1/10$. The integral of $p(x)$ is 1, because some outcome must happen. The integral of $xp(x)$ is $x_{\text{ave}} = 7$, the expected value. Each choice of x is random, but the average is predictable.

This completes a first step in probability theory. The second step comes after more calculus. Decaying probabilities use e^{-x} and e^{-x^2}—then the chance of a large x is very small. Here we end with the expected values of x^n and $1/\sqrt{x}$ and $1/x$, for a random choice between 0 and 1 (so $p(x) = 1$):

$$E(x^n) = \int_0^1 x^n\,dx = \frac{1}{n+1} \qquad E\left(\frac{1}{\sqrt{x}}\right) = \int_0^1 \frac{dx}{\sqrt{x}} = 2 \qquad E\left(\frac{1}{x}\right) = \int_0^1 \frac{dx}{x} = \infty(!)$$

A CONFUSION ABOUT "EXPECTED" CLASS SIZE

A college can advertise an average class size of 29, while most students are in large classes most of the time. I will show quickly how that happens.

Suppose there are 95 classes of 20 students and 5 classes of 200 students. The total enrollment in 100 classes is $1900 + 1000 = 2900$. A random professor has expected class size 29. But a random student sees it differently. The probability is $1900/2900$ of being in a small class and $1000/2900$ of being in a large class. Adding class size times probability gives the expected class size *for the student*:

$$(20)\left(\frac{1900}{2900}\right) + (200)\left(\frac{1000}{2900}\right) = 82 \text{ students in the class.}$$

Similarly, the average waiting time at a restaurant seems like 40 minutes (to the customer). To the hostess, who averages over the whole day, it is 10 minutes. If you came at a random time it *would* be 10, but if you are a random customer it is 40.

Traffic problems could be eliminated by raising the average number of people per car to 2.5, or even 2. But that is virtually impossible. Part of the problem is the difference between (a) the percentage of cars with one person and (b) the percentage of people alone in a car. Percentage (b) is smaller. In practice, most people would be in crowded cars. See Problems $37-38$.

5.6 EXERCISES

Read-through questions

The integrals $\int_0^b v(x)\,dx$ and $\int_b^5 v(x)\,dx$ add to __a__. The integral $\int_1^3 v(x)\,dx$ equals __b__. The reason is __c__. If $v(x) \leqslant x$ then $\int_0^1 v(x)\,dx \leqslant$ __d__. The average value of $v(x)$ on the interval $1 \leqslant x \leqslant 9$ is defined by __e__. It is equal to $v(c)$ at a point $x = c$ which is __f__. The rectangle across this interval with height $v(c)$ has the same area as __g__. The average value of $v(x) = x + 1$ on the interval $1 \leqslant x \leqslant 9$ is __h__.

If x is chosen from $1, 3, 5, 7$ with equal probabilities $\frac{1}{4}$, its expected value (average) is __i__. The expected value of x^2 is __j__. If x is chosen from $1, 2, \ldots, 8$ with probabilities $\frac{1}{8}$, its expected value is __k__. If x is chosen from $1 \leqslant x \leqslant 9$, the chance of hitting an integer is __l__. The chance of falling between x and $x + dx$ is $p(x)\,dx =$ __m__. The expected value $E(x)$ is the integral __n__. It equals __o__.

In 1–6 find the average value of $v(x)$ between a and b, and find all points c where $v_{\text{ave}} = v(c)$.

1 $v = x^4$, $a = -1$, $b = 1$

2 $v = x^5$, $a = -1$, $b = 1$

3 $v = \cos^2 x$, $a = 0$, $b = \pi$

4 $v = \sqrt{x}$, $a = 0$, $b = 4$

5 $v = 1/x^2$, $a = 1$, $b = 2$

6 $v = (\sin x)^9$, $a = -\pi$, $b = \pi$.

7 At $x = 8$, $F(x) = \int_3^x v(t)\,dt + \int_x^5 v(t)\,dt$ is ______.

8 $\int_1^3 x\,dx + \int_3^5 x\,dx - \int_5^1 x\,dx =$ ______.

Are 9–16 true or false? Give a reason or an example.

9 The minimum of $\int_4^x v(t)\,dt$ is at $x=4$.

10 The value of $\int_x^{x+3} v(t)\,dt$ does not depend on x.

11 The average value from $x=0$ to $x=3$ equals

$$\tfrac{1}{3}(v_{\text{ave}} \text{ on } 0 \leqslant x \leqslant 1) + \tfrac{2}{3}(v_{\text{ave}} \text{ on } 1 \leqslant x \leqslant 3).$$

12 The ratio $(f(b)-f(a))/(b-a)$ is the average value of $f(x)$ on $a \leqslant x \leqslant b$.

13 On the symmetric interval $-1 \leqslant x \leqslant 1$, $v(x) - v_{\text{ave}}$ is an odd function.

14 If $l(x) \leqslant v(x) \leqslant u(x)$ then $dl/dx \leqslant dv/dx \leqslant du/dx$.

15 The average of $v(x)$ from 0 to 2 plus the average from 2 to 4 equals the average from 0 to 4.

16 (a) Antiderivatives of even functions are odd functions.
(b) Squares of odd functions are odd functions.

17 What number $\bar{v}$ gives $\int_a^b (v(x) - \bar{v})\,dx = 0$?

18 If $f(2)=6$ and $f(6)=2$ then the average of df/dx from $x=2$ to $x=6$ is ______.

19 (a) The averages of $\cos x$ and $|\cos x|$ from 0 to π are ______.
(b) The average of the numbers $v_1, \ldots, v_n$ is ______ than the average of $|v_1|, \ldots, |v_n|$.

20 (a) Which property of integrals proves $\int_0^1 v(x)\,dx \leqslant \int_0^1 |v(x)|\,dx$?
(b) Which property proves $-\int_0^1 v(x)\,dx \leqslant \int_0^1 |v(x)|\,dx$?

Together these are ***Property*** **8**: $|\int_0^1 v(x)\,dx| \leqslant \int_0^1 |v(x)|\,dx$.

21 What function has v_{ave} (from 0 to x) equal to $\frac{1}{3}v(x)$ at all x? What functions have $v_{\text{ave}} = v(x)$ at all x?

22 (a) If $v(x)$ is increasing, explain from Property **6** why $\int_0^x v(t)\,dt \leqslant xv(x)$ for $x>0$.
(b) Take derivatives of both sides for a second proof.

23 The average of $v(x) = 1/(1+x^2)$ on the interval $[0,b]$ approaches ______ as $b \to \infty$. The average of $V(x) = x^2/(1+x^2)$ approaches ______.

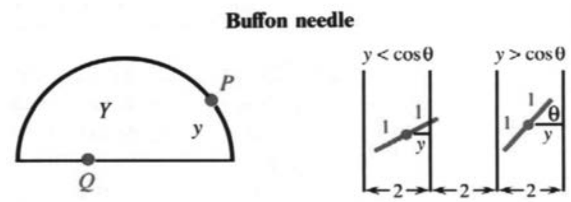
24 If the positive numbers v_n approach zero as $n \to \infty$ prove that their average $(v_1 + \cdots + v_n)/n$ also approaches zero.

25 Find the average distance from $x=a$ to points in the interval $0 \leqslant x \leqslant 2$. Is the formula different if $a<2$?

26 (Computer experiment) Choose random numbers x between 0 and 1 until the average value of x^2 is between .333 and .334. How many values of x^2 are above and below? If possible repeat ten times.

27 A point P is chosen randomly along a semicircle (see figure: equal probability for equal arcs). What is the average distance y from the x axis? The radius is 1.

28 A point Q is chosen randomly between -1 and 1.
(a) What is the average distance Y up to the semicircle?
(b) Why is this different from Problem 27?



29 (A classic way to compute π) A 2″ needle is tossed onto a floor with boards 2″ wide. Find the probability of falling across a crack. (This happens when $\cos\theta > y =$ distance from midpoint of needle to nearest crack. In the rectangle $0 \leqslant \theta \leqslant \pi/2$, $0 \leqslant y \leqslant 1$, shade the part where $\cos\theta > y$ and find the fraction of area that is shaded.)

30 If Buffon's needle has length $2x$ instead of 2, find the probability $P(x)$ of falling across the same cracks.

31 If you roll *three* dice at once, what are the probabilities of each outcome between 3 and 18? What is the expected value?

32 If you choose a random point in the square $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$, what is the chance that its coordinates have $y^2 \leqslant x$?

33 The voltage $V(t) = 220\cos 2\pi t/60$ has frequency 60 hertz and amplitude 220 volts. Find V_{ave} from 0 to t.

34 (a) Show that $v_{\text{even}}(x) = \frac{1}{2}(v(x) + v(-x))$ is always even.
(b) Show that $v_{\text{odd}}(x) = \frac{1}{2}(v(x) - v(-x))$ is always odd.

35 By Problem 34 or otherwise, write $(x+1)^3$ and $1/(x+1)$ as an even function plus an odd function.

36 Prove from the definition of df/dx that it is an odd function if $f(x)$ is even.

37 Suppose four classes have $6, 8, 10$, and 40 students, averaging ______. The chance of being in the first class is ______. The expected class size (for the student) is

$$E(x) = 6\left(\frac{6}{64}\right) + 8\left(\frac{8}{64}\right) + 10\left(\frac{10}{64}\right) + 40\left(\frac{40}{64}\right) = \text{\_\_\_\_\_\_}.$$

38 With groups of sizes $x_1, \ldots, x_n$ adding to G, the average size is ______. The chance of an individual belonging to group 1 is ______. The expected size of his or her group is $E(x) = x_1(x_1/G) + \cdots + x_n(x_n/G)$. * Prove $\Sigma_1^n x_i^2/G \geqslant G/n$.

39 **True or false**, 15 seconds each:
(a) If $f(x) \leqslant g(x)$ then $df/dx \leqslant dg/dx$.
(b) If $df/dx \leqslant dg/dx$ then $f(x) \leqslant g(x)$.
(c) $xv(x)$ is odd if $v(x)$ is even.
(d) If $v_{\text{ave}} \leqslant w_{\text{ave}}$ on all intervals then $v(x) \leqslant w(x)$ at all points.

40 If $v(x) = \begin{cases} 2x & \text{for } x<3 \\ -2x & \text{for } x>3 \end{cases}$ then $f(x) = \begin{cases} x^2 & \text{for } x<3 \\ -x^2 & \text{for } x>3 \end{cases}$.
Thus $\int_0^4 v(x)\,dx = f(4) - f(0) = -16$. Correct the mistake.

41 If $v(x) = |x-2|$ find $f(x)$. Compute $\int_0^5 v(x)\,dx$.

42 Why are there equal areas above and below v_{ave}?

5.7 The Fundamental Theorem and Its Applications

When the endpoints are fixed at a and b, we have a *definite* integral. When the upper limit is a variable point x, we have an *indefinite* integral. More generally: When the endpoints depend in any way on x, ***the integral is a function of*** x. Therefore we can find its derivative. This requires the Fundamental Theorem of Calculus.

The essence of the Theorem is: ***Derivative of integral of*** v ***equals*** v. We also compute the derivative when the integral goes from $a(x)$ to $b(x)$—both limits variable.

Part **2** of the Fundamental Theorem reverses the order: ***Integral of derivative of*** f ***equals*** $f+C$. That will follow quickly from Part **1**, with help from the Mean Value Theorem. It is Part **2** that we use most, since integrals are harder than derivatives.

After the proofs we go to new applications, beyond the standard problem of area under a curve. Integrals can add up rings and triangles and shells—not just rectangles. The answer can be a volume or a probability—not just an area.

THE FUNDAMENTAL THEOREM, PART 1

Start with a continuous function v. Integrate it from a fixed point a to a variable point x. For each x, this integral $f(x)$ is a number. We do not require or expect a formula for $f(x)$—it is the area out to the point x. It is a function of x! The Fundamental Theorem says that this area function has a derivative (another limiting process). ***The derivative*** df/dx ***equals the original*** $v(x)$.

5C (***Fundamental Theorem, Part*** **1**) Suppose $v(x)$ is a continuous function:

$$\textit{If} \quad f(x)=\int_a^x v(t)\,dt \quad \textit{then} \quad df/dx=v(x).$$

The dummy variable is written as t, so we can concentrate on the limits. The value of the integral depends on the limits a and x, not on t.

To find df/dx, start with $\Delta f = f(x+\Delta x)-f(x) =$ *difference of areas*:

$$\Delta f=\int_a^{x+\Delta x} v(t)\,dt-\int_a^x v(t)\,dt=\int_x^{x+\Delta x} v(t)\,dt. \tag{1}$$

Officially, this is Property **1**. The area out to $x+\Delta x$ minus the area out to x equals the small part from x to $x+\Delta x$. Now divide by Δx:

$$\frac{\Delta f}{\Delta x}=\frac{1}{\Delta x}\int_x^{x+\Delta x} v(t)\,dt=\textbf{\textit{average value}}=v(c). \tag{2}$$

This is Property **7**, the Mean Value Theorem for integrals. The average value on this short interval equals $v(c)$. This point c is somewhere between x and $x+\Delta x$ (exact position not known), and we let Δx approach zero. That squeezes c toward x, so $v(c)$ approaches $u(x)$—remember that v is continuous. The limit of equation (2) is the Fundamental Theorem:

$$\frac{\Delta f}{\Delta x}\to\frac{df}{dx} \quad \text{and} \quad v(c)\to v(x) \quad \text{so} \quad \frac{df}{dx}=v(x). \tag{3}$$

If Δx is negative the reasoning still holds. Why assume that $v(x)$ is continuous? Because if v is a step function, then $f(x)$ has a corner where df/dx is not $v(x)$.

We could skip the Mean Value Theorem and simply bound v above and below:

$$\begin{array}{lrcl} \text{for } t \text{ between } x \text{ and } x+\Delta x: & v_{\min} \leqslant & v(t) & \leqslant v_{\max} \\ \text{integrate over that interval:} & v_{\min}\Delta x \leqslant & \Delta f & \leqslant v_{\max}\Delta x \\ \text{divide by } \Delta x: & v_{\min} \leqslant & \Delta f/\Delta x & \leqslant v_{\max} \end{array} \tag{4}$$

As $\Delta x \to 0$, $v_{\min}$ and $v_{\max}$ approach $v(x)$. In the limit df/dx again equals $v(x)$.

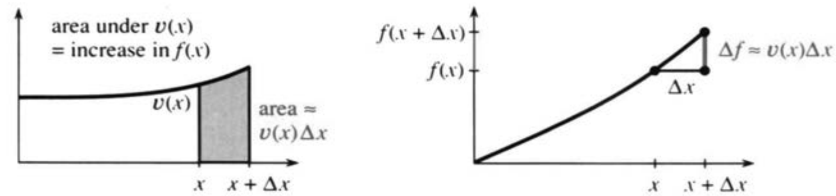


Fig. 5.14 Fundamental Theorem, Part **1**: (thin area Δf)/(base length Δx)$\to$ height $v(x)$.

Graphical meaning The f-graph gives the area under the v-graph. The thin strip in Figure 5.14 has area Δf. ***That area is approximately*** $v(x)$ ***times*** Δx. Dividing by its base, $\Delta f/\Delta x$ is close to the height $v(x)$. When $\Delta x \to 0$ and the strip becomes infinitely thin, the expression "close to" converges to "equals." Then df/dx is the height at $v(x)$.

DERIVATIVES WITH VARIABLE ENDPOINTS

When the upper limit is x, the derivative is $v(x)$. Suppose the *lower limit* is x. The integral goes from x to b, instead of a to x. When x moves, the lower limit moves. The change in area is on the left side of Figure 5.15. *As x goes forward,* ***area is removed***. So there is a minus sign in the derivative of area:

$$\textbf{\textit{The derivative of}} \quad g(x) = \int_x^b v(t)\,dt \quad \textbf{\textit{is}} \quad \frac{dg}{dx} = -v(x). \tag{5}$$

The quickest proof is to reverse b and x, which reverses the sign (Property **3**):

$$g(x) = -\int_b^x v(t)\,dt \quad \text{so by Part } \mathbf{1} \quad \frac{dg}{dx} = -v(x).$$

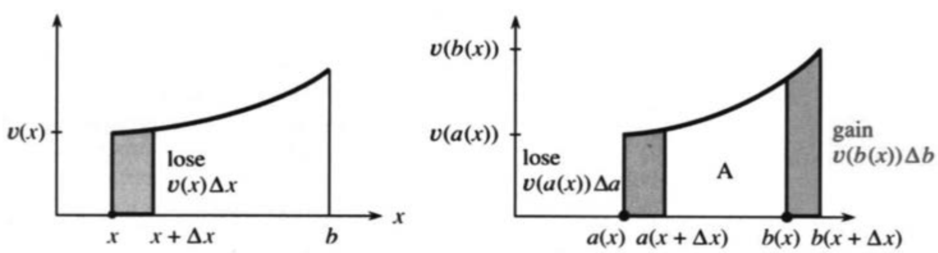


Fig. 5.15 Area from x to b has $dg/dx = -v(x)$. Area $v(b)db$ is added, area $v(a)da$ is lost

The general case is messier but not much harder (it is quite useful). Suppose *both limits* are changing. The upper limit $b(x)$ is not necessarily x, but it depends on x.

The lower limit $a(x)$ can also depend on x (Figure 5.15b). The area A between those limits changes as x changes, and we want dA/dx:

$$\textit{If} \quad A = \int_{a(x)}^{b(x)} v(t)\,dt \quad \textit{then} \quad \frac{dA}{dx} = v(b(x))\frac{db}{dx} - v(a(x))\frac{da}{dx}. \tag{6}$$

The figure shows two thin strips, one added to the area and one subtracted.

First check the two cases we know. When $a = 0$ and $b = x$, we have $da/dx = 0$ and $db/dx = 1$. The derivative according to (6) is $v(x)$ times 1—the Fundamental Theorem. The other case has $a = x$ and $b =$ constant. Then the lower limit in (6) produces $-v(x)$. When the integral goes from $a = 2x$ to $b = x^3$, its derivative is new:

EXAMPLE 1
$$A = \textstyle\int_{2x}^{x^3} \cos t\,dt = \sin x^3 - \sin 2x$$
$$dA/dx = (\cos x^3)(3x^2) - (\cos 2x)(2).$$

That fits with (6), because db/dx is $3x^2$ and da/dx is 2 (with minus sign). It also looks like the chain rule—which it is! To prove (6) we use the letters v and f:

$$A = \int_{a(x)}^{b(x)} v(t)\,dt = f(b(x)) - f(a(x)) \qquad \text{(by Part } \mathbf{2} \text{ below)}$$
$$\frac{dA}{dx} = f'(b(x))\frac{db}{dx} - f'(a(x))\frac{da}{dx} \qquad \text{(by the chain rule)}$$

Since $f' = v$, equation (6) is proved. In the next example the area turns out to be constant, although it seems to depend on x. Note that $v(t) = 1/t$ so $v(3x) = 1/3x$.

EXAMPLE 2 $A = \displaystyle\int_{2x}^{3x} \frac{1}{t}\,dt$ has $\displaystyle\frac{dA}{dx} = \left(\frac{1}{3x}\right)(3) - \left(\frac{1}{2x}\right)(2) = 0.$

Question $A = \displaystyle\int_{-x}^{x} v(t)\,dt$ has $\displaystyle\frac{dA}{dx} = v(x) + v(-x)$. *Why does $v(-x)$ have a plus sign*?

THE FUNDAMENTAL THEOREM, PART 2

We have used a hundred times the Theorem that is now to be proved. It is the key to integration. "*The integral of df/dx is $f(x) + C$.*" The application starts with $v(x)$. We search for an $f(x)$ with this derivative. If $df/dx = v(x)$, the Theorem says that

$$\int v(x)\,dx = \int \frac{df}{dx}\,dx = f(x) + C.$$

We can't rely on knowing formulas for v and f—only the definitions of $\int$ and d/dx.

The proof rests on one extremely special case: df/dx is the ***zero function***. We easily find $f(x) = constant$. The problem is to prove that there are no other possibilities: f *must* be constant. When the slope is zero, the graph *must* be flat. Everybody knows this is true, but intuition is not the same as proof.

Assume that** $df/dx = 0$ **in an interval. If $f(x)$ is not constant, there are points where $f(a) \neq f(b)$. By the Mean Value Theorem, there is a point c where

$$f'(c) = \frac{f(b) - f(a)}{b - a} \quad \text{(this is not zero because } f(a) \neq f(b)\text{)}.$$

But $f'(c) \neq 0$ directly contradicts $df/dx = 0$. Therefore $f(x)$ must be constant.

Note the crucial role of the Mean Value Theorem. A *local* hypothesis ($df/dx = 0$ at each point) yields a *global* conclusion ($f =$ constant in the whole interval). The derivative narrows the field of view, the integral widens it. The Mean Value Theorem connects instantaneous to average, local to global, points to intervals. This special case (the zero function) applies when $A(x)$ and $f(x)$ have the same derivative:

$$\textbf{\textit{If}}\ dA/dx = df/dx\ \textbf{\textit{on an interval}},\ \textbf{\textit{then}}\ A(x) = f(x) + C. \tag{7}$$

Reason: The derivative of $A(x) - f(x)$ is zero. So $A(x) - f(x)$ must be constant.

Now comes the big theorem. It assumes that $v(x)$ is continuous, and integrates using $f(x)$:

> **5D** (***Fundamental Theorem, Part 2***) If $v(x) = \dfrac{df}{dx}$ then $\displaystyle\int_a^b v(x)\,dx = f(b) - f(a)$.

Proof The antiderivative is $f(x)$. But Part **1** gave another antiderivative for the same $v(x)$. It was the integral—constructed from rectangles and now called $A(x)$:

$$A(x) = \int_a^x v(t)\,dt \quad \textbf{\textit{also has}} \quad \frac{dA}{dx} = v(x).$$

Since $A' = v$ and $f' = v$, the special case in equation (7) states that $A(x) = f(x) + C$. That is the essential point: ***The integral from rectangles equals*** $f(x) + C$.

At the lower limit the area integral is $A = 0$. So $f(a) + C = 0$. At the upper limit $f(b) + C = A(b)$. Subtract to find $A(b)$, the definite integral:

$$A(b) = \int_a^b v(x)\,dx = f(b) - f(a).$$

Calculus is beautiful—its Fundamental Theorem is also its most useful theorem.

Another proof of Part **2** starts with $f' = v$ and looks at subintervals:

$$\begin{aligned} f(x_1) - f(a) &= v(x_1^*)(x_1 - a) && \text{(by the Mean Value Theorem)} \\ f(x_2) - f(x_1) &= v(x_2^*)(x_2 - x_1) && \text{(by the Mean Value Theorem)} \\ \ldots &= \ldots && \\ f(b) - f(x_{n-1}) &= v(x_n^*)(b - x_{n-1}) && \text{(by the Mean Value Theorem).} \end{aligned}$$

The left sides add to $f(b) - f(a)$. The sum on the right, as $\Delta x \to 0$, is $\int_a^b v(x)\,dx$.

APPLICATIONS OF INTEGRATION

Up to now the integral has been the area under a curve. There are many other applications, quite different from areas. ***Whenever addition becomes*** "*continuous*," ***we have integrals instead of sums***. Chapter 8 has space to develop more applications, but four examples can be given immediately—which will make the point.

We stay with geometric problems, rather than launching into physics or engineering or biology or economics. All those will come. The goal here is to take a first step away from rectangles.

EXAMPLE 3 (for circles) ***The area*** A ***and circumference*** C ***are related by*** $dA/dr = C$.

The question is why. The area is πr^2. Its derivative $2\pi r$ is the circumference. By the Fundamental Theorem, the integral of C is A. What is missing is the geometrical reason. Certainly πr^2 is the integral of $2\pi r$, but what is the *real* explanation for $A = \int C(r)dr$?

My point is that *the pieces are not rectangles*. We could squeeze rectangles under a circular curve, but their heights would have nothing to do with C. Our intuition has to take a completely different direction, and add up the *thin rings* in Figure 5.16.

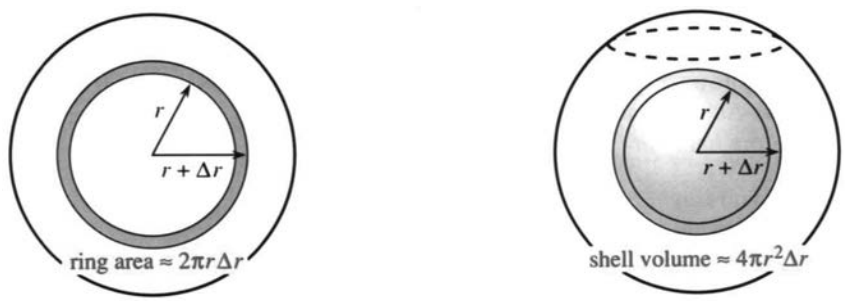


Fig. 5.16 Area of circle = integral over rings. Volume of sphere = integral over shells.

Suppose the ring thickness is Δr. Then the ring area is close to C times Δr. This is precisely the kind of approximation we need, because its error is of higher order $(\Delta r)^2$. *The integral adds ring areas* just as it added rectangular areas:

$$A = \int_0^r C\,dr = \int_0^r 2\pi r\,dr = \pi r^2.$$

That is our first step toward freedom, away from rectangles to rings.

The ring area ΔA can be checked exactly—it is the difference of circles:

$$\Delta A = \pi(r + \Delta r)^2 - \pi r^2 = 2\pi r\,\Delta r + \pi(\Delta r)^2.$$

This is $C\Delta r$ plus a correction. Dividing both sides by $\Delta r \to 0$ leaves $dA/dr = C$.

Finally there is a geometrical reason. The ring unwinds into a thin strip. Its width is Δr and its length is close to C. The inside and outside circles have different perimeters, so this is not a true rectangle—but the area is near $C\Delta r$.

EXAMPLE 4 For a sphere, surface area and volume satisfy $A = dV/dr$.

What worked for circles will work for spheres. The thin rings become *thin shells*. A shell goes from radius r to radius $r + \Delta r$, so its thickness is Δr. We want the volume of the shell, but we don't need it exactly. The surface area is $4\pi r^2$, so the volume is about $4\pi r^2\ \Delta r$. That is close enough!

Again we are correct except for $(\Delta r)^2$. Infinitesimally speaking $dV = A\ dr$:

$$V = \int_0^r A\ dr = \int_0^r 4\pi r^2\ dr = \frac{4}{3}\pi r^3.$$

This is the volume of a sphere. The derivative of V is A, and the shells explain why. Main point: ***Integration is not restricted to rectangles***.

EXAMPLE 5 The distance around a square is $4s$. Why does the area have $dA/ds = 2s$?

The side is s and the area is s^2. Its derivative $2s$ goes only ***half way around the square***. I tried to understand that by drawing a figure. Normally this works, but in the figure dA/ds looks like $4s$. Something is wrong. The bell is ringing so I leave this as an exercise.

EXAMPLE 6 Find the area under $v(x) = \cos^{-1} x$ from $x = 0$ to $x = 1$.

That is a conventional problem, but we have no antiderivative for $\cos^{-1} x$. We could look harder, and find one. However there is another solution—unconventional but correct. ***The region can be filled with horizontal rectangles*** (not vertical rectangles). Figure 5.17b shows a typical strip of length $x = \cos v$ (the curve has $v = \cos^{-1} x$). As the thickness Δv approaches zero, the total area becomes $\int x\, dv$. We are integrating upward, so *the limits are on v not on x*:

$$\text{area} = \int_0^{\pi/2} \cos v \, dv = \sin v \Big]_0^{\pi/2} = 1.$$

The exercises ask you to set up other integrals—not always with rectangles. Archimedes used triangles instead of rings to find the area of a circle.

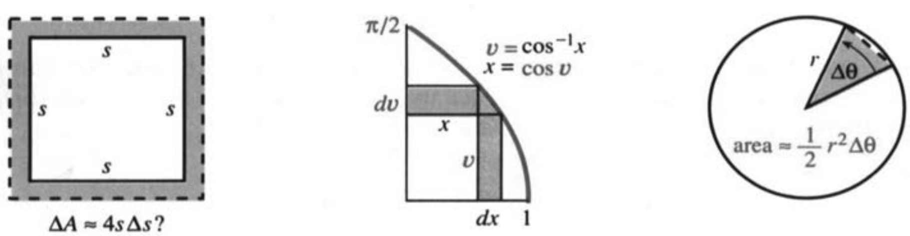


Fig. 5.17 Trouble with a square. Success with horizontal strips and triangles.

5.7 EXERCISES

Read-through questions

The area $f(x) = \int_a^x v(t)\,dt$ is a function of __a__. By Part **1** of the Fundamental Theorem, its derivative is __b__. In the proof, a small change Δx produces the area of a thin __c__. This area Δf is approximately __d__ times __e__. So the derivative of $\int_a^x t^2\,dt$ is __f__.

The integral $\int_x^b t^2\,dt$ has derivative __g__. The minus sign is because __h__. When both limits $a(x)$ and $b(x)$ depend on x, the formula for df/dx becomes __i__ minus __j__. In the example $\int_2^{3x} t\,dt$, the derivative is __k__.

By Part **2** of the Fundamental Theorem, the integral of df/dx is __l__. In the special case when $df/dx = 0$, this says that __m__. From this special case we conclude: If $dA/dx = dB/dx$ then $A(x) =$ __n__. If an antiderivative of $1/x$ is $\ln x$ (whatever that is), then automatically $\int_a^b dx/x =$ __o__.

The square $0 \leqslant x \leqslant s$, $0 \leqslant y \leqslant s$ has area $A =$ __p__. If s is increased by Δs, the extra area has the shape of __q__. That area ΔA is approximately __r__. So $dA/ds =$ __s__.

Find the derivatives of the following functions $F(x)$.

1 $\int_1^x \cos^2 t\,dt$

2 $\int_x^1 \cos 3t\,dt$

3 $\int_0^2 t^n\,dt$

4 $\int_0^2 x^n\,dt$

5 $\int_1^{x^2} u^3\,du$

6 $\int_{-x}^{x/2} v(u)\,du$

7 $\int_x^{x+1} v(t)\,dt$ (a "*running average*" of v)

8 $\dfrac{1}{x}\displaystyle\int_0^x v(t)\,dt$ (the average of v; use product rule)

9 $\dfrac{1}{x}\displaystyle\int_0^x \sin^2 t\,dt$

10 $\dfrac{1}{2}\displaystyle\int_x^{x+2} t^3\,dt$

11 $\int_0^x [\int_0^t v(u)\,du]\,dt$

12 $\int_0^x (df/dt)^2\,dt$

13 $\int_0^x v(t)\,dt + \int_x^1 v(t)\,dt$

14 $\int_0^x v(-t)\,dt$

15 $\int_{-x}^x \sin t^2\,dt$

16 $\int_{-x}^x \sin t\,dt$

17 $\int_0^x u(t)v(t)\,dt$

18 $\int_{a(x)}^{b(x)} 5\,dt$

19 $\displaystyle\int_0^{\sin x} \sin^{-1} t\,dt$

20 $\displaystyle\int_0^{f(x)} \frac{df}{dt}dt$

21 **True or false**

(a) If $df/dx = dg/dx$ then $f(x) = g(x)$.
(b) If $d^2f/dx^2 = d^2g/dx^2$ then $f(x) = g(x) + C$.
(c) If $3 > x$ then the derivative of $\int_3^x v(t)\,dt$ is $-v(x)$.
(d) The derivative of $\int_1^3 v(x)\,dx$ is zero.

22 For $F(x) = \int_x^{2x} \sin t\,dt$, locate $F(\pi + \Delta x) - F(\pi)$ on a sine graph. Where is $F(\Delta x) - F(0)$?

23 Find the function $v(x)$ whose average value between 0 and x is $\cos x$. Start from $\int_0^x v(t)\,dt = x \cos x$.

24 Suppose $df/dx = 2x$. We know that $d(x^2)/dx = 2x$. How do we prove that $f(x) = x^2 + C$?

25 If $\int_{-x}^0 v(t)\,dt = \int_0^x v(t)\,dt$ (equal areas left and right of zero), then $v(x)$ is an ______ function. Take derivatives to prove it.

26 Example 2 said that $\int_{2x}^{3x} dt/t$ does not really depend on x (or t!). Substitute xu for t and watch the limits on u.

27 **True or false**, with reason:

(a) All continuous functions have derivatives.
(b) All continuous functions have antiderivatives.
(c) All antiderivatives have derivatives.
(d) $A(x) = \int_{2x}^{3x} dt/t^2$ has $dA/dx = 0$.

Find $\int_1^x v(t)\,dt$ from the facts in 28–29.

28 $\displaystyle\frac{d(x^n)}{dx} = v(x)$

29 $\displaystyle\int_0^x v(t)\,dt = \frac{x}{x+2}$.

30 What is wrong with Figure 5.17? It seems to show that $dA = 4s\,ds$, which would mean $A = \int 4s\,ds = 2s^2$.

31 The cube $0 \leqslant x, y, z \leqslant s$ has volume $V =$ ______. The three square faces with $x = s$ or $y = s$ or $z = s$ have total area $A =$ ______. If s is increased by Δs, the extra volume has the shape of ______. That volume ΔV is approximately ______. So $dV/ds =$ ______.

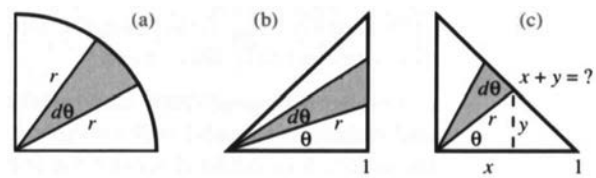
32 The four-dimensional cube $0 \leqslant x, y, z, t \leqslant s$ has hypervolume $H =$ ______. The face with $x = s$ is really a ______. Its volume is $V =$ ______. The total volume of the four faces with $x = s$, $y = s$, $z = s$, or $t = s$ is ______. When s is increased by Δs, the extra hypervolume is $\Delta H \approx$ ______. So $dH/ds =$ ______.

33 The hypervolume of a four-dimensional sphere is $H = \frac{1}{2}\pi^2 r^4$. Therefore the area (volume?) of its three-dimensional surface $x^2 + y^2 + z^2 + t^2 = r^2$ is ______.

34 The area above the parabola $y = x^2$ from $x = 0$ to $x = 1$ is $\frac{2}{3}$. Draw a figure with horizontal strips and integrate.

35 The wedge in Figure (a) has area $\frac{1}{2}r^2 d\theta$. One reason: It is a fraction $d\theta/2\pi$ of the total area πr^2. Another reason: It is close to a triangle with small base $r d\theta$ and height ______. Integrating $\frac{1}{2}r^2 d\theta$ from $\theta = 0$ to $\theta =$ ______ gives the area ______ of a quarter-circle.

36 $A = \int_0^r \sqrt{r^2 - x^2}\,dx$ is also the area of a quarter-circle. Show why, with a graph and thin rectangles. Calculate this integral by substituting $x = r \sin\theta$ and $dx = r\cos\theta\,d\theta$.



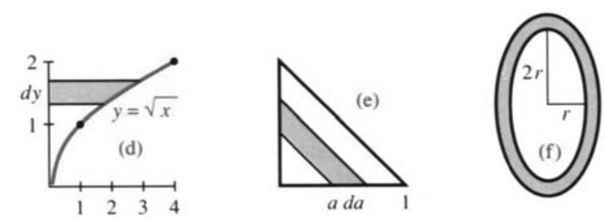
37 The distance r in Figure (b) is related to θ by $r =$ ______. Therefore the area of the thin triangle is $\frac{1}{2}r^2 d\theta =$ ______. Integration to $\theta =$ ______ gives the total area $\frac{1}{2}$.

38 The x and y coordinates in Figure (c) add to $r\cos\theta + r\sin\theta =$ ______. Without integrating explain why

$$\int_0^{\pi/2} \frac{d\theta}{(\cos\theta + \sin\theta)^2} = 1.$$

39 The horizontal strip at height y in Figure (d) has width dy and length $x =$ ______. So the area up to $y = 2$ is ______. What length are the vertical strips that give the same area?

40 Use thin rings to find the area between the circles $r = 2$ and $r = 3$. Draw a picture to show why thin rectangles would be extra difficult.



41 The length of the strip in Figure (e) is approximately ______. The width is ______. Therefore the triangle has area $\int_0^1$ ______ da (do you get $\frac{1}{2}$?).

42 The area of the ellipse in Figure (f) is $2\pi r^2$. Its derivative is $4\pi r$. But this is not the correct perimeter. Where does the usual reasoning go wrong?

43 The derivative of the integral of $v(x)$ is $v(x)$. What is the corresponding statement for sums and differences of the numbers v_j? Prove that statement.

44 The integral of the derivative of $f(x)$ is $f(x)+C$. What is the corresponding statement for sums of differences of f_j? Prove that statement.

45 Does $d^2f/dx^2 = a(x)$ lead to $\int_0^1 (\int_0^x a(t)\,dt)dx = f(1) - f(0)$?

46 The mountain $y = -x^2 + t$ has an area $A(t)$ above the x axis. As t increases so does the area. Draw an xy graph of the mountain at $t = 1$. What line gives dA/dt? Show with words or derivatives that $d^2A/dt^2 > 0$.

5.8 Numerical Integration

This section concentrates on definite integrals. The inputs are $y(x)$ and two endpoints a and b. The output is the integral I. Our goal is to find that number $\int_a^b y(x)\,dx = I$, accurately and in a short time. Normally this goal is achievable—as soon as we have a good method for computing integrals.

Our two approaches so far have weaknesses. The search for an antiderivative succeeds in important cases, and Chapter 7 extends that range—but generally $f(x)$ is not available. The other approach (by rectangles) is in the right direction but too crude. The height is set by $y(x)$ at the right and left end of each small interval. The ***right and left rectangle rules*** add the areas (Δx times y):

$$R_n = (\Delta x)(y_1 + y_2 + \cdots + y_n) \quad \text{and} \quad L_n = (\Delta x)(y_0 + y_1 + \cdots + y_{n-1}).$$

The value of $y(x)$ at the end of interval j is y_j. The extreme left value $y_0 = y(a)$ enters L_n. With n equal intervals of length $\Delta x = (b-a)/n$, the extreme right value is $y_n = y(b)$. It enters R_n. Otherwise the sums are the same—simple to compute, easy to visualize, but very inaccurate.

This section goes from slow methods (*rectangles*) to better methods (*trapezoidal and midpoint*) to good methods (*Simpson and Gauss*). Each improvement cuts down the error. You could discover the formulas without the book, by integrating x and x^2 and x^4. The rule R_n would come out on one side of the answer, and L_n would be on the other side. You would figure out what to do next, to come closer to the exact integral. The book can emphasize one key point:

> ***The quality of a formula depends on how many integrals*** $\int 1\,dx, \int x\,dx, \int x^2\,dx, \ldots,$ ***it computes exactly. If*** $\int x^p\,dx$ ***is the first to be wrong, the order of accuracy*** is p.

By testing the integrals of $1, x, x^2, \ldots$, we decide how accurate the formulas are.

Figure 5.18 shows the rectangle rules R_n and L_n. ***They are already wrong when*** $y = x$. These methods are ***first-order***: $p = 1$. The errors involve the first power of Δx—where we would much prefer a higher power. A larger p in $(\Delta x)^p$ means a smaller error.

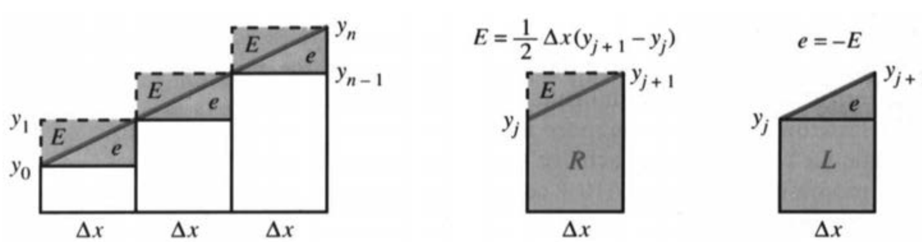


Fig. 5.18 Errors E and e in R_n and L_n are the areas of triangles.

When the graph of $y(x)$ is a straight line, the integral I is known. The error triangles E and e have base Δx. Their heights are the differences $y_{j+1} - y_j$. The areas are $\frac{1}{2}$(base)(height), and the only difference is a minus sign. (L is too low, so the error $L - I$ is negative.) The total error in R_n is the sum of the E's:

$$R_n - I = \tfrac{1}{2}\Delta x(y_1 - y_0) + \cdots + \tfrac{1}{2}\Delta x(y_n - y_{n-1}) = \tfrac{1}{2}\Delta x(y_n - y_0). \quad (1)$$

All y's between y_0 and y_n cancel. Similarly for the sum of the e's:

$$L_n - I = -\tfrac{1}{2}\Delta x(y_n - y_0) = -\tfrac{1}{2}\Delta x[y(b) - y(a)]. \quad (2)$$

The greater the slope of $y(x)$, the greater the error—since rectangles have zero slope.

Formulas (1) and (2) are nice—*but those errors are large*. To integrate $y = x$ from $a = 0$ to $b = 1$, the error is $\frac{1}{2}\Delta x(1-0)$. It takes $500,000$ rectangles to reduce this error to $1/1,000,000$. This accuracy is reasonable, but that many rectangles is unacceptable.

The beauty of the error formulas is that they are "*asymptotically correct*" for all functions. When the graph is curved, the errors don't fit exactly into triangles. But the ratio of predicted error to actual error approaches 1. As $\Delta x \to 0$, the graph is almost straight in each interval—this is linear approximation.

The error prediction $\frac{1}{2}\Delta x[y(b) - y(a)]$ is so simple that we test it on $y(x) = \sqrt{x}$:

$I = \int_0^1 \sqrt{x}\, dx = \frac{2}{3}$

$n =$	1	10	100	1000
error $R_n - I =$	.33	.044	.0048	.00049
error $L_n - I =$	$-.67$	$-.056$	$-.0052$	$-.00051$

The error decreases along each row. So does $\Delta x = .1, .01, .001, .0001$. Multiplying n by 10 divides Δx by 10. The error is also divided by 10 (almost). ***The error is nearly proportional to*** Δx—typical of first-order methods.

The predicted error is $\frac{1}{2}\Delta x$, since here $y(1) = 1$ and $y(0) = 0$. The computed errors in the table come closer and closer to $\frac{1}{2}\Delta x = .5, .05, .005, .0005$. The prediction is the "leading term" in the actual error.

The table also shows a curious fact. Subtracting the last row from the row above gives exact numbers $1, .1, .01$, and .001. This is $(R_n - I) - (L_n - I)$, which is $R_n - L_n$. It comes from an extra rectangle at the right, included in R_n but not L_n. Its height is 1 and its area is $1, .1, .01, .001$.

The errors in R_n and L_n almost cancel. The average $T_n = \frac{1}{2}(R_n + L_n)$ has less error—it is the "trapezoidal rule." First we give the rectangle prediction two final tests:

		$n = 1$	$n = 10$	$n = 100$	$n = 1000$
$\int (x^2 - x)\, dx$:	errors	$1.7 \cdot 10^{-1}$	$1.7 \cdot 10^{-3}$	$1.7 \cdot 10^{-5}$	$1.7 \cdot 10^{-7}$
$\int dx/(10 + \cos 2\pi x)$:	errors	$-1 \cdot 10^{-3}$	$2 \cdot 10^{-14}$	"0"	"0"

Those errors are falling *faster* than Δx. For $y = x^2 - x$ the prediction explains why: $y(0)$ equals $y(1)$. The leading term, with $y(b)$ minus $y(a)$, is *zero*. The exact errors are $\frac{1}{6}(\Delta x)^2$, dropping from 10^{-1} to 10^{-3} to 10^{-5} to 10^{-7}. In these examples L_n is identical to R_n (and also to T_n), because the end rectangles are the same. We will see these $\frac{1}{6}(\Delta x)^2$ errors in the trapezoidal rule.

The last row in the table is more unusual. It shows practically no error. Why do the rectangle rules suddenly achieve such an outstanding success?

The reason is that $y(x) = 1/(10 + \cos 2\pi x)$ is ***periodic***. The leading term in the error is zero, because $y(0) = y(1)$. Also the next term will be zero, because $y'(0) = y'(1)$. Every power of Δx is multiplied by zero, when we integrate over a complete period. So the errors go to zero exponentially fast.

Personal note I tried to integrate $1/(10 + \cos 2\pi x)$ by hand and failed. Then I was embarrassed to discover the answer in my book on applied mathematics. The method was a special trick using complex numbers, which applies over an exact period. Finally I found the antiderivative (quite complicated) in a handbook of integrals, and verified the area $1/\sqrt{99}$.

THE TRAPEZOIDAL AND MIDPOINT RULES

We move to integration formulas that are exact when $y = x$. They have ***second-order accuracy***. The Δx error term disappears. The formulas give the correct area under straight lines. The predicted error is a multiple of $(\Delta x)^2$. That multiple is found by testing $y = x^2$—for which the answers are not exact.

The first formula combines R_n and L_n. To cancel as much error as possible, take the average $\frac{1}{2}(R_n + L_n)$. This yields the ***trapezoidal rule***, which approximates $\int y(x)\,dx$ by T_n:

$$T_n = \tfrac{1}{2}R_n + \tfrac{1}{2}L_n = \Delta x\left(\tfrac{1}{2}y_0 + y_1 + \cdots + y_{n-1} + \tfrac{1}{2}y_n\right). \tag{3}$$

Another way to find T_n is from the area of the "trapezoid" below $y = x$ in Figure 5.19a.

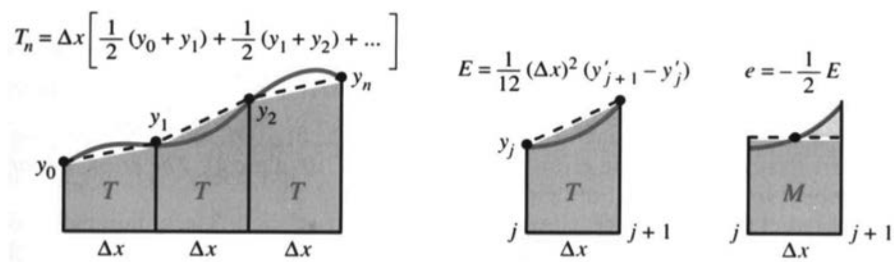


Fig. 5.19 Second-order accuracy: The error prediction is based on $v = x^2$.

The base is Δx and the sides have heights y_{j-1} and y_j. Adding those areas gives $\frac{1}{2}(L_n + R_n)$ in formula (3)—the coefficients of y_j combine into $\frac{1}{2} + \frac{1}{2} = 1$. Only the first and last intervals are missing a neighbor, so the rule has $\frac{1}{2}y_0$ and $\frac{1}{2}y_n$. Because trapezoids (unlike rectangles) fit under a sloping line, T_n is exact when $y = x$.

What is the difference from rectangles? The trapezoidal rule gives weight $\frac{1}{2}\Delta x$ to y_0 and y_n. The rectangle rule R_n gives full weight Δx to y_n (and no weight to y_0). $R_n - T_n$ is exactly the leading error $\frac{1}{2}y_n - \frac{1}{2}y_0$. The change to T_n knocks out that error.

Another important formula is exact for $y(x) = x$. A rectangle has the same area as a trapezoid, if the height of the rectangle is halfway between y_{j-1} and y_j. On a straight line graph that is achieved at the *midpoint* of the interval. By evaluating $y(x)$ at the halfway points $\frac{1}{2}\Delta x, \frac{3}{2}\Delta x, \frac{5}{2}\Delta x, \ldots$, we get much better rectangles. This leads to the ***midpoint rule*** M_n:

$$M_n = \Delta x\left(y_{1/2} + y_{3/2} + \cdots + y_{n-1/2}\right). \tag{4}$$

For $\int_0^4 x\,dx$, trapezoids give $\frac{1}{2}(0) + 1 + 2 + 3 + \frac{1}{2}(4) = 8$. The midpoint rule gives $\frac{1}{2} + \frac{3}{2} + \frac{5}{2} + \frac{7}{2} = 8$, again correct. The rules become different when $y = x^2$, because $y_{1/2}$ is no longer the average of y_0 and y_1. Try both second-order rules on x^2:

$I = \int_0^1 x^2\,dx$	$n =$	1	10	100
	error $T_n - I =$	$1/6$	$1/600$	$1/60000$
	error $M_n - I =$	$-1/12$	$-1/1200$	$-1/120000$

The errors fall by 100 when n is multiplied by 10. The midpoint rule is twice as good ($-1/12$ vs. $1/6$). Since all smooth functions are close to parabolas (quadratic approximation in each interval), the leading errors come from Figure 5.19. The trapezoidal error is exactly $\frac{1}{6}(\Delta x)^2$ when $y(x)$ is x^2 (the 12 in the formula divides the 2 in y'):

$$T_n - I \approx \frac{1}{12}(\Delta x)^2\left[(y_1' - y_0') + \cdots + (y_n' - y_{n-1}')\right] = \frac{1}{12}(\Delta x)^2\left[y_n' - y_0'\right] \quad (5)$$

$$M_n - I \approx -\frac{1}{24}(\Delta x)^2\left[y_n' - y_0'\right] = -\frac{1}{24}(\Delta x)^2\left[y'(b) - y'(a)\right] \quad (6)$$

For exact error formulas, change $y'(b) - y'(a)$ to $(b-a)y''(c)$. The location of c is unknown (as in the Mean Value Theorem). In practice these formulas are not much used—they involve the pth derivative at an unknown location c. The main point about the error is the factor $(\Delta x)^p$.

One crucial fact is easy to overlook in our tests. *Each value of $y(x)$ can be extremely hard to compute*. Every time a formula asks for y_j, a computer calls a subroutine. The goal of numerical integration is to get below the error tolerance, while calling for *a minimum number of values of y*. Second-order rules need about a thousand values for a typical tolerance of 10^{-6}. The next methods are better.

FOURTH-ORDER RULE: SIMPSON

The trapezoidal error is nearly twice the midpoint error ($1/6$ vs. $-1/12$). So a good combination will have twice as much of M_n as T_n. That is ***Simpson's rule***:

$$S_n = \frac{1}{3}T_n + \frac{2}{3}M_n = \frac{1}{6}\Delta x\left[y_0 + 4y_{1/2} + 2y_1 + 4y_{3/2} + 2y_2 + \cdots + 4y_{n-1/2} + y_n\right]. \quad (7)$$

Multiply the midpoint values by $2/3 = 4/6$. The endpoint values are multiplied by $2/6$, except at the far ends a and b (with heights y_0 and y_n). This $1-4-2-4-2-4-1$ pattern has become famous.

Simpson's rule goes deeper than a combination of T and M. It comes from a *parabolic* approximation to $y(x)$ in each interval. When a parabola goes through y_0, $y_{1/2}, y_1$, the area under it is $\frac{1}{6}\Delta x(y_0 + 4y_{1/2} + y_1)$. This is S over the first interval. ***All our rules are constructed this way***: ***Integrate correctly as many powers*** $1, x, x^2, \ldots$ ***as possible***. Parabolas are better than straight lines, which are better than flat pieces. S beats M, which beats R. Check Simpson's rule on powers of x, with $\Delta x = 1/n$:

	$n=1$	$n=10$	$n=100$
error if $y = x^2$	0	0	0
error if $y = x^3$	0	0	0
error if $y = x^4$	$8.33 \cdot 10^{-3}$	$8.33 \cdot 10^{-7}$	$8.33 \cdot 10^{-11}$

Exact answers for x^2 are no surprise. S_n was selected to get parabolas right. But the zero errors for x^3 were not expected. The accuracy has jumped to ***fourth order***, with errors proportional to $(\Delta x)^4$. That explains the popularity of Simpson's rule.

To understand why x^3 is integrated exactly, look at the interval $[-1,1]$. The odd function x^3 has zero integral, and Simpson agrees by symmetry:

$$\int_{-1}^{1} x^3\,dx = \frac{1}{4}x^4\Big]_{-1}^{1} = 0 \quad \text{and} \quad \frac{2}{6}\Big[(-1)^3 + 4(0)^3 + 1^3\Big] = 0. \quad (8)$$

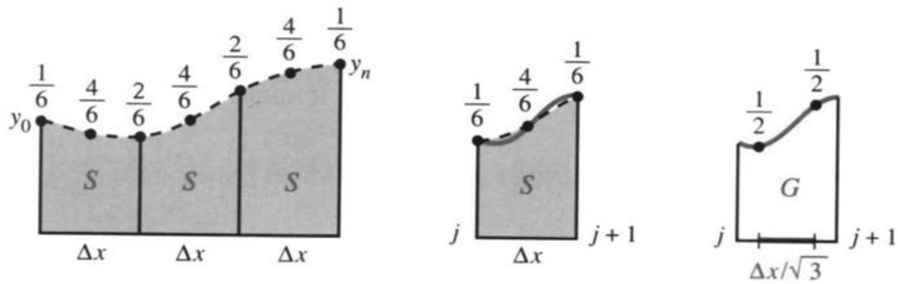


Fig. 5.20 Simpson versus Gauss: $E = c(\Delta x)^4(y'''_{j+1} - y'''_j)$ with $c_S = 1/2880$ and $c_G = -1/4320$.

THE GAUSS RULE (OPTIONAL)

We need a competitor for Simpson, and Gauss can compete with anybody. He calculated integrals in astronomy, and discovered that *two points are enough for a fourth-order method*. From -1 to 1 (a single interval) his rule is

$$\int_{-1}^{1} y(x)\,dx \approx y(-1/\sqrt{3}) + y(1/\sqrt{3}). \tag{9}$$

Those "***Gauss points***" $x = -1/\sqrt{3}$ and $x = 1/\sqrt{3}$ can be found directly. By placing them symmetrically, all odd powers $x, x^3, \ldots$ are correctly integrated. The key is in $y = x^2$, whose integral is $2/3$. The Gauss points $-x_G$ and $+x_G$ get this integral right:

$$\frac{2}{3} = (-x_G)^2 + (x_G)^2, \text{ so } x_G^2 = \frac{1}{3} \quad \text{and} \quad x_G = \pm\frac{1}{\sqrt{3}}.$$

Figure 5.20c shifts to the interval from 0 to Δx. The Gauss points are $(1 \pm 1/\sqrt{3})\Delta x/2$. They are not as convenient as Simpson's (which hand calculators prefer). Gauss is good for thousands of integrations over one interval. Simpson is good when intervals go back to back—then Simpson also uses two y's per interval. For $y = x^4$, you see both errors drop by 10^{-4} in comparing $n = 1$ to $n = 10$:

$I = \int_0^1 x^4\,dx$	Simpson error	$8.33 \cdot 10^{-3}$	$8.33 \cdot 10^{-7}$
	Gauss error	$-5.56 \cdot 10^{-3}$	$-5.56 \cdot 10^{-7}$

DEFINITE INTEGRALS ON A CALCULATOR

It is fascinating to know how numerical integration is actually done. The points are not equally spaced! For an integral from 0 to 1, Hewlett-Packard machines might internally replace x by $3u^2 - 2u^3$ (the limits on u are also 0 and 1). The machine remembers to change dx. For example,

$$\int_0^1 \frac{dx}{\sqrt{x}} \quad \text{becomes} \quad \int_0^1 \frac{6(u-u^2)du}{\sqrt{3u^2-2u^3}} = \int_0^1 \frac{6(1-u)du}{\sqrt{3-2u}}.$$

Algebraically that looks worse—but the infinite value of $1/\sqrt{x}$ at $x = 0$ disappears at $u = 0$. The differential $6(u - u^2)du$ was chosen to vanish at $u = 0$ and $u = 1$. We don't need $y(x)$ at the endpoints—where infinity is most common. In the u variable the integration points are equally spaced—therefore in x they are not.

When a difficult point is *inside* $[a, b]$, break the interval in two pieces. And chop off integrals that go out to infinity. The integral of e^{-x^2} should be stopped by $x = 10$, since the tail is so thin. (It is bad to go too far.) Rapid oscillations are among the toughest—the answer depends on cancellation of highs and lows, and the calculator requires many integration points.

The change from x to u affects periodic functions. I thought equal spacing was good, since $1/(10+\cos 2\pi x)$ was integrated above to enormous accuracy. But there is a danger called ***aliasing***. If $\sin 8\pi x$ is sampled with $\Delta x = 1/8$, it is always zero. A high frequency 8 is confused with a low frequency 0 (its "alias" which agrees at the sample points). With unequal spacing the problem disappears. *Notice how any integration method can be deceived*:

> Ask for the integral of $y=0$ and specify the accuracy. The calculator samples y at $x_1, \ldots, x_k$. (With a PAUSE key, the x's may be displayed.) Then integrate $Y(x) = (x-x_1)^2 \cdots (x-x_k)^2$. That also returns the answer zero (now wrong), because the calculator follows the same steps.

On the calculator you enter the function, the endpoints, and the accuracy. The variable x can be named or not (see the margin). The outputs 4.67077 and 4.7E-5 are the requested integral $\int_1^2 e^x\,dx$ and the estimated error bound. Your input accuracy .00001 guarantees

$$\text{relative error in } y = \left| \frac{\text{true } y - \text{computed } y}{\text{computed } y} \right| \leqslant .00001.$$

The machine estimates accuracy based on its experience in sampling $y(x)$. If you guarantee e^x within $.0000000001$, it thinks you want high accuracy and takes longer.

In consulting for HP, William Kahan chose formulas using $1, 3, 7, 15, \ldots$ sample points. Each new formula uses the samples in the previous formula. The calculator stops when answers are close.

5.8 EXERCISES

Read-through questions

To integrate $y(x)$, divide $[a,b]$ into n pieces of length $\Delta x =$ __a__. R_n and L_n place a __b__ over each piece, using the height at the right or __c__ endpoint: $R_n = \Delta x(y_1 + \cdots + y_n)$ and $L_n =$ __d__. These are __e__ order methods, because they are incorrect for $y =$ __f__. The total error on $[0,1]$ is approximately __g__. For $y = \cos \pi x$ this leading term is __h__. For $y = \cos 2\pi x$ the error is very small because $[0,1]$ is a complete __i__.

A much better method is $T_n = \frac{1}{2}R_n +$ __j__ $= \Delta x[\frac{1}{2}y_0 +$ __k__ $y_1 + \cdots +$ __l__ $y_n]$. This __m__ rule is __n__-order because the error for $y = x$ is __o__. The error for $y = x^2$ from a to b is __p__. The __q__ rule is twice as accurate, using $M_n = \Delta x[$__r__$]$.

Simpson's method is $S_n = \frac{2}{3}M_n +$ __s__. It is __t__-order, because the powers __u__ are integrated correctly. The coefficients of $y_0, y_{1/2}, y_1$ are __v__ times Δx. Over three intervals the weights are $\Delta x/6$ times $1-4-$__w__. Gauss uses __x__ points in each interval, separated by $\Delta x/\sqrt{3}$. For a method of order p the error is nearly proportional to __y__.

1 What is the difference $L_n - T_n$? Compare with the leading error term in (2).

2 If you cut Δx in half, by what factor is the trapezoidal error reduced (approximately)? By what factor is the error in Simpson's rule reduced?

3 Compute R_n and L_n for $\int_0^1 x^3\,dx$ and $n = 1, 2, 10$. Either verify (with computer) or use (without computer) the formula $1^3 + 2^3 + \cdots + n^3 = \frac{1}{4}n^2(n+1)^2$.

4 One way to compute T_n is by averaging $\frac{1}{2}(L_n + R_n)$. Another way is to add $\frac{1}{2}y_0 + y_1 + \cdots + \frac{1}{2}y_n$. Which is more efficient? Compare the number of operations.

5 Test three different rules on $I = \int_0^1 x^4\,dx$ for $n = 2, 4, 8$.

6 Compute π to six places as $4\int_0^1 dx/(1+x^2)$, using any rule.

7 Change Simpson's rule to $\Delta x\left(\frac{1}{4}y_0 + \frac{1}{2}y_{1/2} + \frac{1}{4}y_1\right)$ in each interval and find the order of accuracy p.

8 Demonstrate superdecay of the error when $1/(3 + \sin x)$ is integrated from 0 to 2π.

9 Check that $(\Delta x)^2(y'_{j+1} - y'_j)/12$ is the correct error for $y = 1$ and $y = x$ and $y = x^2$ from the first trapezoid ($j = 0$). Then it is correct for every parabola over every interval.

10 Repeat Problem 9 for the midpoint error $-(\Delta x)^2(y'_{j+1} - y'_j)/24$. Draw a figure to show why the rectangle M has the same area as any trapezoid through the midpoint (including the trapezoid tangent to $y(x)$).

11 In principle $\int_{-\infty}^{\infty} \sin^2 x\,dx/x^2 = \pi$. With a symbolic algebra code or an HP-28S, how many decimal places do you get? Cut off the integral to $\int_{-A}^{A}$ and test large and small A.

12 These four integrals all equal π:

$$\int_0^1 \frac{dx}{\sqrt{x(1-x)}} \quad \int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx \quad \frac{8}{3}\int_0^{\pi} \sin^4 x\,dx \quad \int_0^{\infty} \frac{x^{-1/2}dx}{1+x}$$

(a) Apply the midpoint rule to two of them until $\pi \approx 3.1416$.

(b) Optional: Pick the other two and find $\pi \approx 3$.

13 To compute $\ln 2 = \int_1^2 dx/x = .69315$ with error less than .001, how many intervals should T_n need? Its leading error is $(\Delta x)^2[y'(b) - y'(a)]/12$. Test the actual error with $y = 1/x$.

14 Compare T_n with M_n for $\int_0^1 \sqrt{x}\,dx$ and $n = 1, 10, 100$. The error prediction breaks down because $y'(0) = \infty$.

15 Take $f(x) = \int_0^x y(x)\,dx$ in error formula **3R** to prove that $\int_0^{\Delta x} y(x)dx - y(0)\Delta x$ is exactly $\frac{1}{2}(\Delta x)^2 y'(c)$ some point c.

16 For the periodic function $y(x) = 1/(2 + \cos 6\pi x)$ from -1 to 1, compare T and S and G for $n = 2$.

17 For $I = \int_0^1 \sqrt{1-x^2}\,dx$, the leading error in the trapezoidal rule is ______. Try $n = 2, 4, 8$ to defy the prediction.

18 Change to $x = \sin\theta, \sqrt{1-x^2} = \cos\theta, dx = \cos\theta\,d\theta$, and repeat T_4 on $\int_0^{\pi/2} \cos^2\theta\,d\theta$. What is the predicted error after the change to θ?

19 Write down the three equations $Ay(0) + By\left(\frac{1}{2}\right) + Cy(1) = I$ for the three integrals $I = \int_0^1 1\,dx, \int_0^1 x\,dx, \int_0^1 x^2\,dx$. Solve for A, B, C and name the rule.

20 Can you invent a rule using $Ay_0 + By_{1/4} + Cy_{1/2} + Dy_{3/4} + Ey_1$ to reach higher accuracy than Simpson's?

21 Show that T_n is the only combination of L_n and R_n that has second-order accuracy.

22 Calculate $\int e^{-x^2}\,dx$ with ten intervals from 0 to 5 and 0 to 20 and 0 to 400. The integral from 0 to ∞ is $\frac{1}{2}\sqrt{\pi}$. What is the best point to chop off the infinite integral?

23 The graph of $y(x) = 1/(x^2 + 10^{-10})$ has a sharp spike and a long tail. Estimate $\int_0^1 y\,dx$ from T_{10} and T_{100} (don't expect much). Then substitute $x = 10^{-5}\tan\theta, dx = 10^{-5}\sec^2\theta\,d\theta$ and integrate 10^5 from 0 to $\pi/4$.

24 Compute $\int_0^4 |x - \pi|\,dx$ from T_4 and compare with the divide and conquer method of separating $\int_0^{\pi} |x-\pi|\,dx$ from $\int_{\pi}^4 |x-\pi|\,dx$.

25 Find a, b, c so that $y = ax^2 + bx + c$ equals $1, 3, 7$ at $x = 0, \frac{1}{2}, 1$ (three equations). Check that $\frac{1}{6} \cdot 1 + \frac{4}{6} \cdot 3 + \frac{1}{6} \cdot 7$ equals $\int_0^1 y\, dx$.

26 Find c in $S - I = c(\Delta x)^4 [y'''(1) - y'''(0)]$ by taking $y = x^4$ and $\Delta x = 1$.

27 Find c in $G - I = c(\Delta x)^4 [y'''(1) - y'''(-1)]$ by taking $y = x^4$, $\Delta x = 2$, and $G = (-1/\sqrt{3})^4 + (1/\sqrt{3})^4$.

28 What condition on $y(x)$ makes $L_n = R_n = T_n$ for the integral $\int_a^b y(x)\, dx$?

29 Suppose $y(x)$ is *concave up*. Show from a picture that the trapezoidal answer is too high and the midpoint answer is too low. How does $y'' > 0$ make equation (5) positive and (6) negative?