CHAPTER 11

Vectors and Matrices

This chapter opens up a new part of calculus. It is *multidimensional calculus*, because the subject moves into more dimensions. In the first ten chapters, all functions depended on time t or position x—but not both. We had $f(t)$ or $y(x)$. The graphs were curves in a plane. There was one independent variable (x or t) and one dependent variable (y or f). Now we meet functions $f(x,t)$ that depend on both x and t. Their graphs are ***surfaces*** instead of curves. This brings us to the ***calculus of several variables***.

Start with the surface that represents the function $f(x,t)$ or $f(x,y)$ or $f(x,y,t)$. I emphasize functions, because that is what calculus is about.

EXAMPLE 7 $f(x,t) = \cos(x-t)$ is a traveling wave (cosine curve in motion).

At $t = 0$ the curve is $f = \cos x$. At a later time, the curve moves to the right (Figure 11.1). At each t we get a cross-section of the whole x-t surface. For a wave traveling along a string, the height depends on position as well as time.

A similar function gives a wave going around a stadium. Each person stands up and sits down. Somehow the wave travels.

EXAMPLE 8 $f(x,y) = 3x + y + 1$ is a sloping roof (fixed in time).

The surface is two-dimensional—you can walk around on it. It is flat because $3x + y + 1$ is a linear function. In the y direction the surface goes up at 45°. If y increases by 1, so does f. That slope is 1. In the x direction the roof is steeper (slope 3). There is a direction in between where the roof is steepest (slope $\sqrt{10}$).

EXAMPLE 9 $f(x,y,t) = \cos(x-y-t)$ is an ocean surface with traveling waves.

This surface moves. At each time t we have a new x-y surface. There are three variables, x and y for position and t for time. I can't draw the function, it needs four dimensions! The base coordinates are x, y, t and the height is f. The alternative is a movie that shows the x-y surface changing with t.

At time $t = 0$ the ocean surface is given by $\cos(x-y)$. The waves are in straight lines. The line $x - y = 0$ follows a crest because $\cos 0 = 1$. The top of the next wave is on the parallel line $x - y = 2\pi$, because $\cos 2\pi = 1$. Figure 11.1 shows the ocean surface at a fixed time.

The line $x - y = t$ gives the crest at time t. The water goes up and down (like people in a stadium). ***The wave goes to shore***, ***but the water stays in the ocean***.

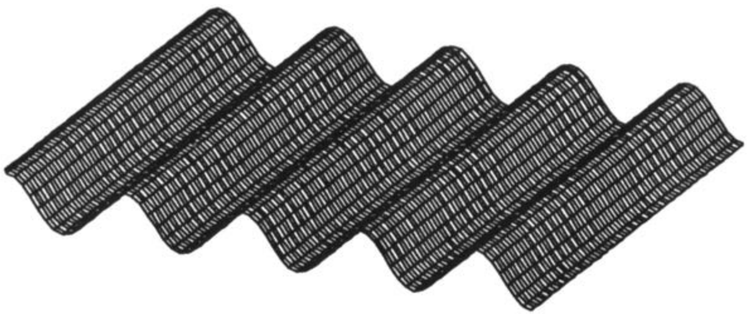


Fig. 11.1 Moving cosine with a small optical illusion—the darker bands seem to go from top to bottom as you turn.

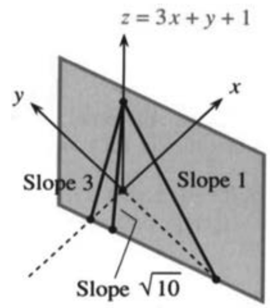


Fig. 11.2 Linear functions give planes.

Of course multidimensional calculus is not only for waves. In business, demand is a function of price and date. In engineering, the velocity and temperature depend on position x and time t. Biology deals with many variables at once (and statistics is always looking for linear relations like $z = x + 2y$). A serious job lies ahead, to carry derivatives and integrals into more dimensions.

11.1 Vectors and Dot Products

In a plane, every point is described by two numbers. We measure across by x and up by y. Starting from the origin we reach the point with coordinates (x, y). I want to describe this movement by a ***vector***—the straight line that starts at $(0,0)$ and ends at (x, y). This vector $\mathbf{v}$ has a ***direction***, which goes from $(0,0)$ to (x, y) and not the other way.

In a picture, the vector is shown by an arrow. In algebra, $\mathbf{v}$ is given by its two components. For a *column vector*, write x above y:

$$\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix} \quad (x \text{ and } y \text{ are the components of } \mathbf{v}). \tag{1}$$

Note that $\mathbf{v}$ is printed in boldface; its components x and y are in lightface.† The vector $-\mathbf{v}$ in the opposite direction changes signs. Adding $\mathbf{v}$ to $-\mathbf{v}$ gives the ***zero vector*** (different from the zero number and also in boldface):

$$-\mathbf{v} = \begin{bmatrix} -x \\ -y \end{bmatrix} \quad \text{and} \quad \mathbf{v} - \mathbf{v} = \begin{bmatrix} x - x \\ y - y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \mathbf{0}. \tag{2}$$

Notice how vector addition or subtraction is done separately on the x's and y's:

$$\mathbf{v} + \mathbf{w} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}. \tag{3}$$

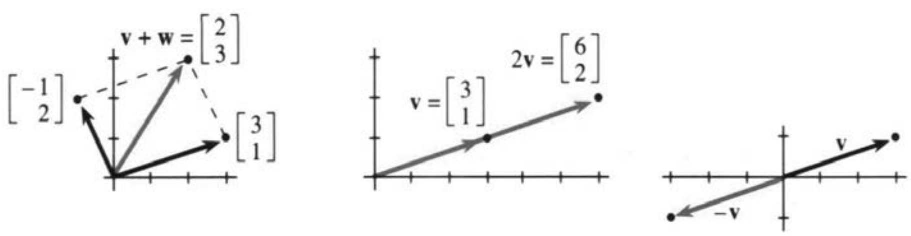


Fig. 11.3 Parallelogram for $\mathbf{v}+\mathbf{w}$, stretching for $2\mathbf{v}$, signs reversed for $-\mathbf{v}$.

The vector $\mathbf{v}$ has components $v_1 = 3$ and $v_2 = 1$. (I write v_1 for the first component and v_2 for the second component. I also write x and y, which is fine for two components.) The vector $\mathbf{w}$ has $w_1 = -1$ and $w_2 = 2$. To add the vectors, add the components. ***To draw this addition, place the start of* w *at the end of* v**. Figure 11.3 shows how $\mathbf{w}$ starts where $\mathbf{v}$ ends.

VECTORS WITHOUT COORDINATES

In that *head-to-tail addition* of $\mathbf{v}+\mathbf{w}$, we did something new. The vector $\mathbf{w}$ was moved away from the origin. Its length and direction were not changed! The new arrow is parallel to the old arrow—only the starting point is different. *The vector is the same as before*.

A vector can be defined without an origin and without x and y axes. The purpose of axes is to give the components—the separate distances x and y. Those numbers

† Another way to indicate a vector is $\vec{v}$. You will recognize vectors without needing arrows.

are necessary for calculations. But x and y coordinates are not necessary for head-to-tail addition $\mathbf{v}+\mathbf{w}$, or for stretching to $2\mathbf{v}$, or for linear combinations $2\mathbf{v}+3\mathbf{w}$. Some applications depend on coordinates, others don't.

Generally speaking, physics works without axes—it is "coordinate-free." A ***velocity*** has direction and magnitude, but it is not tied to a point. A ***force*** also has direction and magnitude, but it can act anywhere—not only at the origin. In contrast, a vector that gives the prices of five stocks is not floating in space. Each component has a meaning—there are five axes, and we know when prices are zero. After examples from geometry and physics (no axes), we return to vectors *with* coordinates.

EXAMPLE 1 (Geometry) Take any four-sided figure in space. Connect the midpoints of the four straight sides. ***Remarkable fact***: ***Those four midpoints lie in the same plane***. More than that, they form a *parallelogram*.

Frankly, this is amazing. Figure 11.4a cannot do justice to the problem, because it is printed on a flat page. Imagine the vectors $\mathbf{A}$ and $\mathbf{D}$ coming upward. $\mathbf{B}$ and $\mathbf{C}$ go down at different angles. Notice how easily we indicate the four sides as vectors, not caring about axes or origin.

I will prove that $\mathbf{V}=\mathbf{W}$. That shows that the midpoints form a parallelogram.

What is $\mathbf{V}$? It starts halfway along $\mathbf{A}$ and ends halfway along $\mathbf{B}$. The small triangle at the bottom shows $\mathbf{V}=\frac{1}{2}\mathbf{A}+\frac{1}{2}\mathbf{B}$. This is vector addition—the tail of $\frac{1}{2}\mathbf{B}$ is at the head of $\frac{1}{2}\mathbf{A}$. Together they equal the shortcut $\mathbf{V}$. For the same reason $\mathbf{W}=\frac{1}{2}\mathbf{C}+\frac{1}{2}\mathbf{D}$. The heart of the proof is to see these relationships.

One step is left. Why is $\frac{1}{2}\mathbf{A}+\frac{1}{2}\mathbf{B}$ equal to $\frac{1}{2}\mathbf{C}+\frac{1}{2}\mathbf{D}$? In other words, why is $\mathbf{A}+\mathbf{B}$ equal to $\mathbf{C}+\mathbf{D}$? (I multiplied by 2.) When the right question is asked, the answer jumps out. A head-to-tail addition $\mathbf{A}+\mathbf{B}$ brings us to the point R. ***Also*** $\mathbf{C}+\mathbf{D}$ ***brings us to*** R. The proof comes down to one line:

$$\mathbf{A}+\mathbf{B}=PR=\mathbf{C}+\mathbf{D}. \text{ Then } \mathbf{V}=\tfrac{1}{2}\mathbf{A}+\tfrac{1}{2}\mathbf{B} \text{ equals } \mathbf{W}=\tfrac{1}{2}\mathbf{C}+\tfrac{1}{2}\mathbf{D}.$$

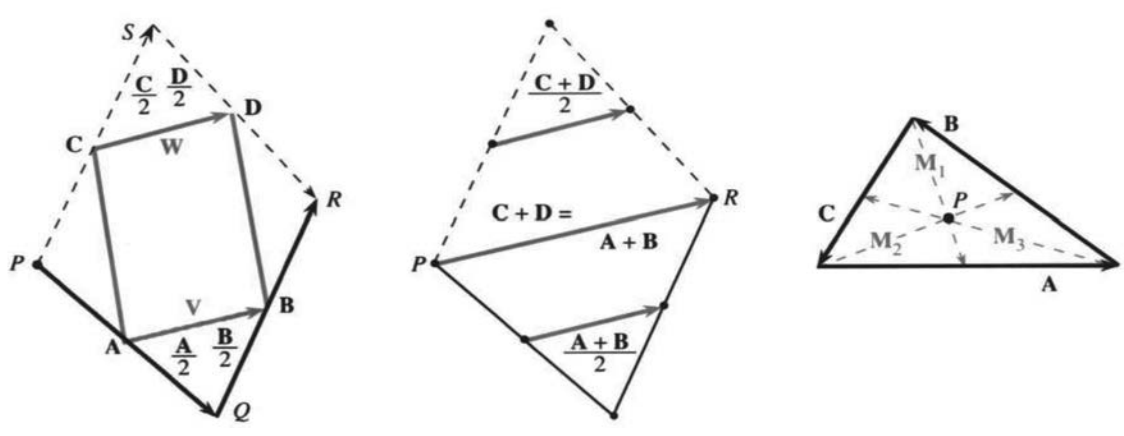


Fig. 11.4 Four midpoints form a parallelogram ($\mathbf{V}=\mathbf{W}$). Three medians meet at P.

EXAMPLE 2 (Also geometry) In any triangle, draw lines from the corners to the midpoints of the opposite sides. To prove by vectors: ***Those three lines meet at a point***. Problem 38 finds the meeting point in Figure 11.4c. Problem 37 says that ***the three vectors add to zero***.

EXAMPLE 3 (Medicine) An electrocardiogram shows the sum of many small vectors, the voltages in the wall of the heart. What happens to this sum—the ***heart vector* V**—in two cases that a cardiologist is watching for?

Case 1. Part of the heart is dead (***infarction***).
Case 2. Part of the heart is abnormally thick (***hypertrophy***).

A heart attack kills part of the muscle. A defective valve, or hypertension, overworks it. In **case 1** the cells die from the cutoff of blood (loss of oxygen). In **case 2** the heart wall can triple in size, from excess pressure. The causes can be chemical or mechanical. The effect we see is electrical.

***The machine is adding small vectors and* "*projecting*" *them in twelve directions*.**

The leads on the arms, left leg, and chest give twelve directions in the body. Each graph shows the component of **V** in one of those directions. Three of the projections—two in the vertical plane, plus lead 2 for front-back—produce the "mean QRS vector" in Figure 11.5. That is the sum **V** when the ventricles start to contract. The left ventricle is larger, so the heart vector normally points down and to the left.

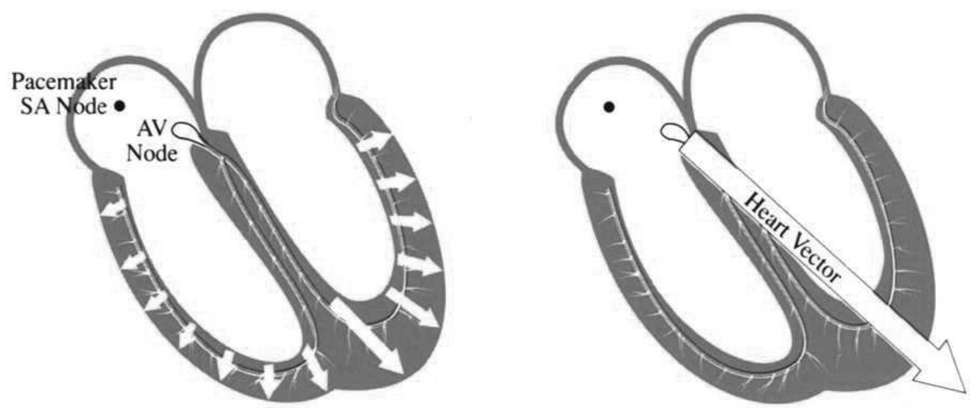


Fig. 11.5 **V** is a sum of small voltage vectors, at the moment of depolarization.

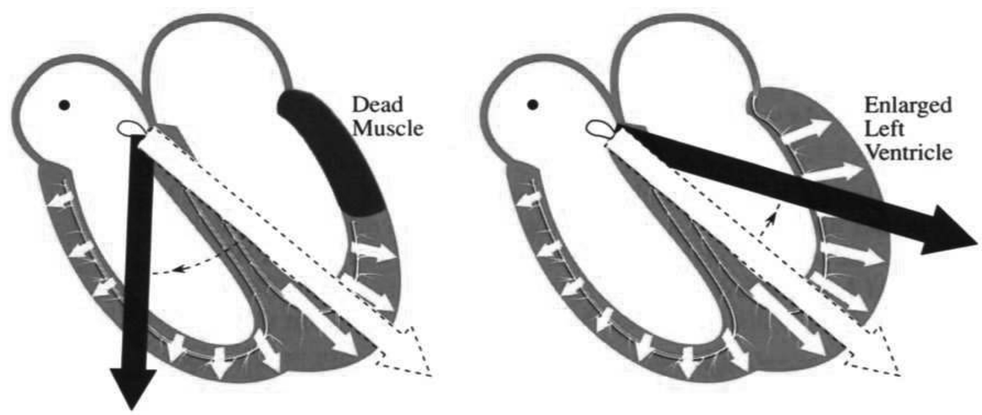


Fig. 11.6 Changes in **V** show dead muscle and overworked muscle.

We come soon to projections, but here the question is about **V** itself. How does the ECG identify the problem?

> ***Case* 1**: ***Heart attack*** The dead cells make no contribution to the electrical potential. Some small vectors are missing. Therefore the sum **V** turns ***away*** from the infarcted part.
> ***Case* 2**: ***Hypertrophy*** The overwork increases the contribution to the potential. Some vectors are larger than normal. Therefore **V** turns ***toward*** the thickened part.

When **V** points in an abnormal direction, the ECG graphs locate the problem. The P, Q, R, S, T waves on separate graphs can all indicate hypertrophy, in different regions of the heart. Infarctions generally occur in the left ventricle, which needs the greatest blood supply. When the supply of oxygen is cut back, that ventricle feels it first. The result can be a heart attack (= myocardial infarction = coronary occlusion). Section 11.2 shows how the projections on the ECG point to the location.

First come the basic facts about vectors—components, lengths, and dot products.

COORDINATE VECTORS AND LENGTH

To compute with vectors we need axes and coordinates. The picture of the heart is "coordinate-free," but calculations require numbers. A vector is known by its components. ***The unit vectors along the axes are* i *and* j *in the plane and* i, j, k *in space***:

$$\textbf{in 2D}: \quad \mathbf{i}=\begin{bmatrix}1\\0\end{bmatrix},\ \mathbf{j}=\begin{bmatrix}0\\1\end{bmatrix} \qquad \textbf{in 3D}: \quad \mathbf{i}=\begin{bmatrix}1\\0\\0\end{bmatrix},\ \mathbf{j}=\begin{bmatrix}0\\1\\0\end{bmatrix},\ \mathbf{k}=\begin{bmatrix}0\\1\\0\end{bmatrix}.$$

Notice how easily we moved into three dimensions! The only change is that vectors have three components. The combinations of **i** and **j** (or **i**, **j**, **k**) produce all vectors **v** in the plane (and all vectors **V** in space):

$$\mathbf{v}=3\mathbf{i}+\mathbf{j}=\begin{bmatrix}3\\1\end{bmatrix} \qquad \mathbf{V}=\mathbf{i}+2\mathbf{j}-2\mathbf{k}=\begin{bmatrix}1\\2\\-2\end{bmatrix}.$$

***Those vectors are also written* $\mathbf{v}=(3,1)$ *and* $\mathbf{V}=(1,2,-2)$**. The components of the vector are also the coordinates of a point. (The vector goes from the origin to the point.) This relation between point and vector is so close that we allow them the same notation: $P=(x,y,z)$ and $\mathbf{v}=(x,y,z)=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$.

The sum $\mathbf{v}+\mathbf{V}$ is totally meaningless. Those vectors live in different dimensions.

From the components we find the ***length***. The length of $(3,1)$ is $\sqrt{3^2+1^2}=\sqrt{10}$. This comes directly from a right triangle. In three dimensions, **V** has a third component to be squared and added. The length of $\mathbf{V}=(x,y,z)$ is $|\mathbf{V}|=\sqrt{x^2+y^2+z^2}$.

Vertical bars indicate length, which takes the place of absolute value. The length of $\mathbf{v}=3\mathbf{i}+\mathbf{j}$ is the distance from the point $(0,0)$ to the point $(3,1)$:

$$|\mathbf{v}|=\sqrt{v_1^2+v_2^2}=\sqrt{10} \qquad |\mathbf{V}|=\sqrt{1^2+2^2+(-2)^2}=3.$$

A unit vector is a vector of length one. Dividing **v** and **V** by their lengths produces unit vectors in the same directions:

$$\frac{\mathbf{v}}{|\mathbf{v}|}=\begin{bmatrix}3/\sqrt{10}\\1/\sqrt{10}\end{bmatrix} \quad\text{and}\quad \frac{\mathbf{V}}{|\mathbf{V}|}=\begin{bmatrix}1/3\\2/3\\-2/3\end{bmatrix} \quad\text{are unit vectors.}$$

11A Each nonzero vector has a positive length $|\mathbf{v}|$. The direction of $\mathbf{v}$ is given by a unit vector $\mathbf{u}=\mathbf{v}/|\mathbf{v}|$. The length times direction equals $\mathbf{v}$.

A unit vector in the plane is determined by its angle θ with the x axis:

$$\mathbf{u}=\begin{bmatrix}\cos\theta\\\sin\theta\end{bmatrix}=(\cos\theta)\mathbf{i}+(\sin\theta)\mathbf{j}\ \text{ is a unit vector}: |\mathbf{u}|^2=\cos^2\theta+\sin^2\theta=1.$$

In 3-space the components of a unit vector are its "direction cosines":

$$\mathbf{U}=(\cos\alpha)\mathbf{i}+(\cos\beta)\mathbf{j}+(\cos\gamma)\mathbf{k}: \quad \alpha,\beta,\gamma=\text{ angles with } x,y,z \text{ axes.}$$

Then $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$. We are doing algebra with numbers while we are doing geometry with vectors. It was the great contribution of Descartes to see how to study algebra and geometry at the same time.

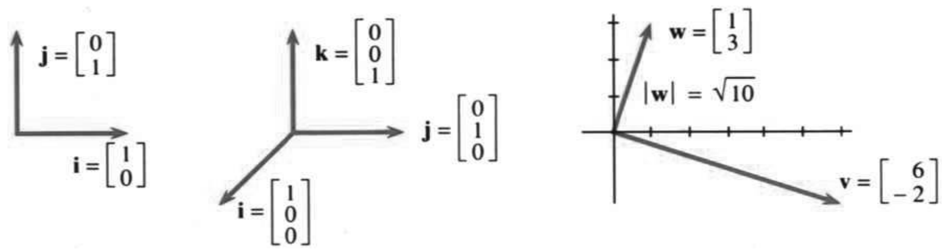


Fig. 11.7 Coordinate vectors $\mathbf{i},\mathbf{j},\mathbf{k}$. Perpendicular vectors $\mathbf{v}\cdot\mathbf{w}=(6)(1)+(-2)(3)=0$.

THE DOT PRODUCT OF TWO VECTORS

There are two basic operations on vectors. First, vectors are added $(\mathbf{v}+\mathbf{w})$. Second, a vector is multiplied by a scalar ($7\mathbf{v}$ or $-2\mathbf{w}$). That leaves a natural question—how do you multiply two vectors? The main part of the answer is—you don't. But there is an extremely important operation that begins with two vectors and produces a number. It is usually indicated by a dot between the vectors, as in $\mathbf{v}\cdot\mathbf{w}$, so it is called the ***dot product***.

DEFINITION 1 *The dot product multiplies the lengths* $|\mathbf{v}|$ *times* $|\mathbf{w}|$ *times a cosine*:

$$\mathbf{v}\cdot\mathbf{w}=|\mathbf{v}||\mathbf{w}|\cos\theta, \qquad \theta=\text{angle between }\mathbf{v}\text{ and }\mathbf{w}.$$

EXAMPLE $\begin{bmatrix}3\\0\end{bmatrix}$ has length 3, $\begin{bmatrix}2\\2\end{bmatrix}$ has length $\sqrt{8}$, the angle is 45°.

The dot product is $|\mathbf{v}||\mathbf{w}|\cos\theta=(3)(\sqrt{8})(1/\sqrt{2})$, which simplifies to 6. The square roots in the lengths are "canceled" by square roots in the cosine. For computing $\mathbf{v}\cdot\mathbf{w}$, a second and much simpler way involves no square roots in the first place.

DEFINITION 2 ***The dot product*** $\mathbf{v}\cdot\mathbf{w}$ ***multiplies component by component and adds***:

$$\mathbf{v}\cdot\mathbf{w} = v_1w_1 + v_2w_2 \qquad \begin{bmatrix}3\\0\end{bmatrix}\cdot\begin{bmatrix}2\\2\end{bmatrix} = (3)(2)+(0)(2) = 6.$$

The first form $|\mathbf{v}||\mathbf{w}|\cos\theta$ is coordinate-free. The second form $v_1w_1+v_2w_2$ computes with coordinates. Remark 4 explains why these two forms are equal.

11B The ***dot product*** or ***scalar product*** or ***inner product*** of three-dimensional vectors is

$$\mathbf{V}\cdot\mathbf{W} = |\mathbf{V}||\mathbf{W}|\cos\theta = V_1W_1 + V_2W_2 + V_3W_3. \tag{4}$$

If the vectors are perpendicular then $\theta = 90°$ and $\cos\theta = 0$ and $\mathbf{V}\cdot\mathbf{W} = 0$.

$$\begin{bmatrix}1\\2\\3\end{bmatrix}\cdot\begin{bmatrix}4\\5\\6\end{bmatrix} = 32\,(\textit{not perpendicular}) \qquad \begin{bmatrix}2\\2\\-1\end{bmatrix}\cdot\begin{bmatrix}-1\\2\\2\end{bmatrix} = 0\,(\textit{perpendicular}).$$

These dot products 32 and 0 equal $|\mathbf{V}||\mathbf{W}|\cos\theta$. In the second one, $\cos\theta$ must be zero. The angle is $\pi/2$ or $-\pi/2$—in either case a right angle. Fortunately the cosine is the same for θ and $-\theta$, so we need not decide the sign of θ.

Remark* 1** When $\mathbf{V} = \mathbf{W}$ the angle is zero but not the cosine! In this case $\cos\theta = 1$ and $\mathbf{V}\cdot\mathbf{V} = |\mathbf{V}|^2$. ***The dot product of* V *with itself is the length squared:

$$\mathbf{V}\cdot\mathbf{V} = (V_1,V_2,V_3)\cdot(V_1,V_2,V_3) = V_1^2+V_2^2+V_3^2 = |\mathbf{V}|^2. \tag{5}$$

***Remark* 2** The dot product of $\mathbf{i} = (1,0,0)$ with $\mathbf{j} = (0,1,0)$ is $\mathbf{i}\cdot\mathbf{j} = 0$. The axes are perpendicular. Similarly $\mathbf{i}\cdot\mathbf{k} = 0$ and $\mathbf{j}\cdot\mathbf{k} = 0$. Those are unit vectors: $\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1$.

***Remark* 3** The dot product has three properties that keep the algebra simple:

1. $\mathbf{V}\cdot\mathbf{W} = \mathbf{W}\cdot\mathbf{V}$ **2.** $(c\mathbf{V})\cdot\mathbf{W} = c(\mathbf{V}\cdot\mathbf{W})$ **3.** $(\mathbf{U}+\mathbf{V})\cdot\mathbf{W} = \mathbf{U}\cdot\mathbf{W}+\mathbf{V}\cdot\mathbf{W}$

When $\mathbf{V}$ is doubled $(c = 2)$ the dot product is doubled. When $\mathbf{V}$ is split into $\mathbf{i},\mathbf{j},\mathbf{k}$ components, the dot product splits in three pieces. The same applies to $\mathbf{W}$, since $\mathbf{V}\cdot\mathbf{W} = \mathbf{W}\cdot\mathbf{V}$. The nine dot products of $\mathbf{i},\mathbf{j},\mathbf{k}$ are zeros and ones, and a giant splitting of both $\mathbf{V}$ and $\mathbf{W}$ gives back the correct $\mathbf{V}\cdot\mathbf{W}$:

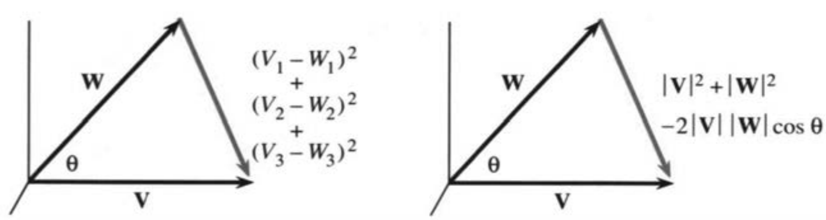


Fig. 11.8 Length squared $= (\mathbf{V}-\mathbf{W})\cdot(\mathbf{V}-\mathbf{W})$, from coordinates and the cosine law.

$$\mathbf{V}\cdot\mathbf{W} = V_1\mathbf{i}\cdot W_1\mathbf{i} + V_2\mathbf{j}\cdot W_2\mathbf{j} + V_3\mathbf{k}\cdot W_3\mathbf{k} + \text{six zeroes} = V_1W_1 + V_2W_2 + V_3W_3.$$

Remark* 4** ***The two forms of the dot product are equal. This comes from computing $|\mathbf{V}-\mathbf{W}|^2$ by coordinates and also by the "law of cosines":

$$\text{with coordinates}: |\mathbf{V}-\mathbf{W}|^2 = (V_1-W_1)^2 + (V_2-W_2)^2 + (V_3-W_3)^2$$
$$\text{from cosine law}: |\mathbf{V}-\mathbf{W}|^2 = |\mathbf{V}|^2 + |\mathbf{W}|^2 - 2|\mathbf{V}||\mathbf{W}|\cos\theta.$$

Compare those two lines. Line 1 contains V_1^2 and V_2^2 and V_3^2. Their sum matches $|\mathbf{V}|^2$ in the cosine law. Also $W_1^2+W_2^2+W_3^2$ matches $|\mathbf{W}|^2$. Therefore the terms containing -2 are the same (you can mentally cancel the -2). *The definitions agree*:

$$-2(V_1W_1+V_2W_2+V_3W_3)\ \textit{equals}\ -2|\mathbf{V}||\mathbf{W}|\cos\theta\ \textit{equals}\ -2\mathbf{V}\cdot\mathbf{W}.$$

The cosine law is coordinate-free. It applies to all triangles (even in n dimensions). Its vector form in Figure 11.8 is $|\mathbf{V}-\mathbf{W}|^2 = |\mathbf{V}|^2 - 2\mathbf{V}\cdot\mathbf{W} + |\mathbf{W}|^2$. This application to $\mathbf{V}\cdot\mathbf{W}$ is its brief moment of glory.

***Remark* 5** The dot product is the best way to compute the cosine of θ:

$$\cos\theta = \frac{\mathbf{V}\cdot\mathbf{W}}{|\mathbf{V}||\mathbf{W}|}. \tag{6}$$

Here are examples of $\mathbf{V}$ and $\mathbf{W}$ with a range of angles from 0 to π:

$\mathbf{i}$ and $3\mathbf{i}$ have the same direction	$\cos\theta = 1$	$\theta = 0$
$\mathbf{i}\cdot(\mathbf{i}+\mathbf{j}) = 1$ is positive	$\cos\theta = 1/\sqrt{2}$	$\theta = \pi/4$
$\mathbf{i}$ and $\mathbf{j}$ are perpendicular: $\mathbf{i}\cdot\mathbf{j} = 0$	$\cos\theta = 0$	$\theta = \pi/2$
$\mathbf{i}\cdot(-\mathbf{i}+\mathbf{j}) = -1$ is negative	$\cos\theta = -1/\sqrt{2}$	$\theta = 3\pi/4$
$\mathbf{i}$ and $-3\mathbf{i}$ have opposite directions	$\cos\theta = -1$	$\theta = \pi$

Remark* 6** ***The Cauchy-Schwarz inequality $|\mathbf{V}\cdot\mathbf{W}| \leqslant |\mathbf{V}||\mathbf{W}|$ ***comes from*** $|\cos\theta| \leqslant 1$.

The left side is $|\mathbf{V}||\mathbf{W}||\cos\theta|$. It never exceeds the right side $|\mathbf{V}||\mathbf{W}|$. This is a key inequality in mathematics, from which so many others follow:

Geometric mean $\sqrt{xy} \leqslant$ ***arithmetic mean*** $\frac{1}{2}(x+y)$ (true for any $x \geqslant 0$ and $y \geqslant 0$).
Triangle inequality $|\mathbf{V}+\mathbf{W}| \leqslant |\mathbf{V}| + |\mathbf{W}|$ ($|\mathbf{V}|, |\mathbf{W}|, |\mathbf{V}+\mathbf{W}|$ are lengths of sides).

These and other examples are in Problems 39 to 44. The Schwarz inequality $|\mathbf{V}\cdot\mathbf{W}| \leqslant |\mathbf{V}||\mathbf{W}|$ becomes an equality when $|\cos\theta| = 1$ and the vectors are _____.

11.1 EXERCISES

Read-through questions

A vector has length and ___a___. If $\mathbf{v}$ has components 6 and -8, its length is $|\mathbf{v}| =$ ___b___ and its direction vector is $\mathbf{u} =$ ___c___. The product of $|\mathbf{v}|$ with $\mathbf{u}$ is ___d___. This vector goes from $(0,0)$ to the point $x =$ ___e___, $y =$ ___f___. A combination of the coordinate vectors $\mathbf{i} =$ ___g___ and $\mathbf{j} =$ ___h___ produces $\mathbf{v} =$ ___i___ $\mathbf{i} +$ ___j___ $\mathbf{j}$.

To add vectors we add their ___k___. The sum of $(6,-8)$ and $(1,0)$ is ___l___. To see $\mathbf{v}+\mathbf{i}$ geometrically, put the ___m___ of $\mathbf{i}$ at the ___n___ of $\mathbf{v}$. The vectors form a ___o___ with diagonal $\mathbf{v}+\mathbf{i}$. (The other diagonal is ___p___.) The vectors $2\mathbf{v}$ and $-\mathbf{v}$ are ___q___ and ___r___. Their lengths are ___s___ and ___t___.

In a space without axes and coordinates, the tail of $\mathbf{V}$ can be placed ___u___. Two vectors with the same ___v___ are the same. If a triangle starts with $\mathbf{V}$ and continues with $\mathbf{W}$, the third side is ___w___. The vector connecting the midpoint of $\mathbf{V}$ to the midpoint of $\mathbf{W}$ is ___x___. That vector is ___y___ the third side. In this coordinate-free form the dot product is $\mathbf{V}\cdot\mathbf{W} =$ ___z___.

Using components, $\mathbf{V}\cdot\mathbf{W} =$ ___A___ and $(1,2,1)\cdot(2,-3,7) =$ ___B___. The vectors are perpendicular if ___C___. The vectors are parallel if ___D___. $\mathbf{V}\cdot\mathbf{V}$ is the same as ___E___. The dot product of $\mathbf{U}+\mathbf{V}$ with $\mathbf{W}$ equals ___F___. The angle between $\mathbf{V}$ and $\mathbf{W}$ has $\cos\theta =$ ___G___. When $\mathbf{V}\cdot\mathbf{W}$ is negative then θ is ___H___. The angle between $\mathbf{i}+\mathbf{j}$ and $\mathbf{i}+\mathbf{k}$ is ___I___. The Cauchy-Schwarz inequality is ___J___, and for $\mathbf{V}=\mathbf{i}+\mathbf{j}$ and $\mathbf{W}=\mathbf{i}+\mathbf{k}$ it becomes $1 \leqslant$ ___K___.

In 1–4 compute $\mathbf{V}+\mathbf{W}$ and $2\mathbf{V}-3\mathbf{W}$ and $|\mathbf{V}|^2$ and $\mathbf{V}\cdot\mathbf{W}$ and $\cos\theta$.

1 $\mathbf{V} = (1,1,1)$, $\mathbf{W} = (-1,-1,-1)$

2 $\mathbf{V} = \mathbf{i}+\mathbf{j}$, $\mathbf{W} = \mathbf{j}-\mathbf{k}$

3 $\mathbf{V} = \mathbf{i}-2\mathbf{j}+\mathbf{k}$, $\mathbf{W} = \mathbf{i}+\mathbf{j}-2\mathbf{k}$

4 $\mathbf{V} = (1,1,1,1)$, $\mathbf{W} = (1,2,3,4)$

5 (a) Find a vector that is perpendicular to (v_1, v_2).
(b) Find two vectors that are perpendicular to (v_1, v_2, v_3).

6 Find two vectors that are perpendicular to $(1,1,0)$ and to each other.

7 What vector is perpendicular to all 2-dimensional vectors? What vector is parallel to all 3-dimensional vectors?

8 In Problems $1-4$ construct unit vectors in the same direction as $\mathbf{V}$.

9 If $\mathbf{v}$ and $\mathbf{w}$ are unit vectors, what is the geometrical meaning of $\mathbf{v}\cdot\mathbf{w}$? What is the geometrical meaning of $(\mathbf{v}\cdot\mathbf{w})\mathbf{v}$? Draw a figure with $\mathbf{v}=\mathbf{i}$ and $\mathbf{w} = (3/5)\mathbf{i} + (4/5)\mathbf{j}$.

10 Write down all unit vectors that make an angle θ with the vector $(1,0)$. Write down *all* vectors at that angle.

11 ***True or false*** in three dimensions:
1. If both $\mathbf{U}$ and $\mathbf{V}$ make a $30°$ angle with $\mathbf{W}$, so does $\mathbf{U}+\mathbf{V}$.
2. If they make a $90°$ angle with $\mathbf{W}$, so does $\mathbf{U}+\mathbf{V}$.
3. If they make a $90°$ angle with $\mathbf{W}$ they are perpendicular: $\mathbf{U}\cdot\mathbf{V} = 0$.

12 From $\mathbf{W} = (1,2,3)$ subtract a multiple of $\mathbf{V} = (1,1,1)$ so that $\mathbf{W}-c\mathbf{V}$ is perpendicular to $\mathbf{V}$. Draw $\mathbf{V}$ and $\mathbf{W}$ and $\mathbf{W}-c\mathbf{V}$.

13 (a) What is the sum $\mathbf{V}$ of the twelve vectors from the center of a clock to the hours?
(b) If the 4 o'clock vector is removed, find $\mathbf{V}$ for the other eleven vectors.
(c) If the vectors to $1,2,3$ are cut in half, find $\mathbf{V}$ for the twelve vectors.

14 (a) By removing one or more of the twelve clock vectors, make the length $|\mathbf{V}|$ as large as possible.
(b) Suppose the vectors start from the top instead of the center (the origin is moved to 12 o'clock, so $\mathbf{v_{12}} = \mathbf{0}$). What is the new sum $\mathbf{V}^*$?

15 Find the angle POQ by vector methods if $P = (1,1,0)$, $O = (0,0,0)$, $Q = (1,2,-2)$.

16 (a) Draw the unit vectors $\mathbf{u_1} = (\cos\theta, \sin\theta)$ and $\mathbf{u_2} = (\cos\phi, \sin\phi)$. By dot products find the formula for $\cos(\theta-\phi)$.
(b) Draw the unit vector $\mathbf{u_3}$ from a $90°$ rotation of $\mathbf{u_2}$. By dot products find the formula for $\sin(\theta+\phi)$.

17 Describe all points (x,y) such that $\mathbf{v} = x\mathbf{i} + y\mathbf{j}$ satisfies
(a) $|\mathbf{v}| = 2$ (b) $|\mathbf{v}-\mathbf{i}| = 2$
(c) $\mathbf{v}\cdot\mathbf{i} = 2$ (d) $\mathbf{v}\cdot\mathbf{i} = |\mathbf{v}|$

18 (Important) If $\mathbf{A}$ and $\mathbf{B}$ are non-parallel vectors from the origin, describe
(a) the endpoints of $t\mathbf{B}$ for all numbers t
(b) the endpoints of $\mathbf{A}+t\mathbf{B}$ for all t
(c) the endpoints of $s\mathbf{A}+t\mathbf{B}$ for all s and t
(d) the vectors $\mathbf{v}$ that satisfy $\mathbf{v}\cdot\mathbf{A} = \mathbf{v}\cdot\mathbf{B}$

19 (a) If $\mathbf{v}+2\mathbf{w} = \mathbf{i}$ and $2\mathbf{v}+3\mathbf{w} = \mathbf{j}$ find $\mathbf{v}$ and $\mathbf{w}$.
(b) If $\mathbf{v} = \mathbf{i}+\mathbf{j}$ and $\mathbf{w} = 3\mathbf{i}+4\mathbf{j}$ then $\mathbf{i} =$ _____ $\mathbf{v}+$ _____ $\mathbf{w}$.

20 If $P = (0,0)$ and $R = (0,1)$ choose Q so the angle PQR is $90°$. All possible Q's lie in a _____.

21 (a) Choose d so that $\mathbf{A} = 2\mathbf{i}+3\mathbf{j}$ is perpendicular to $\mathbf{B} = 9\mathbf{i}+d\mathbf{j}$.
(b) Find a vector $\mathbf{C}$ perpendicular to $\mathbf{A} = \mathbf{i}+\mathbf{j}+\mathbf{k}$ and $\mathbf{B} = \mathbf{i}-\mathbf{k}$.

22 If a boat has velocity $\mathbf{V}$ with respect to the water and the water has velocity $\mathbf{W}$ with respect to the land, then _____. The speed of the boat is not $|\mathbf{V}|+|\mathbf{W}|$ but _____.

23 Find the angle between the diagonal of cube and (a) an edge (b) the diagonal of a face (c) another diagonal of the cube. Choose lines that meet.

24 Draw the triangle PQR in Example 1 (the four-sided figure in space). By geometry not vectors, show that PR is twice as long as $\mathbf{V}$. Similarly $|PR| = 2|\mathbf{W}|$. Also $\mathbf{V}$ is parallel to $\mathbf{W}$ because both are parallel to ______ . So $\mathbf{V} = \mathbf{W}$ as before.

25 (a) If $\mathbf{A}$ and $\mathbf{B}$ are unit vectors, show that they make equal angles with $\mathbf{A}+\mathbf{B}$.

(b) If $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are unit vectors with $\mathbf{A}+\mathbf{B}+\mathbf{C} = 0$, they form a ______ triangle and the angle between any two is ______ .

26 (a) Find perpendicular unit vectors $\mathbf{I}$ and $\mathbf{J}$ in the plane that are different from $\mathbf{i}$ and $\mathbf{j}$.

(b) Find perpendicular unit vectors $\mathbf{I}, \mathbf{J}, \mathbf{K}$ different from $\mathbf{i}, \mathbf{j}, \mathbf{k}$.

27 If $\mathbf{I}$ and $\mathbf{J}$ are perpendicular, take their dot products with $\mathbf{A} = a\mathbf{I} + b\mathbf{J}$ to find a and b.

28 Suppose $\mathbf{I} = (\mathbf{i}+\mathbf{j})/\sqrt{2}$ and $\mathbf{J} = (\mathbf{i}-\mathbf{j})/\sqrt{2}$. Check $\mathbf{I}\cdot\mathbf{J} = 0$ and write $\mathbf{A} = 2\mathbf{i}+3\mathbf{j}$ as a combination $a\mathbf{I}+b\mathbf{J}$. (Best method: use a and b from Problem 27. Alternative: Find $\mathbf{i}$ and $\mathbf{j}$ from $\mathbf{I}$ and $\mathbf{J}$ and substitute into $\mathbf{A}$.)

29 (a) Find the position vector OP and the velocity vector PQ when the point P moves around the unit circle (see figure) with speed 1. (b) Change to speed 2.

30 The sum $(\mathbf{A}\cdot\mathbf{i})^2 + (\mathbf{A}\cdot\mathbf{j})^2 + (\mathbf{A}\cdot\mathbf{k}^2)$ equals ______ .

31 In the semicircle find $\mathbf{C}$ and $\mathbf{D}$ in terms of $\mathbf{A}$ and $\mathbf{B}$. Prove that $\mathbf{C}\cdot\mathbf{D} = 0$ (they meet at right angles).

35 The vector from the earth's center to Seattle is $a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$.

(a) Along the circle at the latitude of Seattle, what two functions of a, b, c stay constant? $\mathbf{k}$ goes to the North Pole.

(b) On the circle at the longitude of Seattle—the meridian—what two functions of a, b, c stay constant?

(c) Extra credit: Estimate a, b, c in your present position. The $0°$ meridian through Greenwich has $b = 0$.

36 If $|\mathbf{A}+\mathbf{B}|^2 = |\mathbf{A}|^2 + |\mathbf{B}|^2$, prove that $\mathbf{A}$ is perpendicular to $\mathbf{B}$.

37 In Figure 11.4, the medians go from the corners to the midpoints of the opposite sides. Express $\mathbf{M_1}, \mathbf{M_2}, \mathbf{M_3}$ in terms of $\mathbf{A}, \mathbf{B}, \mathbf{C}$. Prove that $\mathbf{M}_1 + \mathbf{M}_2 + \mathbf{M}_3 = 0$. What relation holds between $\mathbf{A}, \mathbf{B}, \mathbf{C}$?

38 The point $\frac{2}{3}$ of the way along is the same for all three medians. This means that $\mathbf{A} + \frac{2}{3}\mathbf{M}_3 = \frac{2}{3}\mathbf{M}_2 =$ ______ . Prove that those three vectors are equal.

39 (a) Verify the Schwarz inequality $|\mathbf{V}\cdot\mathbf{W}| \leqslant |\mathbf{V}||\mathbf{W}|$ for $\mathbf{V} = \mathbf{i}+2\mathbf{j}+2\mathbf{k}$ and $\mathbf{W} = 2\mathbf{i}+2\mathbf{j}+\mathbf{k}$.

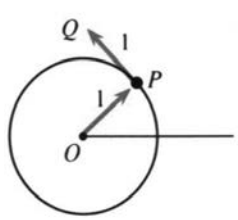
(b) What does the inequality become when $\mathbf{V} = (\sqrt{x}, \sqrt{y})$ and $\mathbf{W} = (\sqrt{y}, \sqrt{x})$?

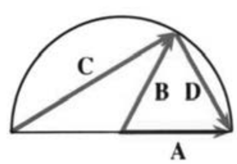
40 By choosing the right vector $\mathbf{W}$ in the Schwarz inequality, show that $(V_1+V_2+V_3)^2 \leqslant 3(V_1^2+V_2^2+V_2^3)$. What is $\mathbf{W}$?

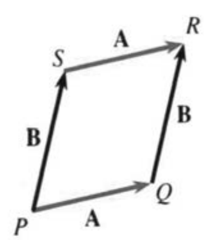
41 The Schwarz inequality for $a\mathbf{i}+b\mathbf{j}$ and $c\mathbf{i}+d\mathbf{j}$ says that $(a^2+b^2)(c^2+d^2) \geqslant (ac+bd)^2$. Multiply out to show that the difference is $\geqslant 0$.

42 The vectors $\mathbf{A}, \mathbf{B}, \mathbf{C}$ form a triangle if $\mathbf{A}+\mathbf{B}+\mathbf{C} = \mathbf{0}$. The **triangle inequality** $|\mathbf{A}+\mathbf{B}| \leqslant |\mathbf{A}| + |\mathbf{B}|$ says that any one side length is less than ______ . The proof comes from Schwarz:

$$|\mathbf{A}+\mathbf{B}|^2 = \mathbf{A}\cdot\mathbf{A} + 2\mathbf{A}\cdot\mathbf{B} + \mathbf{B}\cdot\mathbf{B}$$
$$\leqslant |\mathbf{A}|^2 + \text{______} + |\mathbf{B}|^2 = (|\mathbf{A}| + |\mathbf{B}|^2).$$







32 The diagonal PR has $|PR|^2 = (\mathbf{A}+\mathbf{B})\cdot(\mathbf{A}+\mathbf{B}) = \mathbf{A}\cdot\mathbf{A} + \mathbf{A}\cdot\mathbf{B} + \mathbf{B}\cdot\mathbf{A} + \mathbf{B}\cdot\mathbf{B}$. Add $|QS|^2$ from the other diagonal to prove the parallelogram law: $|PR|^2 + |QS|^2 =$ sum of squares of the four side lengths.

33 If $(1,2,3), (3,4,7)$, and $(2,1,2)$ are corners of a parallelogram, find all possible fourth corners.

34 The diagonals of the parallelogram are $\mathbf{A}+\mathbf{B}$ and ______ . If they have the same length, prove that $\mathbf{A}\cdot\mathbf{B} = 0$ and the region is a ______ .

43 **True or false,** with reason or example:

(a) $|\mathbf{V}+\mathbf{W}|^2$ is never larger than $|\mathbf{V}|^2 + |\mathbf{W}|^2$

(b) In a real triangle $|\mathbf{V}+\mathbf{W}|$ never equals $|\mathbf{V}| + |\mathbf{W}|$

(c) $\mathbf{V}\cdot\mathbf{W}$ equals $\mathbf{W}\cdot\mathbf{V}$

(d) The vectors perpendicular to $\mathbf{i}+\mathbf{j}+\mathbf{k}$ lie along a line.

44 If $\mathbf{V} = \mathbf{i}+2\mathbf{k}$ choose $\mathbf{W}$ so that $\mathbf{V}\cdot\mathbf{W} = |\mathbf{V}||\mathbf{W}|$ and $|\mathbf{V}+\mathbf{W}| = |\mathbf{V}| + |\mathbf{W}|$.

45 A methane molecule has a carbon atom at $(0,0,0)$ and hydrogen atoms at $(1,1,-1)$, $(1,-1,1)$, $(-1,1,1)$, and $(-1,-1,-1)$. Find

(a) the distance between hydrogen atoms

(b) the angle between vectors going out from the carbon atom to the hydrogen atoms.

46 (a) Find a vector **V** at a 45° angle with **i** and **j**.

(b) Find **W** that makes a 60° angle with **i** and **j**.

(c) Explain why no vector makes a 30° angle with **i** and **j**.

11.2 Planes and Projections

The most important "curves" are straight lines. The most important functions are linear. Those sentences take us back to the beginning of the book—the graph of $mx+b$ is a line. The goal now is to move into three dimensions, where ***graphs are surfaces.*** Eventually the surfaces will be curved. But calculus starts with the flat surfaces that correspond to straight lines:

What are the most important surfaces? ***Planes***.
What are the most important functions? ***Still linear***.

The geometrical idea of a plane is turned into algebra, by finding ***the equation of a plane***. Not just a general formula, but the particular equation of a particular plane.

A line is determined by one point (x_0, y_0) and the slope m. The point-slope equation is $y - y_0 = m(x - x_0)$. That is a linear equation, it is satisfied when $y = y_0$ and $x = x_0$, and dy/dx is m. For a plane, we start again with a particular point—which is now (x_0, y_0, z_0). But the slope of a plane is not so simple. Many planes climb at a 45° angle—with "slope 1"—and more information is needed.

The direction of a plane is described by a vector **N**. The vector is not *in* the plane, but *perpendicular* to the plane. In the plane, there are many directions. Perpendicular to the plane, there is only one direction. A vector in that perpendicular direction is a ***normal vector***.

The normal vector **N** can point "up" or "down". The length of **N** is not crucial (we often make it a unit vector and call it **n**). Knowing **N** and the point $P_0 = (x_0, y_0, z_0)$, we know the plane (Figure 11.9). For its equation we switch to algebra and use the dot product—which is the key to perpendicularity.

N is described by its components (a, b, c). In other words **N** is $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$. ***This vector is perpendicular to every direction in the plane***. A typical direction goes from

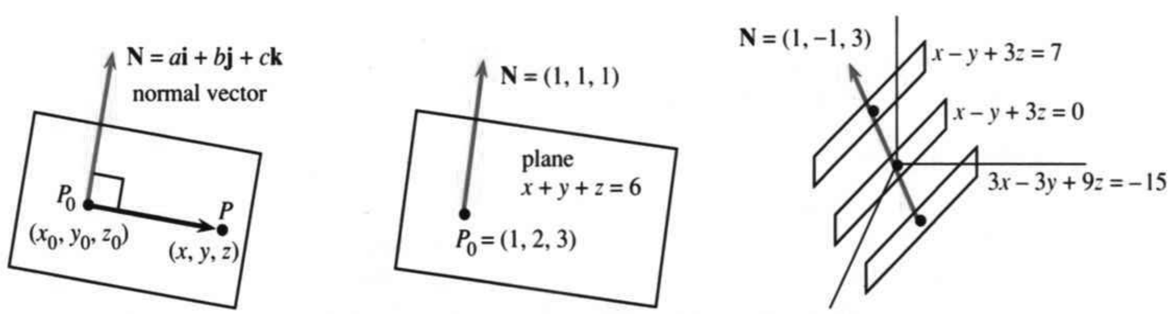


Fig. 11.9 The normal vector to a plane. Parallel planes have the same N.

P_0 to another point $P = (x, y, z)$ in the plane. The vector from P_0 to P has components $(x - x_0, y - y_0, z - z_0)$. This vector lies in the plane, so *its dot product with* **N** *is zero*:

11C The plane through P_0 perpendicular to $\mathbf{N} = (a, b, c)$ has the equation

$$(a, b, c) \cdot (x - x_0,\ y - y_0,\ z - z_0) = 0 \quad \text{or}$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0. \qquad (1)$$

The point P lies on the plane when its coordinates x, y, z satisfy this equation.

EXAMPLE 1 The plane through $P_0 = (1,2,3)$ perpendicular to $\mathbf{N} = (1,1,1)$ has the equation $(x-1)+(y-2)+(z-3) = 0$. That can be rewritten as $x+y+z = 6$.

Notice three things. First, P_0 lies on the plane because $1+2+3 = 6$. Second, $\mathbf{N} = (1,1,1)$ can be recognized from the x,y,z coefficients in $x+y+z = 6$. Third, we could change $\mathbf{N}$ to $(2,2,2)$ and we could change P_0 to $(8,2,-4)$—because $\mathbf{N}$ is still perpendicular and P_0 is still in the plane: $8+2-4 = 6$.

The new normal vector $\mathbf{N} = (2,2,2)$ produces $2(x-1)+2(y-2)+2(z-3) = 0$. That can be rewritten as $2x+2y+2z = 12$. Same normal direction, same plane.

The new point $P_0 = (8,2,-4)$ produces $(x-8)+(y-2)+(z+4) = 0$. That is another form of $x+y+z = 6$. All we require is a perpendicular $\mathbf{N}$ and a point P_0 in the plane.

EXAMPLE 2 The plane through $(1,2,4)$ with the same $\mathbf{N} = (1,1,1)$ has a different equation: $(x-1)+(y-2)+(z-4) = 0$. This is $x+y+z = 7$ (instead of 6). ***These planes with* 7 *and* 6 *are parallel*.**

Starting from $a(x-x_0)+b(y-y_0)+c(z-z_0) = 0$, we often move $ax_0+by_0+cz_0$ to the right hand side—and call this constant d:

11D With the P_0 terms on the right side, the equation of the plane is $\mathbf{N}\cdot\mathbf{P} = d$:

$$ax+by+cz = ax_0+by_0+cz_0 = d. \tag{2}$$

A different d gives a ***parallel plane***; $d = 0$ gives a ***plane through the origin***.

EXAMPLE 3 The plane $x-y+3z = 0$ goes through the origin $(0,0,0)$. The normal vector is read directly from the equation: $\mathbf{N} = (1,-1,3)$. The equation is satisfied by $P_0 = (1,1,0)$ and $P = (1,4,1)$. Subtraction gives a vector $\mathbf{V} = (0,3,1)$ that is in the plane, and $\mathbf{N}\cdot\mathbf{V} = 0$.

The parallel planes $x-y+3z = d$ have the same $\mathbf{N}$ but different d's. These planes miss the origin because d is not zero ($x = 0, y = 0, z = 0$ on the left side needs $d = 0$ on the right side). Note that $3x-3y+9z = -15$ is parallel to both planes. $\mathbf{N}$ is changed to $3\mathbf{N}$ in Figure 11.9, but its direction is not changed.

EXAMPLE 4 ***The angle between two planes is the angle between their normal vectors.***

The planes $x-y+3z = 0$ and $3y+z = 0$ are perpendicular, because $(1,-1,3)\cdot(0,3,1) = 0$. The planes $z = 0$ and $y = 0$ are also perpendicular, because $(0,0,1)\cdot(0,1,0) = 0$. (Those are the xy plane and the xz plane.) The planes $x+y = 0$ and $x+z = 0$ make a $60°$ angle, because $\cos 60° = (1,1,0)\cdot(1,0,1)/\sqrt{2}\sqrt{2} = \frac{1}{2}$.

The cosine of the angle between two planes is $|\mathbf{N}_1\cdot\mathbf{N}_2|/|\mathbf{N}_1||\mathbf{N}_2|$. See Figure 11.10.

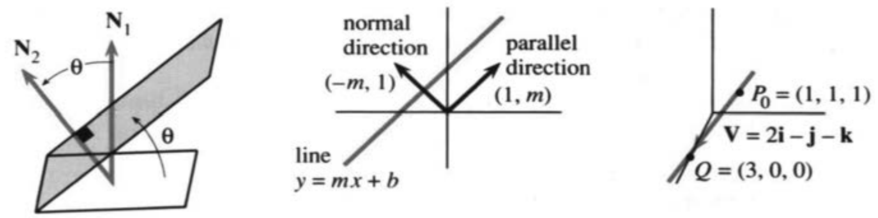


Fig. 11.10 Angle between planes = angle between normals. Parallel and perpendicular to a line. *A line in space through P_0 and Q.*

***Remark* 1** We gave the "point-slope" equation of a line (using m), and the "point-normal" equation of a plane (using **N**). What is the normal vector **N** to a line?

The vector $\mathbf{V} = (1, m)$ is parallel to the line $y = mx + b$. The line goes across by 1 and up by m. ***The perpendicular vector is*** $\mathbf{N} = (-m, 1)$. The dot product $\mathbf{N}\cdot\mathbf{V}$ is $-m + m = 0$. Then the point-normal equation matches the point-slope equation:

$$-m(x - x_0) + 1(y - y_0) = 0 \text{ is the same as } y - y_0 = m(x - x_0). \qquad (3)$$

Remark* 2** What is the point-slope equation for a plane? The difficulty is that a plane has different slopes in the x and y directions. The function $f(x, y) = m(x - x_0) + M(y - y_0)$ has ***two derivatives m and M.

This remark has to stop. In Chapter 13, "slopes" become "***partial derivatives***."

A LINE IN SPACE

In three dimensions, a line is not as simple as a plane. ***A line in space needs two equations***. Each equation gives a plane, and the line is the ***intersection of two planes***.

The equations $x + y + z = 3$ ***and*** $2x + 3y + z = 6$ ***determine a line***.

Two points on that line are $P_0 = (1, 1, 1)$ and $Q = (3, 0, 0)$. They satisfy both equations so they lie on both planes. Therefore they are on the line of intersection. The direction of that line, subtracting coordinates of P_0 from Q, is along the vector $\mathbf{V} = 2\mathbf{i} - \mathbf{j} - \mathbf{k}$.

The line goes through $P_0 = (1, 1, 1)$ ***in the direction of*** $\mathbf{V} = 2\mathbf{i} - \mathbf{j} - \mathbf{k}$.

Starting from $(x_0, y_0, z_0) = (1, 1, 1)$***, add on any multiple*** $t\mathbf{V}$. Then $x = 1 + 2t$ and $y = 1 - t$ and $z = 1 - t$. Those are the components of the vector equation $\mathbf{P} = \mathbf{P}_0 + t\mathbf{V}$—which produces the line.

Here is the problem. The line needs two equations—or a vector equation with a *parameter* t. Neither form is as simple as $ax + by + cz = d$. Some books push ahead anyway, to give full details about both forms. After trying this approach, I believe that those details should wait. Equations with parameters are the subject of Chapter 12, and a line in space is the first example. Vectors and planes give plenty to do here—especially when a vector is projected onto another vector or a plane.

PROJECTION OF A VECTOR

What is the projection of a vector **B** onto another vector **A**? One part of **B** goes ***along*** **A**—that is the projection. The other part of **B** is ***perpendicular*** to **A**. We now compute these two parts, which are **P** and $\mathbf{B} - \mathbf{P}$.

In geometry, projections involve $\cos\theta$. In algebra, we use the dot product (which is closely tied to $\cos\theta$). In applications, the vector **B** might be a *velocity* **V** or a *force* **F**:

> An airplane flies northeast, and a 100-mile per hour wind blows due east. What is the projection of $\mathbf{V} = (100, 0)$ in the flight direction **A**?
>
> Gravity makes a ball roll down the surface $2x + 2y + z = 0$. What are the projections of $\mathbf{F} = (0, 0, -mg)$ in the plane and perpendicular to the plane?

The component of **V** along **A** is the push from the wind (tail wind). The other component of **V** pushes sideways (crosswind). Similarly the force parallel to the surface makes the ball move. Adding the two components brings back **V** or **F**.

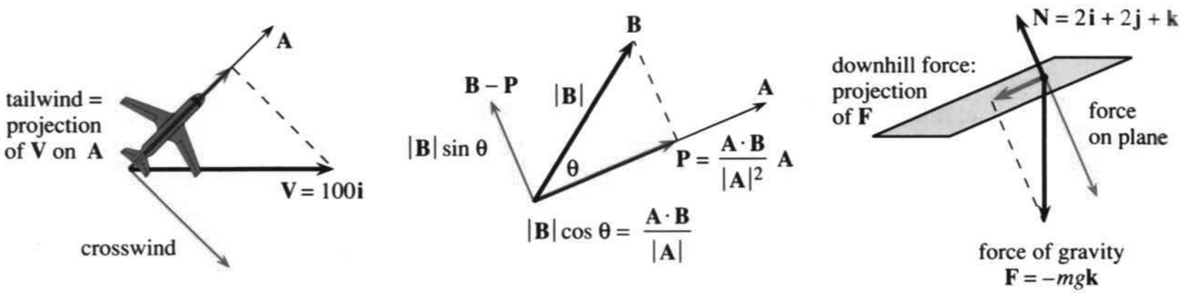


Fig. 11.11 Projections along **A** of wind velocity **V** and force **F** and vector **B**.

***We now compute the projection of* B *onto* A**. Call this projection **P**. Since its direction is known—**P** is along **A**—we can describe **P** in two ways:

1) *Give the* **length** *of* **P** *along* **A**
2) *Give the* **vector P** *as a multiple of* **A**.

Figure 11.11b shows the projection **P** and its length. The hypotenuse is $|\mathbf{B}|$. The length is $|\mathbf{P}| = |\mathbf{B}|\cos\theta$. The perpendicular component $\mathbf{B} - \mathbf{P}$ has length $|\mathbf{B}|\sin\theta$. The cosine is positive for angles less than 90°. The cosine (and **P**!) are zero when **A** and **B** are perpendicular. $|\mathbf{B}|\cos\theta$ is negative for angles greater than 90°, and the projection points along – **A** (the length is $|\mathbf{B}||\cos\theta|$). Unless the angle is 0° or 30° or 45° or 60° or 90°, we don't want to compute cosines—and we don't have to. The dot product does it automatically:

$$|\mathbf{A}|\,|\mathbf{B}|\cos\theta = \mathbf{A}\cdot\mathbf{B} \textit{ so the length of } \mathbf{P} \textit{ along } \mathbf{A} \textit{ is } |\mathbf{B}|\cos\theta = \frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{A}|}. \qquad (4)$$

Notice that the length of **A** cancels out at the end of (4). If **A** is doubled, **P** is unchanged. But if **B** is doubled, the projection is doubled.

What is the vector **P**? Its length along **A** is $\mathbf{A}\cdot\mathbf{B}/|\mathbf{A}|$. If **A** is a unit vector, then $|\mathbf{A}| = 1$ and the projection is $\mathbf{A}\cdot\mathbf{B}$ times **A**. Generally **A** is not a unit vector, until we divide by $|\mathbf{A}|$. ***Here is the projection* P *of* B *along* A**:

$$\mathbf{P} = (\textit{length of } \mathbf{P})(\textit{unit vector}) = \left(\frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{A}|}\right)\left(\frac{\mathbf{A}}{|\mathbf{A}|}\right) = \frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{A}|^2}\mathbf{A}. \qquad (5)$$

EXAMPLE 5 For the wind velocity $\mathbf{V} = (100, 0)$ and flying direction $\mathbf{A} = (1, 1)$, find $\mathbf{P}$. Here $\mathbf{V}$ points east, $\mathbf{A}$ points northeast. The projection of $\mathbf{V}$ onto $\mathbf{A}$ is $\mathbf{P}$:

$$\text{length } |\mathbf{P}| = \frac{\mathbf{A}\cdot\mathbf{V}}{|\mathbf{A}|} = \frac{100}{\sqrt{2}} \qquad \text{vector } \mathbf{P} = \frac{\mathbf{A}\cdot\mathbf{V}}{|\mathbf{A}|^2}\mathbf{A} = \frac{100}{2}(1,1) = (50, 50).$$

EXAMPLE 6 Project $\mathbf{F} = (0, 0, -mg)$ onto the plane with normal $\mathbf{N} = (2, 2, 1)$. The projection of $\mathbf{F}$ along $\mathbf{N}$ is *not* the answer. But compute that first:

$$\frac{\mathbf{F}\cdot\mathbf{N}}{|\mathbf{N}|} = -\frac{mg}{3} \qquad \mathbf{P} = \frac{\mathbf{F}\cdot\mathbf{N}}{|\mathbf{N}|^2}\mathbf{N} = -\frac{mg}{9}(2, 2, 1).$$

$\mathbf{P}$ is the component of $\mathbf{F}$ *perpendicular* to the plane. It does *not* move the ball. The in-plane component is the difference $\mathbf{F} - \mathbf{P}$. Any vector $\mathbf{B}$ has two projections, along $\mathbf{A}$ and perpendicular:

The projection $\mathbf{P} = \dfrac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{A}|^2}\mathbf{A}$ ***is perpendicular to the remaining component*** $\mathbf{B} - \mathbf{P}$.

EXAMPLE 7 Express $\mathbf{B} = \mathbf{i} - \mathbf{j}$ as the sum of a vector $\mathbf{P}$ parallel to $\mathbf{A} = 3\mathbf{i} + \mathbf{j}$ and a vector $\mathbf{B} - \mathbf{P}$ perpendicular to $\mathbf{A}$. Note $\mathbf{A}\cdot\mathbf{B} = 2$.

Solution $\quad \mathbf{P} = \dfrac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{A}|^2}\mathbf{A} = \dfrac{2}{10}\mathbf{A} = \dfrac{6}{10}\mathbf{i} + \dfrac{2}{10}\mathbf{j}$. Then $\mathbf{B} - \mathbf{P} = \dfrac{4}{10}\mathbf{i} - \dfrac{12}{10}\mathbf{j}$.

Check: $\mathbf{P}\cdot(\mathbf{B} - \mathbf{P}) = \left(\frac{6}{10}\right)\left(\frac{4}{10}\right) - \left(\frac{2}{10}\right)\left(\frac{12}{10}\right) = 0$. These projections of $\mathbf{B}$ are perpendicular.

Pythagoras: $|\mathbf{P}|^2 + |\mathbf{B} - \mathbf{P}|^2$ equals $|\mathbf{B}|^2$. Check that too: $0.4 + 1.6 = 2.0$.

Question When is $\mathbf{P} = \mathbf{0}$? Answer When $\mathbf{A}$ and $\mathbf{B}$ are perpendicular.

EXAMPLE 8 ***Find the nearest point to the origin on the plane*** $x + 2y + 2z = 5$.

The shortest distance from the origin is along the normal vector $\mathbf{N}$. The vector $\mathbf{P}$ to the nearest point (Figure 11.12) is t times $\mathbf{N}$, for some unknown number t. We find t by requiring $\mathbf{P} = t\,\mathbf{N}$ to lie on the plane.

The plane $x + 2y + 2z = 5$ has normal vector $\mathbf{N} = (1, 2, 2)$. Therefore $\mathbf{P} = t\,\mathbf{N} = (t, 2t, 2t)$. To lie on the plane, this must satisfy $x + 2y + 2z = 5$:

$$t + 2(2t) + 2(2t) = 5 \quad \text{or} \quad 9t = 5 \quad \text{or} \quad t = \tfrac{5}{9}. \tag{6}$$

Then $\mathbf{P} = \frac{5}{9}\mathbf{N} = (\frac{5}{9}, \frac{10}{9}, \frac{10}{9})$. That locates the nearest point. The distance is $\frac{5}{9}|\mathbf{N}| = \frac{5}{3}$. This example is important enough to memorize, with letters not numbers:

11E On the plane $ax + by + cz = d$, the nearest point to $(0, 0, 0)$ is

$$P = \frac{(da, db, dc)}{a^2 + b^2 + c^2}. \qquad \text{The distance is} \qquad \frac{|d|}{\sqrt{a^2 + b^2 + c^2}}. \tag{7}$$

The steps are the same. $\mathbf{N}$ has components a, b, c. The nearest point on the plane is a multiple (ta, tb, tc). It lies on the plane if $a(ta) + b(tb) + c(tc) = d$.

Thus $t = d/(a^2+b^2+c^2)$. The point $(ta, tb, tc) = t\mathbf{N}$ is in equation (7). The distance to the plane is $|t\mathbf{N}| = |d|/|\mathbf{N}|$.

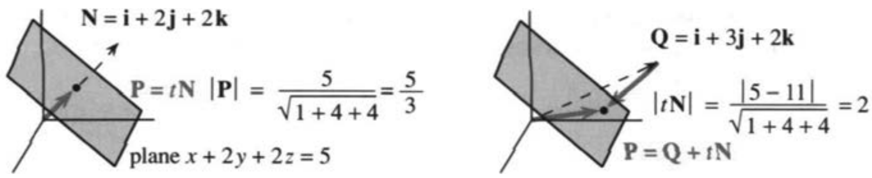


Fig. 11.12 Vector to the nearest point P is a multiple $t\mathbf{N}$. The distance is in (7) and (9).

Question How far is the plane from an arbitrary point $Q = (x_1, y_1, z_1)$?

Answer *The vector from Q to P is our multiple t*$\mathbf{N}$. In vector form $\mathbf{P} = \mathbf{Q} + t\mathbf{N}$. This reaches the plane if $\mathbf{P}\cdot\mathbf{N} = d$, and again we find t:

$$(\mathbf{Q}+t\mathbf{N})\cdot\mathbf{N} = d \quad \text{yields} \quad t = (d - \mathbf{Q}\cdot\mathbf{N})/|\mathbf{N}|^2. \tag{8}$$

This new term $\mathbf{Q}\cdot\mathbf{N}$ enters the distance from Q to the plane:

$$\textbf{\textit{distance}} = |t\mathbf{N}| = |d - \mathbf{Q}\cdot\mathbf{N}|/|\mathbf{N}| = |d - ax_1 - by_1 - cz_1|/\sqrt{a^2+b^2+c^2}. \tag{9}$$

When the point is on the plane, that distance is zero—because $ax_1 + by_1 + cz_1 = d$. When $\mathbf{Q}$ is $\mathbf{i}+3\mathbf{j}+2\mathbf{k}$, the figure shows $\mathbf{Q}\cdot\mathbf{N} = 11$ and distance $= 2$.

PROJECTIONS OF THE HEART VECTOR

An electrocardiogram has leads to your right arm–left arm–left leg. *You produce the voltage*. The machine amplifies and records the readings. There are also six chest leads, to add a front-back dimension that is monitored across the heart. We will concentrate on the big “Einthoven triangle,” named after the inventor of the ECG.

The graphs show voltage variations plotted against time. The first graph plots the voltage difference between the arms. Lead II connects the left leg to the right arm. Lead III completes the triangle, which has roughly equal sides (especially if you are a little lopsided). So the projections are based on 60° and 120° angles.

The heart vector **V** is the sum of many small vectors—all moved to the same origin. **V** is the net effect of action potentials from the cells—small dipoles adding to a single dipole. The pacemaker ($S{-}A$ node) starts the impulse. The atria depolarize to give the P wave on the graphs. This is actually a P *loop* of the heart vector—the graphs only show its projections. The impulse reaches the *AV* node, pauses, and moves quickly through the ventricles. This produces the QRS complex—the large sharp movement on the graph.

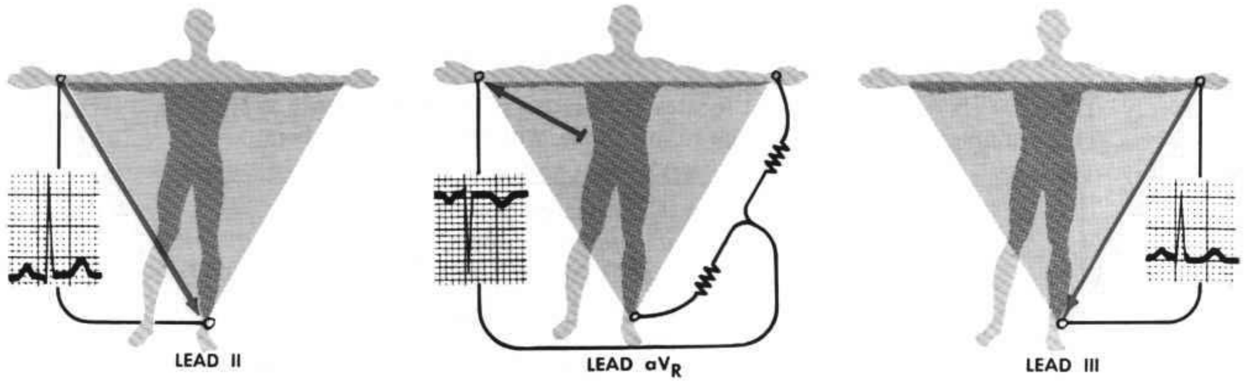


Fig. A The graphs show the component of the moving heart vector along each lead. These figures art reproduced with permission from the CIBA Collection of Medical Illustrations by Frank H. Netter, M.D. Copyright 1978 CIBA-GEIGY, all rights reserved.

The total QRS ***interval should not exceed*** $1/10$ ***second*** ($2\frac{1}{2}$ spaces on the printout). $\mathbf{V}$ points first toward the right shoulder. This direction is opposite to the leads, so the tracings go slightly down. That is the Q wave, small and negative. Then the heart vector sweeps toward the left leg. In positions 3 and 4, its projection on lead I (between the arms) is strongly positive. The R wave is this first upward deflection in each lead. Closing the loop, the S wave is negative (best seen in leads I and aVR).

Question 1 How many graphs from the arms and leg are really independent?

Answer Only two! In a plane, the heart vector $\mathbf{V}$ has two components. If we know two projections, we can compute the others. (The ECG does that for us.) Different vectors show better in different projections. A mathematician would use $90°$ angles, with an electrode at your throat.

Question 2 How are the voltages related? What is the aVR lead?

Answer Project the heart vector $\mathbf{V}$ onto the sides of the triangle:

The lead vectors have $\mathbf{L_I} - \mathbf{L_{II}} + \mathbf{L_{III}} = \mathbf{0}$— they form a triangle.

The projections have $\mathbf{V_I} - \mathbf{V_{II}} + \mathbf{V_{III}} = \mathbf{V}\cdot\mathbf{L_I} - \mathbf{V}\cdot\mathbf{L_{II}} + \mathbf{V}\cdot\mathbf{L_{III}} = 0$.

The aVR lead is $-\frac{1}{2}\mathbf{L_I} - \frac{1}{2}\mathbf{L_{II}}$. It is pure algebra (no wire). By vector addition it points toward the electrode on the right arm. Its length is $\sqrt{3}$ if the other lengths are 2.

Including aVL and aVF to the left arm and foot, there are ***six leads intersecting at equal angles***. Visualize them going out from a single point (the origin in the chest).

Question 3 If the heart vector is $\mathbf{V} = 2\mathbf{i} - \mathbf{j}$, what voltage differences are recorded?

Answer The leads around the triangle have length 2. The machine projects $\mathbf{V}$:

Lead I is the horizontal vector $2\mathbf{i}$. So $\mathbf{V}\cdot\mathbf{L_I} = 4$.
Lead II is the $-60°$ vector $\mathbf{i} - \sqrt{3}\mathbf{j}$. So $\mathbf{V}\cdot\mathbf{L_{II}} = 2 + \sqrt{3}$.
Lead III is the $-120°$ vector $-\mathbf{i} - \sqrt{3}\mathbf{j}$. So $\mathbf{V}\cdot\mathbf{L_{III}} = -2 + \sqrt{3}$.

The first and third add to the second. The largest R waves are in leads I and II. In aVR the projection of $\mathbf{V}$ will be negative (Problem 46), and will be labeled an S wave.

Question 4 What about the *potential* (not just its differences). Is it zero at the center?

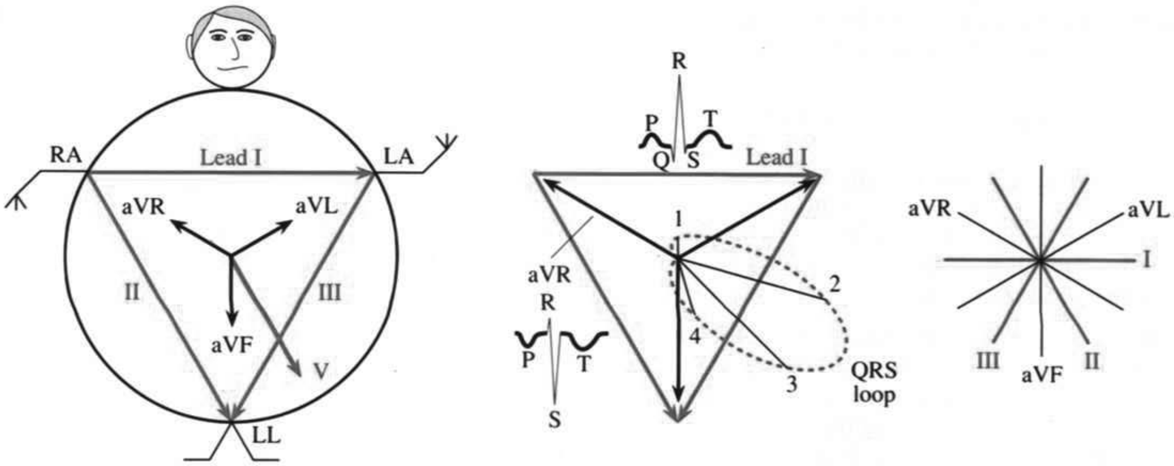


Fig. B Heart vector goes around the QRS loop. Projections are spikes on the ECG.

Answer ***It is zero if we say so***. The potential contains an arbitrary constant C. (It is like an indefinite integral. Its differences are like definite integrals.) Cardiologists define a "central terminal" where the potential is zero.

The average of **V** over a loop is the *mean heart vector* **H**. This average requires $\int \mathbf{V}\,dt$, by Chapter 5. With no time to integrate, the doctor looks for a lead where the area under the QRS complex is zero. Then the direction of **H** (the *axis*) is perpendicular to that lead. There is so much to say about calculus in medicine.

11.2 EXERCISES

Read-through questions

A plane in space is determined by a point $P_0 = (x_0, y_0, z_0)$ and a __a__ vector **N** with components (a, b, c). The point $P = (x, y, z)$ is on the plane if the dot product of **N** with __b__ is zero. (*That answer was not P!*) The equation of this plane is $a($__c__$) + b($__d__$) + c($__e__$) = 0$. The equation is also written as $ax + by + cz = d$, where d equals __f__. A parallel plane has the same __g__ and a different __h__. A plane through the origin has $d =$ __i__.

The equation of the plane through $P_0 = (2, 1, 0)$ perpendicular to $\mathbf{N} = (3, 4, 5)$ is __j__ . A second point in the plane is $P = (0, 0,$ __k__$)$. The vector from P_0 to P is __l__, and it is __m__ to **N**. (Check by dot product.) The plane through $P_0 = (2, 1, 0)$ perpendicular to the z axis has $\mathbf{N} =$ __n__ and equation __o__.

The component of **B** in the direction of **A** is __p__, where θ is the angle between the vectors. This is $\mathbf{A} \cdot \mathbf{B}$ divided by __q__. The projection vector **P** is $|\mathbf{B}| \cos\theta$ times a __r__ vector in the direction of **A**. Then $\mathbf{P} = (|\mathbf{B}| \cos\theta)(\mathbf{A}/|\mathbf{A}|)$ simplifies to __s__. When **B** is doubled, **P** is __t__. When **A** is doubled, **P** is __u__. If **B** reverses direction then **P** __v__. If **A** reverses direction then **P** __w__.

When **B** is a velocity vector, **P** represents the __x__. When **B** is a force vector, **P** is __y__. The component of **B** perpendicular to **A** equals __z__. The shortest distance from $(0, 0, 0)$ to the plane $ax + by + cz = d$ is along the __A__ vector. The distance is __B__ and the closest point on the plane is $P =$ __C__ The distance from $\mathbf{Q} = (x_1, y_1, z_1)$ to the plane is __D__.

Find two points P and P_0 on the planes 1–6 and a normal vector N. Verify that $\mathbf{N} \cdot (P - P_0) = 0$.

1 $x + 2y + 3z = 0$ **2** $x + 2y + 3z = 6$ **3** the yz plane

4 the plane through $(0, 0, 0)$ perpendicular to $\mathbf{i} + \mathbf{j} - \mathbf{k}$

5 the plane through $(1, 1, 1)$ perpendicular to $\mathbf{i} + \mathbf{j} - \mathbf{k}$

6 the plane through $(0, 0, 0)$ and $(1, 0, 0)$ and $(0, 1, 1)$.

Find an $x - y - z$ equation for planes 7–10.

7 The plane through $P_0 = (1, 2, -1)$ perpendicular to $\mathbf{N} = \mathbf{i} + \mathbf{j}$

8 The plane through $P_0 = (1, 2, -1)$ perpendicular to $\mathbf{N} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$

9 The plane through $(1, 0, 1)$ parallel to $x + 2y + z = 0$

10 The plane through (x_0, y_0, z_0) parallel to $x+y+z=1$.

11 When is a plane with normal vector **N** parallel to the vector **V**? When is it perpendicular to **V**?

12 (a) If two planes are perpendicular (front wall and side wall), is every line in one plane perpendicular to every line in the other?

(b) If a third plane is perpendicular to the first, it might be (parallel) (perpendicular) (at a 45° angle) to the second.

13 Explain why a plane cannot

(a) contain $(1,2,3)$ and $(2,3,4)$ and be perpendicular to $\mathbf{N}=\mathbf{i}+\mathbf{j}$

(b) be perpendicular to $\mathbf{N}=\mathbf{i}+\mathbf{j}$ and parallel to $\mathbf{V}=\mathbf{i}+\mathbf{k}$

(c) contain $(1,0,0), (0,1,0), (0,0,1)$, and $(1,1,1)$

(d) contain $(1,1,-1)$ if it has $\mathbf{N}=\mathbf{i}+\mathbf{j}-\mathbf{k}$ (*maybe it can*)

(e) go through the origin and have the equation $ax+by+cz=1$.

14 The equation $3x+4y+7z-t=0$ yields a hyperplane in four dimensions. Find its normal vector **N** and two points P, Q on the hyperplane. Check $(P-Q)\cdot\mathbf{N}=0$.

15 The plane through (x,y,z) perpendicular to $a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$ goes through $(0,0,0)$ if ______. The plane goes through (x_0, y_0, z_0) if ______.

16 A curve in three dimensions is the intersection of ______ surfaces. A line in four dimensions is the intersection of ______ hyperplanes.

17 (angle between planes) Find the cosine of the angle between $x+2y+2z=0$ and (a) $x+2z=0$ (b) $x+2z=5$ (c) $x=0$.

18 **N** is perpendicular to a plane and **V** is along a line. Draw the angle θ between the plane and the line, and explain why $\mathbf{V}\cdot\mathbf{N}/|\mathbf{V}||\mathbf{N}|$ is $\sin\theta$ not $\cos\theta$. Find the angle between the xy plane and $\mathbf{V}=\mathbf{i}+\mathbf{j}+\sqrt{2}\mathbf{k}$.

In 19–26 find the projection P of B along A. Also find $|\mathbf{P}|$.

19 $\mathbf{A}=(4,2,4)$, $\mathbf{B}=(1,-1,0)$

20 $\mathbf{A}=(1,-1,0)$, $\mathbf{B}=(4,2,4)$

21 **B** = unit vector at 60° angle with **A**

22 **B** = vector of length 2 at 60° angle with **A**

23 $\mathbf{B}=-\mathbf{A}$

24 $\mathbf{A}=\mathbf{i}+\mathbf{j}$, $\mathbf{B}=\mathbf{i}+\mathbf{k}$

25 **A** is perpendicular to $x-y+z=0$, $\mathbf{B}=\mathbf{i}+\mathbf{j}$.

26 **A** is perpendicular to $x-y+z=5$, $\mathbf{B}=\mathbf{i}+\mathbf{j}+5\mathbf{k}$.

27 The force $\mathbf{F}=3\mathbf{i}-4\mathbf{k}$ acts at the point $(1,2,2)$. How much force pulls toward the origin? How much force pulls vertically down? Which direction does a mass move under the force **F**?

28 The projection of **B** along **A** is **P** = ______.The projection of **B** perpendicular to **A** is ______. Check the dot product of the two projections.

29 $\mathbf{P}=(x,y,z)$ is on the plane $ax+by+cz=5$ if $\mathbf{P}\cdot\mathbf{N}=|\mathbf{P}||\mathbf{N}|\cos\theta=5$. Since the largest value of $\cos\theta$ is 1, the smallest value of $|\mathbf{P}|$ is ______. This is the distance between ______.

30 If the air speed of a jet is 500 and the wind speed is 50, what information do you need to compute the jet's speed over land? What is that speed?

31 How far is the plane $x+y-z=1$ from $(0,0,0)$ and also from $(1,1,-1)$? Find the nearest points.

32 Describe all points at a distance 1 from the plane $x+2y+2z=3$.

33 The shortest distance from $Q=(2,1,1)$ to the plane $x+y+z=0$ is along the vector ______. The point $\mathbf{P}=\mathbf{Q}+t\mathbf{N}=(2+t,1+t,1+t)$ lies on the plane if $t=$ ______. Then **P** = ______ and the shortest distance is ______. (This distance is not $|\mathbf{P}|$.)

34 The plane through $(1,1,1)$ perpendicular to $\mathbf{N}=\mathbf{i}+2\mathbf{j}+2\mathbf{k}$ is a distance ______ from $(0,0,0)$.

35 (*Distance between planes*) $2x-2y+z=1$ is parallel to $2x-2y+z=3$ because ______. Choose a vector **Q** on the first plane and find t so that $\mathbf{Q}+t\mathbf{N}$ lies on the second plane. The distance is $|t\mathbf{N}|=$ ______.

36 The distance between the planes $x+y+5z=7$ and $3x+2y+z=1$ is zero because ______.

In Problems 37–41 all points and vectors are in the xy plane.

37 The line $3x+4y=10$ is perpendicular to the vector **N** = ______. On the line, the closest point to the origin is $P=t\mathbf{N}$. Find t and P and $|P|$.

38 Draw the line $x+2y=4$ and the vector $\mathbf{N}=\mathbf{i}+2\mathbf{j}$. The closest point to $Q=(3,3)$ is $P=Q+t\mathbf{N}$. Find t. Find P.

39 A new way to find P in Problem 37: minimize $x^2+y^2=x^2+\left(\frac{10}{4}-\frac{3}{4}x\right)^2$. By calculus find the best x and y.

40 To catch a drug runner going from $(0,0)$ to $(4,0)$ at 8 meters per second, you must travel from $(0,3)$ to $(4,0)$ at ______ meters per second. The projection of your velocity vector onto his velocity vector will have length ______.

41 Show by vectors that the distance from (x_1, y_1) to the line $ax+by=d$ is $|d-ax_1-by_1|/\sqrt{a^2+b^2}$.

42 It takes three points to determine a plane. So why does $ax+by+cz=d$ contain four numbers a,b,c,d? When does $ex+fy+gz=1$ represent the same plane?

43 (projections by calculus) The dot product of $\mathbf{B}-t\mathbf{A}$ with itself is $|\mathbf{B}|^2-2t\mathbf{A}\cdot\mathbf{B}+t^2|\mathbf{A}|^2$. (a) This has a minimum at $t=$ ______. (b) Then $t\mathbf{A}$ is the projection of ______. A figure showing **B**, $t\mathbf{A}$, and $\mathbf{B}-t\mathbf{A}$ is worth 1000 words.

44 From their equations, how can you tell if two planes are
(a) parallel (b) perpendicular (c) at a 45° angle?

Problems 45–48 are about the ECG and heart vector.

45 The aVR lead is $-\frac{1}{2}\mathbf{L_I} - \frac{1}{2}\mathbf{L_{II}}$. Find the aVL and aVF leads toward the left arm and foot. Show that aVR+aVL+aVF $= \mathbf{0}$. They go out from the center at 120° angles.

46 Find the projection on the aVR lead of $\mathbf{V} = 2\mathbf{i} - \mathbf{j}$ in Question 3.

47 If the potentials are $\varphi_{\text{RA}} = 1$ (right arm) and $\varphi_{\text{LA}} = 2$ and $\varphi_{\text{LL}} = -3$, find the heart vector $\mathbf{V}$. The *differences* in potential are the projections of $\mathbf{V}$.

48 If $\mathbf{V}$ is perpendicular to a lead $\mathbf{L}$, the reading on that lead is ______. If $\int \mathbf{V}(t)\,dt$ is perpendicular to lead $\mathbf{L}$, why is the *area* under the reading zero ?

11.3 Cross Products and Determinants

After saying that vectors are not multiplied, we offered the dot product. Now we contradict ourselves further, by defining the cross product. Where $\mathbf{A}\cdot\mathbf{B}$ was a number, ***the cross product*** $\mathbf{A}\times\mathbf{B}$ ***is a vector***. It has length and direction:

The length is $|\mathbf{A}||\mathbf{B}||\sin\theta|$. ***The direction is perpendicular to*** $\mathbf{A}$ ***and*** $\mathbf{B}$.

The cross product (also called vector product) is defined in three dimensions only. $\mathbf{A}$ and $\mathbf{B}$ lie on a plane through the origin. $\mathbf{A}\times\mathbf{B}$ is along the normal vector $\mathbf{N}$, perpendicular to that plane. We still have to say whether it points "up" or "down" along $\mathbf{N}$.

The length of $\mathbf{A}\times\mathbf{B}$ depends on $\sin\theta$, where $\mathbf{A}\cdot\mathbf{B}$ involved $\cos\theta$. The dot product rewards vectors for being parallel $(\cos 0 = 1)$. The cross product is largest when $\mathbf{A}$ is perpendicular to $\mathbf{B}$ $(\sin\pi/2 = 1)$. At every angle

$$|\mathbf{A}\cdot\mathbf{B}|^2 + |\mathbf{A}\times\mathbf{B}|^2 = |\mathbf{A}|^2|\mathbf{B}|^2\cos^2\theta + |\mathbf{A}|^2|\mathbf{B}|^2\sin^2\theta = |\mathbf{A}|^2|\mathbf{B}|^2. \qquad (1)$$

That will be a bridge from geometry to algebra. ***This section goes from definition to formula to volume to determinant***. Equations (6) and (14) are the key formulas for $\mathbf{A}\times\mathbf{B}$.

Notice that $\mathbf{A}\times\mathbf{A} = \mathbf{0}$. (This is the zero vector, not the zero number.) When $\mathbf{B}$ is parallel to $\mathbf{A}$, the angle is zero and the sine is zero. Parallel vectors have $\mathbf{A}\times\mathbf{B} = \mathbf{0}$. Perpendicular vectors have $\sin\theta = 1$ and $|\mathbf{A}\times\mathbf{B}| = |\mathbf{A}||\mathbf{B}| =$ area of rectangle with sides $\mathbf{A}$ and $\mathbf{B}$.

Here are four examples that lead to the cross product $\mathbf{A}\times\mathbf{B}$.

EXAMPLE 1 (From geometry) Find the area of a parallelogram and a triangle.

Vectors $\mathbf{A}$ and $\mathbf{B}$, going out from the origin, form two sides of a triangle. They produce the parallelogram in Figure 11.13, which is twice as large as the triangle.

The area of a parallelogram is base times height (perpendicular height not sloping height). The base is $|\mathbf{A}|$. The height is $|\mathbf{B}||\sin\theta|$. We take absolute values because height and area are not negative. Then the area is the length of the cross product:

$$\textbf{\textit{area of parallelogram}} = |\mathbf{A}||\mathbf{B}||\sin\theta| = |\mathbf{A}\times\mathbf{B}|. \qquad (2)$$

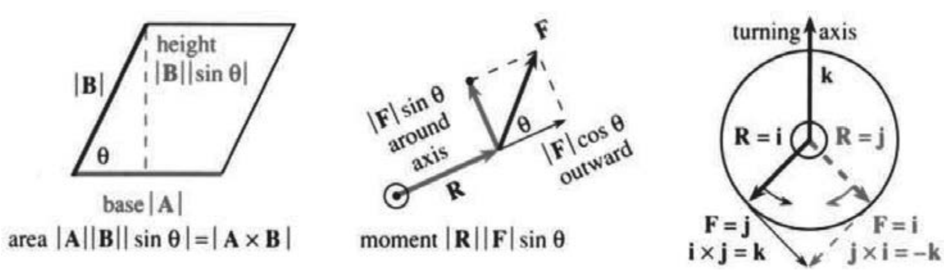


Fig. 11.13 Area $|\mathbf{A}\times\mathbf{B}|$ and moment $|\mathbf{R}\times\mathbf{F}|$. Cross products are perpendicular to the page.

EXAMPLE 2 (From physics) The torque vector $\mathbf{T} = \mathbf{R}\times\mathbf{F}$ produces rotation.

The force $\mathbf{F}$ acts at the point (x, y, z). When $\mathbf{F}$ is parallel to the position vector $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, the force pushes outward (*no turning*). When $\mathbf{F}$ is perpendicular to $\mathbf{R}$, the force creates *rotation*. For in-between angles there is an outward force

$|\mathbf{F}|\cos\theta$ and a turning force $|\mathbf{F}|\sin\theta$. The turning force times the distance $|\mathbf{R}|$ is the ***moment*** $|\mathbf{R}||\mathbf{F}|\sin\theta$.

The moment gives the magnitude and sign of the ***torque vector*** $\mathbf{T}=\mathbf{R}\times\mathbf{F}$. The direction of $\mathbf{T}$ is along the axis of rotation, at right angles to $\mathbf{R}$ and $\mathbf{F}$.

EXAMPLE 3 Does the cross product go up or down? *Use the right-hand rule*.

Forces and torques are probably just fine for physicists. Those who are not natural physicists want to see something turn.† We can visualize a record or compact disc rotating around its axis—which comes up through the center.

At a point on the disc, you give a push. When the push is outward (hard to do), nothing turns. Rotation comes from force "around" the axis. The disc can turn either way—depending on the angle between force and position. A sign convention is necessary, and it is the ***right-hand rule***:

$\mathbf{A}\times\mathbf{B}$ ***points along your right thumb when the fingers curl from*** $\mathbf{A}$ ***toward*** $\mathbf{B}$.

This rule is simplest for the vectors $\mathbf{i},\mathbf{j},\mathbf{k}$ in Figure 11.14—which is all we need.

Suppose the fingers curl from $\mathbf{i}$ to $\mathbf{j}$. The thumb points along $\mathbf{k}$. ***The*** x-y-z ***axes form a*** "**right-handed triple**." Since $|\mathbf{i}|=1$ and $|\mathbf{j}|=1$ and $\sin\pi/2=1$, the length of $\mathbf{i}\times\mathbf{j}$ is 1. ***The cross product is*** $\mathbf{i}\times\mathbf{j}=\mathbf{k}$. The disc turns counterclockwise—its angular velocity is up—when the force acts at $\mathbf{i}$ in the direction $\mathbf{j}$.

Figure 11.14b reverses $\mathbf{i}$ and $\mathbf{j}$. The force acts at $\mathbf{j}$ and its direction is $\mathbf{i}$. The disc turns clockwise (the way records and compact discs actually turn). When the fingers curl from $\mathbf{j}$ to $\mathbf{i}$, the thumb points *down*. Thus $\mathbf{j}\times\mathbf{i}=-\mathbf{k}$. This is a special case of an amazing rule:

$$\textit{The cross product is anticommutative}: \mathbf{B}\times\mathbf{A}=-(\mathbf{A}\times\mathbf{B}). \qquad (3)$$

That is quite remarkable. Its discovery by Hamilton produced an intellectual revolution in 19th century algebra, which had been totally accustomed to $AB=BA$. This commutative law is old and boring for numbers (it is new and boring for dot products). Here we see its *opposite* for vector products $\mathbf{A}\times\mathbf{B}$. Neither law holds for matrix products.

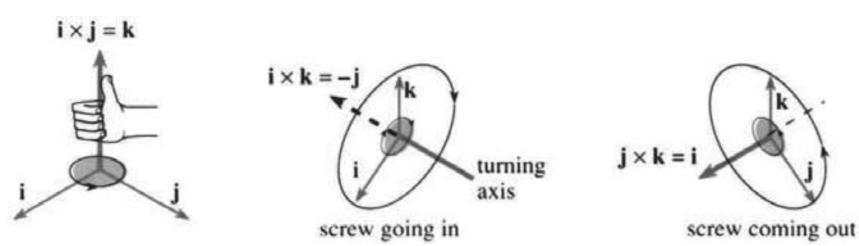


Fig. 11.14 $\mathbf{i}\times\mathbf{j}=\mathbf{k}=-(\mathbf{j}\times\mathbf{i})$ $\quad\mathbf{i}\times\mathbf{k}=-\mathbf{j}=-(\mathbf{k}\times\mathbf{i})$ $\quad\mathbf{j}\times\mathbf{k}=\mathbf{i}=-(\mathbf{k}\times\mathbf{j})$.

EXAMPLE 4 A screw goes into a wall or out, following the right-hand rule.

The disc was in the xy plane. So was the force. (We are not breaking records here.) The axis was up and down. To see the cross product more completely we need to turn a screw into a wall.

Figure 11.14b shows the xz plane as the wall. The screw is in the y direction. By turning from x toward z we drive the screw *into* the wall—which is the *negative* y

†Everybody is a natural mathematician. That is the axiom behind this book.

direction. In other words $\mathbf{i}\times\mathbf{k}$ equals *minus* $\mathbf{j}$. We turn the screw clockwise to make it go in. To take out the screw, twist from $\mathbf{k}$ toward $\mathbf{i}$. Then $\mathbf{k}\times\mathbf{i}$ equals *plus* $\mathbf{j}$.

To summarize: $\mathbf{k}\times\mathbf{i}=\mathbf{j}$ and $\mathbf{j}\times\mathbf{k}=\mathbf{i}$ have plus signs because **kij** and **jki** are in the same "***cyclic order***" as **ijk**. (*Anticyclic is minus*.) The z-x-y and y-z-x axes form righthanded triples like x-y-z.

THE FORMULA FOR THE CROSS PRODUCT

We begin the algebra of $\mathbf{A}\times\mathbf{B}$. It is essential for computation, and it comes out beautifully. The square roots in $|\mathbf{A}||\mathbf{B}||\sin\theta|$ will disappear in formula (6) for $\mathbf{A}\times\mathbf{B}$. (The square roots also disappeared in $\mathbf{A}\cdot\mathbf{B}$, which is $|\mathbf{A}||\mathbf{B}|\cos\theta$. But $|\mathbf{A}||\mathbf{B}|\tan\theta$ would be terrible.) Since $\mathbf{A}\times\mathbf{B}$ is a vector we need to find *three components*.

Start with the two-dimensional case. The vectors $a_1\mathbf{i}+a_2\mathbf{j}$ and $b_1\mathbf{i}+b_2\mathbf{j}$ are in the xy plane. Their cross product must go in the z direction. Therefore $\mathbf{A}\times\mathbf{B}=\underline{\quad ?\quad}\mathbf{k}$ and there is only one nonzero component. It must be $|\mathbf{A}||\mathbf{B}|\sin\theta$ (with the correct sign), but we want a better formula. There are two clean ways to compute $\mathbf{A}\times\mathbf{B}$, either by algebra (***a***) or by a bridge (***b***) to the dot product and geometry:

$$(\boldsymbol{a})\quad (a_1\mathbf{i}+a_2\mathbf{j})\times(b_1\mathbf{i}+b_2\mathbf{j})=a_1b_1\mathbf{i}\times\mathbf{i}+a_1b_2\mathbf{i}\times\mathbf{j}+a_2b_1\mathbf{j}\times\mathbf{i}+a_2b_2\mathbf{j}\times\mathbf{j}. \tag{4}$$

On the right are $\mathbf{0}$, $a_1b_2\mathbf{k}$, $-a_2b_1\mathbf{k}$ and $\mathbf{0}$. ***The cross product is*** $(a_1b_2-a_2b_1)\mathbf{k}$.

(***b***) Rotate $\mathbf{B}=b_1\mathbf{i}+b_2\mathbf{j}$ clockwise through 90° into $\mathbf{B}^*=b_2\mathbf{i}-b_1\mathbf{j}$. Its length is unchanged (and $\mathbf{B}\cdot\mathbf{B}^*=0$). Then $|\mathbf{A}||\mathbf{B}^*|\sin\theta$ equals $|\mathbf{A}||\mathbf{B}^*\cos\theta$, which is $\mathbf{A}\cdot\mathbf{B}^*$:

$$|\mathbf{A}||\mathbf{B}|\sin\theta=\mathbf{A}\cdot\mathbf{B}^*=\begin{bmatrix}a_1\\a_2\end{bmatrix}\cdot\begin{bmatrix}b_2\\-b_1\end{bmatrix}=a_1b_2-a_2b_1. \tag{5}$$

> **11F** In the xy plane, $\mathbf{A}\times\mathbf{B}$ equals $(a_1b_2-a_2b_1)\mathbf{k}$. The parallelogram with sides $\mathbf{A}$ and $\mathbf{B}$ has area $|a_1b_2-a_2b_1|$. The triangle OAB has area $\frac{1}{2}|a_1b_2-a_2b_1|$.

EXAMPLE 5 For $\mathbf{A}=\mathbf{i}+2\mathbf{j}$ and $\mathbf{B}=4\mathbf{i}+5\mathbf{j}$ the cross product is $(1\cdot5-2\cdot4)\mathbf{k}=-3\mathbf{k}$. Area of parallelogram $=3$, area of triangle $=3/2$. The minus sign in $\mathbf{A}\times\mathbf{B}=-3\mathbf{k}$ is absent in the areas.

Note Splitting $\mathbf{A}\times\mathbf{B}$ into four separate cross products is correct, but it does not follow easily from $|\mathbf{A}||\mathbf{B}|\sin\theta$. Method (***a***) is not justified until Remark 1 below. An algebraist would change the definition of $\mathbf{A}\times\mathbf{B}$ to start with the distributive law (splitting rule) and the anticommutative law:

$$\mathbf{A}\times(\mathbf{B}+\mathbf{C})=(\mathbf{A}\times\mathbf{B})+(\mathbf{A}\times\mathbf{C})\quad\text{and}\quad\mathbf{A}\times\mathbf{B}=-(\mathbf{B}\times\mathbf{A}).$$

THE CROSS PRODUCT FORMULA (3 COMPONENTS)

We move to three dimensions. The goal is to compute all three components of $\mathbf{A}\times\mathbf{B}$ (not just the length). Method (***a***) splits each vector into its $\mathbf{i},\mathbf{j},\mathbf{k}$ components, making nine separate cross products:

$$(a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})\times(b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k})=a_1b_2(\mathbf{i}\times\mathbf{i})+a_1b_2(\mathbf{i}\times\mathbf{j})+\text{ seven more terms.}$$

Remember $\mathbf{i}\times\mathbf{i}=\mathbf{j}\times\mathbf{j}=\mathbf{k}\times\mathbf{k}=\mathbf{0}$. Those three terms disappear. The other six terms come in pairs, and ***please notice the cyclic pattern***:

FORMULA $\mathbf{A}\times\mathbf{B}=(a_2b_3-a_3b_2)\mathbf{i}+(a_3b_1-a_1b_3)\mathbf{j}+(a_1b_2-a_2b_1)\mathbf{k}$. (6)

The $\mathbf{k}$ component is the 2×2 answer, when $a_3=b_3=0$. The $\mathbf{i}$ component involves indices 2 and 3, $\mathbf{j}$ involves 3 and 1, $\mathbf{k}$ involves 1 and 2. The cross product formula is written as a "determinant" in equation (14) below—many people use that form to compute $\mathbf{A}\times\mathbf{B}$.

EXAMPLE 6 $(\mathbf{i}+2\mathbf{j}+3\mathbf{k})\times(4\mathbf{i}+5\mathbf{j}+6\mathbf{k})=(2\cdot 6-3\cdot 5)\mathbf{i}+(3\cdot 4-1\cdot 6)\mathbf{j}+(1\cdot 5-$
$2\cdot 4)\mathbf{k}$. The $\mathbf{i},\mathbf{j},\mathbf{k}$ components give $\mathbf{A}\times\mathbf{B}=-3\mathbf{i}+6\mathbf{j}-3\mathbf{k}$. Never add the $-3,6,$ and -3.

Remark* 1** The three-dimensional formula (6) is still to be matched with $\mathbf{A}\times\mathbf{B}$ from geometry. One way is to rotate $\mathbf{B}$ into $\mathbf{B}^*$ as before, staying in the plane of $\mathbf{A}$ and $\mathbf{B}$. Fortunately there is an easier test. The vector in equation (6) satisfies all four geometric requirements on $\mathbf{A}\times\mathbf{B}$: ***perpendicular to* A**, ***perpendicular to* B**, ***correct length, right-hand rule. The length is checked in Problem 16—here is the zero dot product with $\mathbf{A}$:

$$\mathbf{A}\cdot(\mathbf{A}\times\mathbf{B})=a_1(a_2b_3-a_3b_2)+a_2(a_3b_1-a_1b_3)+a_3(a_1b_2-a_2b_1)=0. \quad (7)$$

***Remark* 2** (Optional) There is a wonderful extension of the Pythagoras formula $a^2+b^2=c^2$. Instead of sides of a triangle, we go to *areas of projections* on the yz, xz, and xy planes. $3^2+6^2+3^2$ is the square of the parallelogram area in Example 6.

For triangles these areas are cut in half. Figure 11.15a shows three projected triangles of area $\frac{1}{2}$. Its Pythagoras formula is $\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2$ = (area of PQR)^2.

EXAMPLE 7 $P=(1,0,0)$, $Q=(0,1,0)$, $R=(0,0,1)$ lie in a plane. Find its equation.

Idea for any P,Q,R: ***Find vectors* A *and* B *in the plane. Compute the normal*** $\mathbf{N}=\mathbf{A}\times\mathbf{B}$.

Solution The vector from P to Q has components $-1,1,0$. It is $\mathbf{A}=\mathbf{j}-\mathbf{i}$ (subtract to go from P to Q). Similarly the vector from P to R is $\mathbf{B}=\mathbf{k}-\mathbf{i}$. Since $\mathbf{A}$ and $\mathbf{B}$ are in the plane of Figure 11.15, $\mathbf{N}=\mathbf{A}\times\mathbf{B}$ is perpendicular:

$$(\mathbf{j}-\mathbf{i})\times(\mathbf{k}-\mathbf{i})=(\mathbf{j}\times\mathbf{k})-(\mathbf{i}\times\mathbf{k})-(\mathbf{j}\times\mathbf{i})+(\mathbf{i}\times\mathbf{i})=\mathbf{i}+\mathbf{j}+\mathbf{k}. \quad (8)$$

The normal vector is $\mathbf{N}=\mathbf{i}+\mathbf{j}+\mathbf{k}$. ***The equation of the plane is*** $1x+1y+1z=d$.

With the right choice $d=1$, this plane contains P, Q, R. The equation is $x+y+z=1$.

EXAMPLE 8 What is the area of this same triangle PQR?

Solution The area is half of the cross-product length $|\mathbf{A}\times\mathbf{B}|=|\mathbf{i}+\mathbf{j}+\mathbf{k}|=\sqrt{3}$.

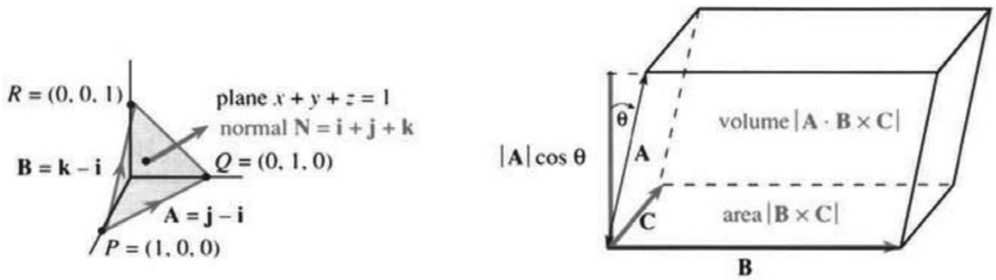


Fig. 11.15 Area of PQR is $\sqrt{3}/2$. $\mathbf{N}$ is $PQ \times PR$. ***Volume of box is*** $|\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})|$.

DETERMINANTS AND VOLUMES

We are close to good algebra. The two plane vectors $a_1\mathbf{i}+a_2\mathbf{j}$ and $b_1\mathbf{i}+b_2\mathbf{j}$ are the sides of a parallelogram. Its area is $a_1b_2-a_2b_1$, possibly with a sign change. There is a special way to write these four numbers—in a "***square matrix***." There is also a name for the combination that leads to area. It is the "***determinant of the matrix***":

$$\textit{The matrix is } \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}, \textit{ its determinant is } \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1.$$

This is a 2 by 2 matrix (notice brackets) and a 2 by 2 determinant (notice vertical bars). The matrix is an array of four numbers and the determinant is one number:

$$\text{Examples of determinants: } \begin{vmatrix} 2 & 1 \\ 4 & 3 \end{vmatrix} = 6-4=2, \quad \begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} = 0, \quad \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1.$$

The second has no area because $\mathbf{A}=\mathbf{B}$. The third is a unit square $(\mathbf{A}=\mathbf{i},\mathbf{B}=\mathbf{j})$.

Now move to three dimensions, where determinants are most useful. The parallelogram becomes a parallelepiped. The word "box" is much shorter, and we will use it, but remember that *the box is squashed*. (Like a rectangle squashed to a parallelogram, the angles are generally not 90°.) The three edges from the origin are $\mathbf{A}=(a_1,a_2,a_3),\mathbf{B}=(b_1,b_2,b_3),\mathbf{C}=(c_1,c_2,c_3)$. Those edges are at right angles only when $\mathbf{A}\cdot\mathbf{B}=\mathbf{A}\cdot\mathbf{C}=\mathbf{B}\cdot\mathbf{C}=0$.

Question: ***What is the volume of the box***? The right-angle case is easy—it is length times width times height. The volume is $|\mathbf{A}|$ times $|\mathbf{B}|$ times $|\mathbf{C}|$, when the angles are 90°. For a squashed box (Figure 11.15) we need the perpendicular height, not the sloping height.

There is a beautiful formula for volume. $\mathbf{B}$ and $\mathbf{C}$ give a parallelogram in the base, and $|\mathbf{B}\times\mathbf{C}|$ is the base area. This cross product points straight up. The third vector $\mathbf{A}$ points up at an angle—its perpendicular height is $|\mathbf{A}|\cos\theta$. Thus the volume is area $|\mathbf{B}\times\mathbf{C}|$ times $|\mathbf{A}|$ times $\cos\theta$. ***The volume is the dot product of*** $\mathbf{A}$ ***with*** $\mathbf{B}\times\mathbf{C}$.

11G The ***triple scalar product*** is $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$. ***Volume of box*** $=|\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})|$.

Important: $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$ is a number, not a vector. This volume is zero when $\mathbf{A}$ is in the same plane as $\mathbf{B}$ and $\mathbf{C}$ (the box is totally flattened). Then $\mathbf{B}\times\mathbf{C}$ is perpendicular to **A** and their dot product is zero.

$$\textit{Useful facts}: \quad \mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\times\mathbf{B})\cdot\mathbf{C} = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}).$$

All those come from the same box, with different sides chosen as base—but no change in volume. Figure 11.15 has **B** and **C** in the base but it can be **A** and **B** or **A** and **C**. ***The triple product*** $\mathbf{A}\cdot(\mathbf{C}\times\mathbf{B})$ ***has opposite sign***, since $\mathbf{C}\times\mathbf{B} = -(\mathbf{B}\times\mathbf{C})$. This order **ACB** is not cyclic like **ABC** and **CAB** and **BCA**.

To compute this triple product $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$, we take $\mathbf{B}\times\mathbf{C}$ from equation (6):

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = a_1(b_2c_3-b_3c_2)+a_2(b_3c_1-b_1c_3)+a_3(b_1c_2-b_2c_1). \tag{9}$$

The numbers a_1,a_2,a_3 multiply 2 by 2 determinants to give a 3 by 3 determinant! There are three terms with plus signs (like $a_1b_2c_3$). The other three have minus signs (like $-a_1b_3c_2$). The plus terms have indices $123, 231, 312$ in cyclic order. The minus terms have anticyclic indices $132, 213, 321$. Again there is a special way to write the nine components of $\mathbf{A},\mathbf{B},\mathbf{C}$—as a "3 by 3 matrix." The combination in (9), which gives volume, is a "3 by 3 determinant:"

$$\textbf{matrix} = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}, \quad \textbf{determinant} = \mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

A single number is produced out of nine numbers, by formula (9). The nine numbers are multiplied three at a time, as in $a_1b_1c_2$—except this product is not allowed. ***Each row and column must be represented once***. This gives the six terms in the determinant:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{matrix} a_1b_2c_3+a_2b_3c_1+a_3b_1c_2 \\ -a_1b_3c_2-a_2b_1c_3-a_3b_2c_1 \end{matrix} \tag{10}$$

The trick is in the $\pm$ signs. Products down to the right are "*plus*":

$$\begin{vmatrix} 2 & 1 & \underline{1} \\ \underline{1} & 2 & 1 \\ 1 & \underline{1} & 2 \end{vmatrix} = \begin{matrix} 2\cdot2\cdot2+1\cdot1\cdot1\cdot+\underline{1}\cdot\underline{1}\cdot\underline{1} \\ -2\cdot1\cdot1-\ 1\cdot1\cdot2\ -1\cdot2\cdot1 \end{matrix} = \begin{matrix} 8+1+1 \\ -2-2-2 \end{matrix} = 4.$$

With practice the six products like $2\cdot2\cdot2$ are done in your head. Write down only $8+1+1-2-2-2=4$. This is the determinant and the volume.

Note the special case when the vectors are $\mathbf{i},\mathbf{j},\mathbf{k}$. The box is a unit cube:

$$\textbf{volume of cube} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{matrix} 1+0+0 \\ -0-0-0 \end{matrix} = 1.$$

If $\mathbf{A},\mathbf{B},\mathbf{C}$ ***lie in the same plane, the volume is zero***. A zero determinant is the test to see whether three vectors lie in a plane. Here row $\mathbf{A} =$ row $\mathbf{B} -$ row $\mathbf{C}$:

$$\begin{vmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = \begin{matrix} 0\cdot1\cdot1+1\cdot0\cdot(-1)+(-1)\cdot(-1)\cdot0 \\ -0\cdot0\cdot0-1\cdot(-1)\cdot1-(-1)\cdot1\cdot(-1) \end{matrix} = 0. \tag{11}$$

Zeros in the matrix simplify the calculation. All three products with plus signs—down to the right—are zero. The only two nonzero products cancel each other.

If the three -1's are changed to $+1$'s, the determinant is -2. The determinant can be negative when all nine entries are positive! A negative determinant only means that the rows $\mathbf{A}, \mathbf{B}, \mathbf{C}$ form a "left-handed triple." This extra information from the sign—right-handed vs. left-handed—is free and useful, but the volume is the absolute value.

The determinant yields the volume also in higher dimensions. In physics, four dimensions give space-time. Ten dimensions give superstrings. Mathematics uses all dimensions. The 64 numbers in an 8 by 8 matrix give the volume of an eight-dimensional box—with $8! = 40,320$ terms instead of $3! = 6$. Under pressure from my class I omit the formula.

Question When is the point (x, y, z) on the plane through the origin containing $\mathbf{B}$ and $\mathbf{C}$? For the vector $\mathbf{A} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ to lie in that plane, the volume $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$ must be zero. The equation of the plane is ***determinant = zero***.

Follow this example for $\mathbf{B} = \mathbf{j} - \mathbf{i}$ and $\mathbf{C} = \mathbf{k} - \mathbf{i}$ to find the plane parallel to $\mathbf{B}$ and $\mathbf{C}$:

$$\begin{vmatrix} x & y & z \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = \begin{matrix} x\cdot 1\cdot 1 + y\cdot 0\cdot(-1) + z\cdot 0\cdot(-1) \\ -x\cdot 0\cdot 0 - y\cdot 1\cdot(-1) - z\cdot 1\cdot(-1) \end{matrix} = 0. \qquad (12)$$

This equation is $x + y + z = 0$. The normal vector $\mathbf{N} = \mathbf{B}\times\mathbf{C}$ has components $1, 1, 1$.

THE CROSS PRODUCT AS A DETERMINANT

There is a connection between 3 by 3 and 2 by 2 determinants that you have to see. The numbers in the top row multiply determinants from the other rows:

$$\begin{vmatrix} \underline{a_1} & a_2 & a_3 \\ b_1 & \underline{b_2} & \underline{b_3} \\ c_1 & \underline{c_2} & \underline{c_3} \end{vmatrix} = \underline{a_1}\begin{vmatrix} \underline{b_2} & \underline{b_3} \\ \underline{c_2} & \underline{c_3} \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}. \qquad (13)$$

The highlighted product $a_1(b_2c_3 - b_3c_2)$ gives two of the six terms. ***All six products contain an a and b and c from different columns***. There are $3! = 6$ different orderings of columns $1, 2, 3$. Note how a_3 multiplies a determinant from columns 1 and 2.

Equation (13) is identical with equations (9) and (10). We are meeting the same six terms in different ways. The new feature is the minus sign in front of a_2—and the common mistake is to forget that sign. In a 4 by 4 determinant, $a_1, -a_2, a_3, -a_4$ would multiply 3 by 3 determinants.

Now comes a key step. We write $\mathbf{A}\times\mathbf{B}$ as a determinant. The vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ go in the top row, the components of $\mathbf{A}$ and $\mathbf{B}$ go in the other rows. ***The "determinant" is exactly $\mathbf{A}\times\mathbf{B}$***:

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \underline{a_1} & a_2 & \underline{a_3} \\ \underline{b_1} & b_2 & \underline{b_3} \end{vmatrix} = \mathbf{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} \underline{a_1} & \underline{a_3} \\ \underline{b_1} & \underline{b_3} \end{vmatrix} + \mathbf{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}. \qquad (14)$$

This time we highlighted the $\mathbf{j}$ component with its minus sign. There is no great mathematics in formula (14)—it is probably illegal to mix $\mathbf{i}, \mathbf{j}, \mathbf{k}$ with six numbers but it works. This is the good way to remember and compute $\mathbf{A}\times\mathbf{B}$. In the example $(\mathbf{j}-\mathbf{i})\times(\mathbf{k}-\mathbf{i})$ from equation (8), those two vectors go into the last two rows:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \underline{\mathbf{k}} \\ \underline{-1} & \underline{1} & 0 \\ \underline{-1} & \underline{0} & 1 \end{vmatrix} = \mathbf{i}\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - \mathbf{j}\begin{vmatrix} -1 & 0 \\ -1 & 1 \end{vmatrix} + \underline{\mathbf{k}}\begin{vmatrix} \underline{-1} & \underline{1} \\ \underline{-1} & \underline{0} \end{vmatrix} = \mathbf{i}+\mathbf{j}+\underline{\mathbf{k}}.$$

The $\mathbf{k}$ component is highlighted, to see $a_1b_2-a_2b_1$ again. Note the change from equation (11), which had $0,1,-1$ in the top row. That triple product was a number (zero). This cross product is a vector $\mathbf{i}+\mathbf{j}+\mathbf{k}$.

Review question 1 With the $\mathbf{i},\mathbf{j},\mathbf{k}$ row changed to $3,4,5$, what is the determinant?
Answer $3\cdot 1+4\cdot 1+5\cdot 1=12$. That triple product is the volume of a box.

Review question 2 When is $\mathbf{A}\times\mathbf{B}=\mathbf{0}$ and when is $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})=0$? Zero vector, zero number.
Answer When $\mathbf{A}$ and $\mathbf{B}$ are on the same line. When $\mathbf{A},\mathbf{B},\mathbf{C}$ are in the same plane.

Review question 3 Does the parallelogram area $|\mathbf{A}\times\mathbf{B}|$ equal a 2 by 2 determinant?
Answer If $\mathbf{A}$ and $\mathbf{B}$ lie in the xy plane, *yes*. Generally *no*.

Review question 4 What are the *vector* triple products $(\mathbf{A}\times\mathbf{B})\times\mathbf{C}$ and $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$?
Answer Not computed yet. These are two new vectors in Problem 47.

Review question 5 Find the plane through the origin containing $\mathbf{A}=\mathbf{i}+\mathbf{j}+2\mathbf{k}$ and $\mathbf{B}=\mathbf{i}+\mathbf{k}$. Find the cross product of those same vectors $\mathbf{A}$ and $\mathbf{B}$.
Answer The position vector $\mathbf{P}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ is perpendicular to $\mathbf{N}=\mathbf{A}\times\mathbf{B}$:

$$\mathbf{P}\cdot(\mathbf{A}\times\mathbf{B})=\begin{vmatrix} x & y & z \\ 1 & 1 & 2 \\ 1 & 0 & 1 \end{vmatrix} = x+y-z=0. \quad \mathbf{A}\times\mathbf{B}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 2 \\ 1 & 0 & 1 \end{vmatrix} = \mathbf{i}+\mathbf{j}-\mathbf{k}.$$

11.3 EXERCISES

Read-through questions

The cross product $\mathbf{A}\times\mathbf{B}$ is a __a__ whose length is __b__. Its direction is __c__ to $\mathbf{A}$ and $\mathbf{B}$. That length is the area of a __d__, whose base is $|\mathbf{A}|$ and whose height is __e__. When $\mathbf{A}=a_1\mathbf{i}+a_2\mathbf{j}$ and $\mathbf{B}=b_1\mathbf{i}+b_2\mathbf{j}$, the area is __f__. This equals a 2 by 2 __g__. In general $|\mathbf{A}\cdot\mathbf{B}|^2+|\mathbf{A}\times\mathbf{B}|^2=$ __h__.

The rules for cross product are $\mathbf{A}\times\mathbf{A}=$ __i__ and $\mathbf{A}\times\mathbf{B}=-($__j__$)$ and $\mathbf{A}\times(\mathbf{B}+\mathbf{C})=\mathbf{A}\times\mathbf{B}+$ __k__. In particular $\mathbf{A}\times\mathbf{B}$ needs the __l__-hand rule to decide its direction. If the fingers curl from $\mathbf{A}$ towards $\mathbf{B}$ (not more than $180°$), then __m__ points __n__. By this rule $\mathbf{i}\times\mathbf{j}=$ __o__ and $\mathbf{i}\times\mathbf{k}=$ __p__ and $\mathbf{j}\times\mathbf{k}=$ __q__.

The vectors $a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$ and $b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$ have cross product __r__$\mathbf{i}+$__s__$\mathbf{j}+$__t__$\mathbf{k}$. The vectors $\mathbf{A}=\mathbf{i}+\mathbf{j}+\mathbf{k}$ and $\mathbf{B}=\mathbf{i}+\mathbf{j}$ have $\mathbf{A}\times\mathbf{B}=$ __u__. (*This is also the* 3 *by* 3 *determinant* __v__.) Perpendicular to the plane containing $(0,0,0),(1,1,1),(1,1,0)$ is the normal vector $\mathbf{N}=$ __w__. The area of the triangle with those three vertices is __x__, which is half the area of the parallelogram with fourth vertex at __y__.

Vectors, $\mathbf{A},\mathbf{B},\mathbf{C}$ from the origin determine a __z__. Its volume $|\mathbf{A}\cdot($__A__$)|$ comes from a 3 by 3 __B__. There are six terms, __C__ with a plus sign and __D__ with minus. In every term each row and __E__ is represented once. The rows $(1,0,0),(0,0,1)$, and

$(0,1,0)$ have determinant $=$ __F__. That box is a __G__, but its sides form a __H__-handed triple in the order given.

If $\mathbf{A},\mathbf{B},\mathbf{C}$ lie in the same plane then $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$ is __I__. For $\mathbf{A}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ the first row contains the letters __J__. So the plane containing $\mathbf{B}$ and $\mathbf{C}$ has the equation __K__ $=0$. When $\mathbf{B}=\mathbf{i}+\mathbf{j}$ and $\mathbf{C}=\mathbf{k}$ that equation is __L__. $\mathbf{B}\times\mathbf{C}$ is __M__.

A 3 by 3 determinant splits into __N__ 2 by 2 determinants. They come from rows 2 and 3, and are multiplied by the entries in row 1. With $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ in row 1, this determinant equals the __O__ product. Its $\mathbf{j}$ component is __P__, including the __Q__ sign which is easy to forget.

Compute the cross products 1–8 from formula (6) or the determinant (14). Do one example both ways.

1 $(\mathbf{i}\times\mathbf{j})\times\mathbf{k}$

2 $(\mathbf{i}\times\mathbf{j})\times\mathbf{i}$

3 $(2\mathbf{i}+3\mathbf{j})\times(\mathbf{i}+\mathbf{k})$

4 $(2\mathbf{i}+3\mathbf{j}+\mathbf{k})\times(2\mathbf{i}+3\mathbf{j}-\mathbf{k})$

5 $(2\mathbf{i}+3\mathbf{j}+\mathbf{k})\times(\mathbf{i}-\mathbf{j}-\mathbf{k})$

6 $(\mathbf{i}+\mathbf{j}-\mathbf{k})\times(\mathbf{i}-\mathbf{j}+\mathbf{k})$

7 $(\mathbf{i}+2\mathbf{j}+3\mathbf{k})\times(4\mathbf{i}-9\mathbf{j})$

8 $(\mathbf{i}\cos\theta+\mathbf{j}\sin\theta)\times(\mathbf{i}\sin\theta-\mathbf{j}\cos\theta)$

9 When are $|\mathbf{A}\times\mathbf{B}|=|\mathbf{A}||\mathbf{B}|$ and $|\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})|=|\mathbf{A}||\mathbf{B}||\mathbf{C}|$?

10 **True or false**:

(a) $\mathbf{A}\times\mathbf{B}$ never equals $\mathbf{A}\cdot\mathbf{B}$.

(b) If $\mathbf{A}\times\mathbf{B}=\mathbf{0}$ and $\mathbf{A}\cdot\mathbf{B}=0$, then either $\mathbf{A}=\mathbf{0}$ or $\mathbf{B}=\mathbf{0}$.

(c) If $\mathbf{A}\times\mathbf{B}=\mathbf{A}\times\mathbf{C}$ and $\mathbf{A}\neq\mathbf{0}$, then $\mathbf{B}=\mathbf{C}$.

In 11–16 find $|\mathbf{A}\times\mathbf{B}|$ by equation (1) and then by computing $\mathbf{A}\times\mathbf{B}$ and its length.

11 $\mathbf{A}=\mathbf{i}+\mathbf{j}+\mathbf{k}$, $\mathbf{B}=\mathbf{i}$

12 $\mathbf{A}=\mathbf{i}+\mathbf{j}$, $\mathbf{B}=\mathbf{i}-\mathbf{j}$

13 $\mathbf{A}=-\mathbf{B}$

14 $\mathbf{A}=\mathbf{i}+\mathbf{j}$, $\mathbf{B}=\mathbf{j}+\mathbf{k}$

15 $\mathbf{A}=a_1\mathbf{i}+a_2\mathbf{j}$, $\mathbf{B}=b_1\mathbf{i}+b_2\mathbf{j}$

16 $\mathbf{A}=(a_1,a_2,a_3)$, $\mathbf{B}=(b_1,b_2,b_3)$

In Problem 16 (the general case), equation (1) proves that the length from equation (6) is correct.

17 **True or false**, by testing on $\mathbf{A}=\mathbf{i}$, $\mathbf{B}=\mathbf{j}$, $\mathbf{C}=\mathbf{k}$:

(a) $\mathbf{A}\times(\mathbf{A}\times\mathbf{B})=\mathbf{0}$

(b) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})=(\mathbf{A}\times\mathbf{B})\cdot\mathbf{C}$

(c) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})=\mathbf{C}\cdot(\mathbf{B}\times\mathbf{A})$

(d) $(\mathbf{A}-\mathbf{B})\times(\mathbf{A}+\mathbf{B})=2(\mathbf{A}\times\mathbf{B})$.

18 (a) From $\mathbf{A}\times\mathbf{B}=-(\mathbf{B}\times\mathbf{A})$ deduce that $\mathbf{A}\times\mathbf{A}=\mathbf{0}$.

(b) Split $(\mathbf{A}+\mathbf{B})\times(\mathbf{A}+\mathbf{B})$ into four terms, to deduce that $(\mathbf{A}\times\mathbf{B})=-(\mathbf{B}\times\mathbf{A})$.

What are the normal vectors to the planes 19–22 ?

19 $(2,1,0)\cdot(x,y,z)=4$

20 $3x+4z=5$

21 $\begin{vmatrix} x & y & z \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix}=2$

22 $\begin{vmatrix} x & y & z \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{vmatrix}=0$

Find N and the equation of the plane described in 23–29.

23 Contains the points $(2,1,1)$, $(1,2,1)$, $(1,1,2)$

24 Contains the points $(0,1,2)$, $(1,2,3)$, $(2,3,4)$

25 Through $(0,0,0)$, $(1,1,1)$, (a,b,c) [What if $a=b=c$?]

26 Parallel to $\mathbf{i}+\mathbf{j}$ and $\mathbf{k}$

27 $\mathbf{N}$ makes a 45° angle with $\mathbf{i}$ and $\mathbf{j}$

28 $\mathbf{N}$ makes a 60° angle with $\mathbf{i}$ and $\mathbf{j}$

29 $\mathbf{N}$ makes a 90° angle with $\mathbf{i}$ and $\mathbf{j}$

30 The triangle with sides $\mathbf{i}$ and $\mathbf{j}$ is ______ as large as the parallelogram with those sides. The tetrahedron with edges $\mathbf{i},\mathbf{j},\mathbf{k}$ is ______ as large as the box with those edges. Extra credit: In four dimensions the "simplex" with edges $\mathbf{i},\mathbf{j},\mathbf{k},\mathbf{l}$ has volume $=$ ______ .

31 If the points (x,y,z), $(1,1,0)$, and $(1,2,1)$ lie on a plane through the origin, what determinant is zero? What equation does this give for the plane?

32 Give an example of a right-hand triple and left-hand triple. Use vectors other than just $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$.

33 When $\mathbf{B}=3\mathbf{i}+\mathbf{j}$ is rotated 90° clockwise in the xy plane it becomes $\mathbf{B}^*=$ ______. When rotated 90° counterclockwise it is ______ . When rotated 180° it is ______ .

34 From formula (6) verify that $\mathbf{B}\cdot(\mathbf{A}\times\mathbf{B})=0$.

35 Compute

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 6 \end{vmatrix}, \quad \begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix}, \quad \begin{vmatrix} 1 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 1 \end{vmatrix}.$$

36 Which of the following are equal to $\mathbf{A}\times\mathbf{B}$? $(\mathbf{A}+\mathbf{B})\times\mathbf{B}$, $(-\mathbf{B})\times(-\mathbf{A})$, $|\mathbf{A}||\mathbf{B}||\sin\theta|$, $(\mathbf{A}+\mathbf{C})\times(\mathbf{B}-\mathbf{C})$, $\frac{1}{2}(\mathbf{A}-\mathbf{B})\times(\mathbf{A}+\mathbf{B})$.

37 Compare the six terms on both sides to prove that

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

The matrix is "*transposed*"—same determinant.

38 Compare the six terms to prove that

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = -b_1\begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} + b_2\begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} - b_3\begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix}.$$

This is an "expansion on row 2." Note minus signs.

39 Choose the signs and 2 by 2 determinants in

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \pm c_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \pm c_2 \_\_\_\_\_\_ \pm c_3 \_\_\_\_\_\_ .$$

40 Show that $(\mathbf{A}\times\mathbf{B})+(\mathbf{B}\times\mathbf{C})+(\mathbf{C}\times\mathbf{A})$ is perpendicular to $\mathbf{B}-\mathbf{A}$ and $\mathbf{C}-\mathbf{B}$ and $\mathbf{A}-\mathbf{C}$.

Problems 41–44 compute the areas of triangles.

41 The triangle *PQR* in Example 7 has squared area $(\sqrt{3}/2)^2 = (\frac{1}{2})^2+(\frac{1}{2})^2+(\frac{1}{2})^2$, from the 3D version of Pythagoras in Remark 2. Find the area of *PQR* when $P=(a,0,0), Q=(0,b,0)$, and $R=(0,0,c)$. Check with $\frac{1}{2}|\mathbf{A}\times\mathbf{B}|$.

42 A triangle in the xy plane has corners at $(a_1,b_1),(a_2,b_2)$ and (a_3,b_3). Its area A is half the area of a parallelogram. Find two sides of the parallelogram and explain why

$$A=\tfrac{1}{2}|(a_2-a_1)(b_3-b_1)-(a_3-a_1)(b_2-b_1)|.$$

43 By Problem 42 find the area A of the triangle with corners $(2,1)$ and $(4,2)$ and $(1,2)$. Where is a fourth corner to make a parallelogram?

44 Lifting the triangle of Problem 42 up to the plane $z=1$ gives corners $(a_1,b_1,1),(a_2,b_2,1),(a_3,b_3,1)$. The area of the triangle times $\frac{1}{3}$ is the volume of the upside-down pyramid from $(0,0,0)$ to these corners. This pyramid volume is $\frac{1}{6}$ the box volume, so $\frac{1}{3}$ (area of triangle) $=\frac{1}{6}$(volume of box):

$$\textbf{\textit{area of triangle}} = \frac{1}{2}\begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix}.$$

Find the area A in Problem 43 from this determinant.

45 (1) The projections of $\mathbf{A}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$ and $\mathbf{B}=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$ onto the xy plane are ______.

(2) The parallelogram with sides **A** and **B** projects to a parallelogram with area ______.

(3) General fact: The projection onto the plane normal to the unit vector **n** has area $(\mathbf{A}\times\mathbf{B})\cdot\mathbf{n}$. Verify for $\mathbf{n}=\mathbf{k}$.

46 (a) For $\mathbf{A}=\mathbf{i}+\mathbf{j}-4\mathbf{k}$ and $\mathbf{B}=-\mathbf{i}+\mathbf{j}$, compute $(\mathbf{A}\times\mathbf{B})\cdot\mathbf{i}$ and $(\mathbf{A}\times\mathbf{B})\cdot\mathbf{j}$ and $(\mathbf{A}\times\mathbf{B})\cdot\mathbf{k}$. By Problem 45 those are the areas of projections onto the yz and xz and xy planes.

(b) Square and add those areas to find $|\mathbf{A}\times\mathbf{B}|^2$. This is the Pythagoras formula in space (Remark 2).

47 (a) The triple cross product $(\mathbf{A}\times\mathbf{B})\times\mathbf{C}$ is in the plane of **A** and **B**, because it is perpendicular to the cross product ______.

(b) Compute $(\mathbf{A}\times\mathbf{B})\times\mathbf{C}$ when $\mathbf{A}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$, $\mathbf{B}=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$, $\mathbf{C}=\mathbf{i}$.

(c) Compute $(\mathbf{A}\cdot\mathbf{C})\mathbf{B}-(\mathbf{B}\cdot\mathbf{C})\mathbf{A}$ when $\mathbf{C}=\mathbf{i}$. The answers in (b) and (c) should agree. This is also true if $\mathbf{C}=\mathbf{j}$ or $\mathbf{C}=\mathbf{k}$ or $\mathbf{C}=c_1\mathbf{i}+c_2\mathbf{j}+c_3\mathbf{k}$. That proves the tricky formula

$$(\mathbf{A}\times\mathbf{B})\times\mathbf{C}=(\mathbf{A}\cdot\mathbf{C})\mathbf{B}-(\mathbf{B}\cdot\mathbf{C})\mathbf{A}. \qquad (*)$$

48 Take the dot product of equation $(*)$ with **D** to prove

$$(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D})=(\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D})-(\mathbf{B}\cdot\mathbf{C})(\mathbf{A}\cdot\mathbf{D}).$$

49 The plane containing $P=(0,1,1)$ and $Q=(1,0,1)$ and $R=(1,1,0)$ is perpendicular to the cross product $\mathbf{N}=$ ______. Find the equation of the plane and the area of triangle *PQR*.

50 Let $P=(1,0,-1), Q=(1,1,1), R=(2,2,1)$. Choose S so that *PQRS* is a parallelogram and compute its area. Choose T,U,V so that *OPQRSTUV* is a box (parallelepiped) and compute its volume.

11.4 Matrices and Linear Equations

We are moving from geometry to algebra. Eventually we get back to calculus, where functions are nonlinear—but linear equations come first. In Chapter 1, $y = mx + b$ produced a line. Two equations produce two lines. If they cross, the intersection point solves both equations—and we want to find it.

Three equations in three variables x, y, z produce three planes. Again they go through one point (*usually*). Again the problem is to find that intersection point —which solves the three equations.

The ultimate problem is to solve n equations in n unknowns. There are n hyperplanes in n-dimensional space, which meet at the solution. We need a test to be sure they meet. We also want the solution. These are the objectives of ***linear algebra***, which joins with calculus at the center of pure and applied mathematics.†

Like every subject, linear algebra requires a good notation. To state the equations and solve them, we introduce a "matrix." ***The problem will be*** $A\mathbf{u} = \mathbf{d}$. ***The solution will be*** $\mathbf{u} = A^{-1}\mathbf{d}$. It remains to understand where the equations come from, where the answer comes from, and what the matrices A and A^{-1} stand for.

TWO EQUATIONS IN TWO UNKNOWNS

Linear algebra has no reason to choose one variable as special. The equation $y - y_0 = m(x - x_0)$ separates y from x. A better equation for a line is $ax + by = d$. (A vertical line like $x = 5$ appears when $b = 0$. The first form did not allow slope $m = \infty$.) This section studies two lines:

$$\begin{aligned} a_1 x + b_1 y &= d_1 \\ a_2 x + b_2 y &= d_2. \end{aligned} \tag{1}$$

By solving both equations at once, we are asking (x, y) to lie on both lines. The practical question is: Where do the lines cross? The mathematician's question is: Does a solution exist and is it unique?

To understand everything is not possible. There are parts of life where you never know what is going on (until too late). But two equations in two unknowns can have no mysteries. There are three ways to write the system—by ***rows***, by ***columns***, and by ***matrices***. Please look at all three, since setting up a problem is generally harder and more important than solving it. After that comes the concession to the real world: we compute x and y.

EXAMPLE 1 How do you invest \$5000 to earn \$400 a year interest, if a money market account pays 5% and a deposit account pays 10%?

Set up equations by rows: With x dollars at 5% the interest is $.05x$. With y dollars at 10% the interest is $.10y$. One row for principal, another row for interest:

$$\begin{aligned} x + \quad y &= 5000 \\ .05x + .10y &= \;\; 400. \end{aligned} \tag{2}$$

†Linear algebra dominates some applications while calculus governs others. Both are essential. A fuller treatment is presented in the author's book *Linear Algebra and Its Applications* (Harcourt Brace Jovanovich, 3rd edition 1988), and in many other texts.

Same equations by columns: The left side of (2) contains x times one vector plus y times another vector. The right side is a third vector. The equation by columns is

$$x\begin{bmatrix}1\\.05\end{bmatrix}+y\begin{bmatrix}1\\.10\end{bmatrix}=\begin{bmatrix}5000\\400\end{bmatrix}. \tag{3}$$

Same equations by matrices: Look again at the left side. There are two unknowns x and y, which go into a vector **u**. They are multiplied by the four numbers 1, .05, 1, and .10, which go into a *two by two matrix* A. The left side becomes ***a matrix times a vector***:

$$A\mathbf{u}=\begin{bmatrix}1&1\\.05&.10\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}5000\\400\end{bmatrix}. \tag{4}$$

Now you see where the "rows" and "columns" came from. They are the rows and columns of a matrix. The rows entered the separate equations (2). The columns entered the vector equation (3). The matrix-vector multiplication $A\mathbf{u}$ is defined so that all these equations are the same:

$A\mathbf{u}$ by rows: $$\begin{bmatrix}a_1&b_1\\a_2&b_2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}a_1x+b_1y\\a_2x+b_2y\end{bmatrix}$$ (each row is a dot product)

$A\mathbf{u}$ by columns: $$\begin{bmatrix}a_1&b_1\\a_2&b_2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=x\begin{bmatrix}a_1\\a_2\end{bmatrix}+y\begin{bmatrix}b_1\\b_2\end{bmatrix}$$ (combination of column vectors)

A is the ***coefficient matrix***. The unknown vector is **u**. The known vector on the right side, with components 5000 and 400, is **d**. The matrix equation is $A\mathbf{u}=\mathbf{d}$.

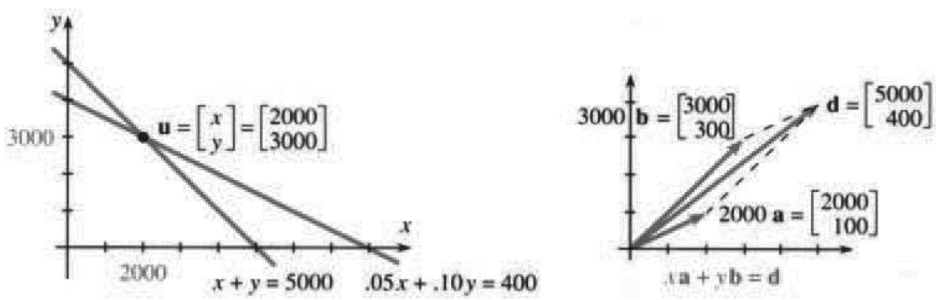


Fig. 11.16 Each row of $A\mathbf{u}=\mathbf{d}$ gives a line. Each column gives a vector.

This notation $A\mathbf{u}=\mathbf{d}$ continues to apply when there are more equations and more unknowns. The matrix A has a ***row for each equation*** (usually m rows). It has a ***column for each unknown*** (usually n columns). For 2 equations in 3 unknowns it is a 2 by 3 matrix (therefore rectangular). For 6 equations in 6 unknowns the matrix is 6 by 6 (therefore square). The best way to get familiar with matrices is to work with them. Note also the pronunciation: "matri*sees*" and never "matrixes."

Answer to the practical question The solution is $x=2000$, $y=3000$. That is the intersection point in the row picture (Figure 11.16). It is also the correct combination in the column picture. The matrix equation checks both at once, because matrices are multiplied by rows *or* by columns. The product either way is **d**:

$$\begin{bmatrix}1&1\\.05&.10\end{bmatrix}\begin{bmatrix}2000\\3000\end{bmatrix}=\begin{bmatrix}2000+\quad 3000\\(.05)2000+(.10)3000\end{bmatrix}=\begin{bmatrix}5000\\400\end{bmatrix}=\mathbf{d}.$$

Singular case In the row picture, the lines cross at the solution. But there is a case that gives trouble. ***When the lines are parallel***, they never cross and there is *no* solution. When the lines are the same, there is an *infinity* of solutions:

$$\textit{parallel lines}\quad \begin{aligned}2x+y&=0\\2x+y&=1\end{aligned}\qquad \textit{same line}\quad \begin{aligned}2x+\ \ y&=0\\4x+2y&=0\end{aligned}\tag{5}$$

This trouble also appears in the column picture. The columns are vectors **a** and **b**. The equation $A\mathbf{u}=\mathbf{d}$ is the same as $x\mathbf{a}+y\mathbf{b}=\mathbf{d}$. We are asked to find the combination of **a** and **b** (with coefficients x and y) that produces **d**. In the singular case **a** and **b** lie along the same line (Figure 11.17). No combination can produce **d**, unless it happens to lie on this line.

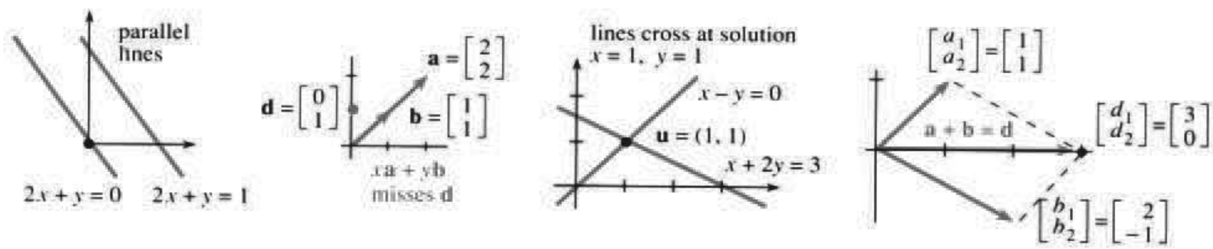


Fig. 11.17 Row and column pictures: *singular* (no solution) and *nonsingular* $(x=y=1)$.

The investment problem is *non*singular, and $2000\,\mathbf{a}+3000\,\mathbf{b}$ equals **d**. We also drew

EXAMPLE 2: The matrix A multiplies $\mathbf{u}=(1,1)$ to solve $x+2y=3$ and $x-y=0$:

$$A\mathbf{u}=\begin{bmatrix}1&2\\1&-1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}1+2\\1-1\end{bmatrix}=\begin{bmatrix}3\\0\end{bmatrix}.\quad \text{By columns}\begin{bmatrix}1\\1\end{bmatrix}+\begin{bmatrix}2\\-1\end{bmatrix}=\begin{bmatrix}3\\0\end{bmatrix}.$$

The crossing point is $(1,1)$ in the row picture. The solution is $x=1, y=1$ in the column picture (Figure 11.17b). Then 1 times **a** plus 1 times **b** equals the right side **d**.

SOLUTION BY DETERMINANTS

Up to now we just wrote down the answer. The real problem is to find x and y when they are unknown. We solve two equations with letters not numbers:

$$\begin{aligned}a_1x+b_1y&=d_1\\a_2x+b_2y&=d_2.\end{aligned}$$

The key is to eliminate x. Multiply the first equation by a_2 and the second equation by a_1. Subtract the first from the second and the x's disappear:

$$(a_1b_2-a_2b_1)y=(a_1d_2-a_2d_1).\tag{6}$$

To eliminate y, subtract b_1, times the second equation from b_2 times the first:

$$(b_2a_1-b_1a_2)x=(b_2d_1-b_1d_2).\tag{7}$$

What you see in those parentheses are 2 by 2 determinants! Remember from Section 11.3:

The determinant of $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ ***is the number*** $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1.$

This number appears on the left side of (6) and (7). The right side of (7) is also a determinant—but it has d's in place of a's. The right side of (6) has d's in place of b's. So x and y are ***ratios of determinants***, given by Cramer's Rule:

11H ***Cramer's Rule*** The solution is $x = \dfrac{\begin{vmatrix} d_1 & b_1 \\ d_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \quad y = \dfrac{\begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}.$

The investment example is solved by three determinants from the three columns:

$$\begin{vmatrix} 1 & 1 \\ .05 & .10 \end{vmatrix} = .05 \qquad \begin{vmatrix} 5000 & 1 \\ 400 & .10 \end{vmatrix} = 100 \qquad \begin{vmatrix} 1 & 5000 \\ .05 & 400 \end{vmatrix} = 150.$$

Cramer's Rule has $x = 100/.05 = 2000$ and $y = 150/.05 = 3000$. This is the solution. The singular case is when *the determinant of A is zero*—and we can't divide by it.

11I Cramer's Rule breaks down when det $A = 0$—which is the singular case. Then the lines in the row picture are parallel, and one column is a multiple of the other column.

EXAMPLE 3 The lines $2x + y = 0$, $2x + y = 1$ are parallel. The determinant is zero:

$$\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ has det } A = \begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} = 0.$$

The lines in Figure 11.17a don't meet. Notice the columns: $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ is a multiple of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

One final comment on 2 by 2 systems. They are small enough so that all solution methods apply. Cramer's Rule uses ***determinants***. Larger systems use ***elimination*** (3 by 3 matrices are on the borderline). A third solution (the same solution!) comes from the ***inverse matrix*** A^{-1}, to be described next. But the inverse is more a symbol for the answer than a new way of computing it, because to find A^{-1} we still use determinants or elimination.

THE INVERSE OF A MATRIX

The symbol A^{-1} is pronounced "*A inverse*." It stands for a matrix—the one that solves $A\mathbf{u} = \mathbf{d}$. I think of A as a matrix that takes $\mathbf{u}$ to $\mathbf{d}$. Then A^{-1} is a matrix that takes $\mathbf{d}$ back to $\mathbf{u}$. If $A\mathbf{u} = \mathbf{d}$ then $\mathbf{u} = A^{-1}\mathbf{d}$ (provided the inverse exists). This is exactly like functions and inverse functions: $g(x) = y$ and $x = g^{-1}(y)$. Our goal is to find A^{-1} when we know A.

The first approach will be very direct. Cramer's Rule gave formulas for x and y, the components of $\mathbf{u}$. From that rule we can read off A^{-1}, *assuming that* $D = a_1b_2 - a_2b_1$ *is not zero*. D is det A and we divide by it:

$$\textbf{\textit{Cramer}}: \mathbf{u} = \frac{1}{D}\begin{bmatrix} b_2d_1 - b_1d_2 \\ -a_2d_1 + a_1d_2 \end{bmatrix} \quad \textbf{This is } A^{-1}\mathbf{d} = \frac{1}{D}\begin{bmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{bmatrix}\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \tag{8}$$

The matrix on the right (including $1/D$ in all four entries) is A^{-1}. Notice the sign pattern and the subscript pattern. The inverse exists if D is not zero—this is important. Then the solution comes from a matrix-vector multiplication, A^{-1} times $\mathbf{d}$. We repeat the rules for that multiplication:

DEFINITION A matrix M times a vector $\mathbf{v}$ equals a vector of dot products:

$$M\mathbf{v} = \begin{bmatrix} \text{row 1} \\ \text{row 2} \end{bmatrix}\begin{bmatrix} \mathbf{v} \end{bmatrix} = \begin{bmatrix} (\text{row 1}) \cdot \mathbf{v} \\ (\text{row 2}) \cdot \mathbf{v} \end{bmatrix}. \tag{9}$$

Equation (8) follows this rule with $M = A^{-1}$ and $\mathbf{v} = \mathbf{d}$. Look at Example 1:

$$A = \begin{bmatrix} 1 & 1 \\ .05 & .10 \end{bmatrix}, \quad \det A = .05, \quad A^{-1} = \frac{1}{.05}\begin{bmatrix} .10 & -1 \\ -.05 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -20 \\ -1 & 20 \end{bmatrix}.$$

There stands the inverse matrix. It multiplies $\mathbf{d}$ to give the solution $\mathbf{u}$:

$$A^{-1}\mathbf{d} = \begin{bmatrix} 2 & -20 \\ -1 & 20 \end{bmatrix}\begin{bmatrix} 5000 \\ 400 \end{bmatrix} = \begin{bmatrix} (2)(5000) - (20)(400) \\ (-1)(5000) + (20)(400) \end{bmatrix} = \begin{bmatrix} 2000 \\ 3000 \end{bmatrix}.$$

The formulas work perfectly, but you have to see a direct way to reach $A^{-1}\mathbf{d}$. ***Multiply both sides of*** $A\mathbf{u} = \mathbf{d}$ ***by*** A^{-1}. The multiplication "cancels" A on the left side, and leaves $\mathbf{u} = A^{-1}\mathbf{d}$. This approach comes next.

MATRIX MULTIPLICATION

To understand the power of matrices, we must multiply them. The product of A^{-1} with $A\mathbf{u}$ is a matrix times a vector. But that multiplication can be done another way. First A^{-1} multiplies A, a matrix times a matrix. The product $A^{-1}A$ is another matrix (a very special matrix). Then this new matrix multiplies $\mathbf{u}$.

The matrix-matrix rule comes directly from the matrix-vector rule. Effectively, a vector $\mathbf{v}$ is a matrix V with only one column. When there are more columns, M times V splits into separate matrix-vector multiplications, side by side:

DEFINITION A matrix M times a matrix V equals a matrix of dot products:

$$MV = \begin{bmatrix} \text{row 1} \\ \text{row 2} \end{bmatrix}\begin{bmatrix} \mathbf{v_1} & \mathbf{v_2} \end{bmatrix} = \begin{bmatrix} (\text{row 1}) \cdot \mathbf{v_1} & (\text{row 1}) \cdot \mathbf{v_2} \\ (\text{row 2}) \cdot \mathbf{v_1} & (\text{row 2}) \cdot \mathbf{v_2} \end{bmatrix}. \tag{10}$$

EXAMPLE 4 $$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1\cdot5+2\cdot7 & 1\cdot6+2\cdot8 \\ 3\cdot5+4\cdot7 & 3\cdot6+4\cdot8 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}.$$

EXAMPLE 5 Multiplying A^{-1} times A produces the "***identity matrix***" $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$:

$$A^{-1}A = \frac{\begin{bmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{bmatrix}}{D} \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} = \frac{\begin{bmatrix} a_1b_2 - a_2b_1 & 0 \\ 0 & -a_2b_1 + a_1b_2 \end{bmatrix}}{D} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{11}$$

This identity matrix is denoted by I. It has 1's on the diagonal and 0's off the diagonal. It acts like the number 1. ***Every vector satisfies*** $I\mathbf{u} = \mathbf{u}$.

11J (***Inverse matrix and identity matrix***) $AA^{-1} = I$ and $A^{-1}A = I$ and $I\mathbf{u} = \mathbf{u}$:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad A^{-1} = \frac{1}{D}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}. \tag{12}$$

Note the placement of a, b, c, d. With these letters D is $ad - bc$.

The next section moves to three equations. The algebra gets more complicated (and 4 by 4 is worse). It is not easy to write out A^{-1}. So we stay longer with the 2 by 2 formulas, where each step can be checked. Multiplying $A\mathbf{u} = \mathbf{d}$ by the inverse matrix gives $A^{-1}A\mathbf{u} = A^{-1}\mathbf{d}$—and the left side is $I\mathbf{u} = \mathbf{u}$.

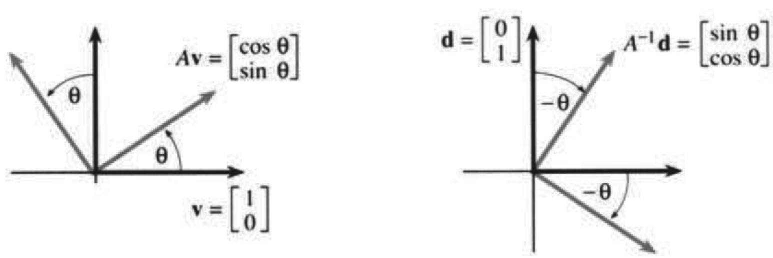


Fig. 11.18 Rotate $\mathbf{v}$ forward into $A\mathbf{v}$. Rotate $\mathbf{d}$ backward into $A^{-1}\mathbf{d}$.

EXAMPLE 6 $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ rotates every $\mathbf{v}$ to $A\mathbf{v}$, through the angle θ.

***Question* 1** Where is the vector $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ rotated to ?

***Question* 2** What is A^{-1} ?

***Question* 3** Which vector $\mathbf{u}$ is rotated into $\mathbf{d} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$?

***Solution* 1** $\mathbf{v}$ rotates into $A\mathbf{v} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$.

***Solution* 2** $\det A = 1$ so $A^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} =$ rotation through $-\theta$.

***Solution* 3** If $A\mathbf{u} = \mathbf{d}$ then $\mathbf{u} = A^{-1}\mathbf{d} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \sin\theta \\ \cos\theta \end{bmatrix}$.

Historical note I was amazed to learn that it was Leibniz (again!) who proposed the notation we use for matrices. ***The entry in row** i **and column** j **is** a_{ij}*. The identity matrix has $a_{11} = a_{22} = 1$ and $a_{12} = a_{21} = 0$. This is in a linear algebra book by Charles Dodgson—better known to the world as Lewis Carroll, the author of *Alice in Wonderland*. I regret to say that he preferred his own notation $i \rfloor j$ instead of a_{ij}. "I have turned the symbol toward the left, to avoid all chance of confusion with $\int$." It drove his typesetter mad.

PROJECTION ONTO A PLANE = LEAST SQUARES FITTING BY A LINE

We close with a genuine application. It starts with three-dimensional vectors $\mathbf{a}, \mathbf{b}, \mathbf{d}$ and leads to a 2 by 2 system. One good feature: $\mathbf{a}, \mathbf{b}, \mathbf{d}$ can be n-dimensional with no change in the algebra. In practice that happens. Second good feature: There is a calculus problem in the background. The example is ***to fit points by a straight line***.

There are three ways to state the problem, and they look different:

1. Solve $x\mathbf{a} + y\mathbf{b} = \mathbf{d}$ as well as possible (three equations, two unknowns x and y).
2. Project the vector $\mathbf{d}$ onto the plane of the vectors $\mathbf{a}$ and $\mathbf{b}$.
3. Find the closest straight line ("***least squares***") to three given points.

Figure 11.19 shows a three-dimensional vector $\mathbf{d}$ above the plane of $\mathbf{a}$ and $\mathbf{b}$. Its projection onto the plane is $\mathbf{p} = x\mathbf{a} + y\mathbf{b}$. The numbers x and y are unknown, and our goal is to find them. The calculation will use the dot product, which is always the key to right angles.

The difference $\mathbf{d} - \mathbf{p}$ is the "*error*." There has to be an error, because no combination of $\mathbf{a}$ and $\mathbf{b}$ can produce $\mathbf{d}$ exactly. (Otherwise $\mathbf{d}$ is in the plane.) The projection $\mathbf{p}$ is the closest point to $\mathbf{d}$, and it is governed by one fundamental law: ***The error is perpendicular to the plane***. That makes the error perpendicular to both vectors $\mathbf{a}$ and $\mathbf{b}$:

$$\mathbf{a} \cdot (x\mathbf{a} + y\mathbf{b} - \mathbf{d}) = 0 \quad \text{and} \quad \mathbf{b} \cdot (x\mathbf{a} + y\mathbf{b} - \mathbf{d}) = 0. \tag{13}$$

Rewrite those as two equations for the two unknown numbers x and y:

$$\begin{aligned} (\mathbf{a}\cdot\mathbf{a})x + (\mathbf{a}\cdot\mathbf{b})y &= \mathbf{a}\cdot\mathbf{d} \\ (\mathbf{b}\cdot\mathbf{a})x + (\mathbf{b}\cdot\mathbf{b})y &= \mathbf{b}\cdot\mathbf{d}. \end{aligned} \tag{14}$$

These are the famous ***normal equations*** in statistics, to compute x and y and $\mathbf{p}$.

EXAMPLE 7 For $\mathbf{a} = (1,1,1)$ and $\mathbf{b} = (1,2,3)$ and $\mathbf{d} = (0,5,4)$, solve equation (14):

$$\begin{aligned} 3x + 6y &= 9 \\ 6x + 14y &= 22 \end{aligned} \quad \text{gives} \quad \begin{aligned} x &= -1 \\ y &= 2 \end{aligned} \quad \text{so} \quad \mathbf{p} = -\mathbf{a} + 2\mathbf{b} = (1,3,5) = \textit{projection}.$$

Notice the three equations that we are not solving (we can't): $x\mathbf{a} + y\mathbf{b} = \mathbf{d}$ is

$$\begin{aligned} x + y &= 0 \\ x + 2y &= 5 \\ x + 3y &= 4 \end{aligned} \quad \text{with the 3 by 2 matrix } A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}. \tag{15}$$

For $\mathbf{d} = (0, 5, 4)$ there is no solution; $\mathbf{d}$ is not in the plane of $\mathbf{a}$ and $\mathbf{b}$. For $\mathbf{p} = (1, 3, 5)$ there is a solution, $x = -1$ and $y = 2$. The vector $\mathbf{p}$ is in the plane. The error $\mathbf{d} - \mathbf{p}$ is $(-1, 2, -1)$. This error is perpendicular to the columns $(1, 1, 1)$ and $(1, 2, 3)$, so it is perpendicular to their plane.

SAME EXAMPLE (*written as a line-fitting problem*) Fit the points $(1, 0)$ and $(2, 5)$ and $(3, 4)$ as closely as possible ("least squares") by a straight line.

Two points determine a line. The example asks the line $f = x + yt$ to go through *three* points. That gives the three equations in (15), which can't be solved with two unknowns. We have to settle for the closest line, drawn in Figure 11.19b. This line is computed again below, by calculus.

Notice that the closest line has heights $1, 3, 5$ where the data points have heights $0, 5, 4$. Those are the numbers in $\mathbf{p}$ and $\mathbf{d}$! The heights $1, 3, 5$ fit onto a line; the heights $0, 5, 4$ do not. In the first figure, $\mathbf{p} = (1, 3, 5)$ is in the plane and $\mathbf{d} = (0, 5, 4)$ is not. Vectors in the plane lead to heights that lie on a line.

Notice another coincidence. The coefficients $x = -1$ and $y = 2$ give the projection $-\mathbf{a} + 2\mathbf{b}$. They also give the closest line $f = -1 + 2t$. All numbers appear in both figures.

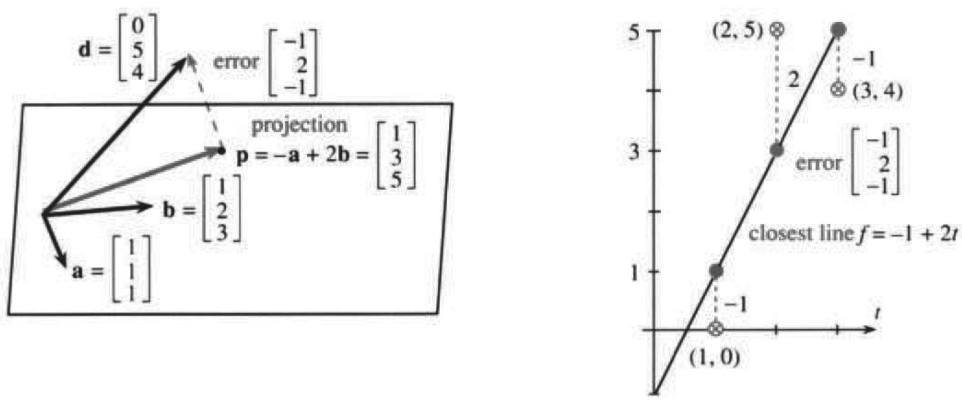


Fig. 11.19 Projection onto plane is $(1, 3, 5)$ with coefficients $-1, 2$. Closest line has heights $1, 3, 5$ with coefficients $-1, 2$. Error in both pictures is $-1, 2, -1$.

Remark Finding the closest line is a *calculus problem*: ***Minimize a sum of squares***. The numbers x and y that minimize E give the least squares solution:

$$E(x, y) = (x + y - 0)^2 + (x + 2y - 5)^2 + (x + 3y - 4)^2. \qquad (16)$$

Those are the three errors in equation (15), squared and added. They are also the three errors in the straight line fit, between the line and the data points. The projection minimizes the error (by geometry), the normal equations (14) minimize the error (by algebra), and now calculus minimizes the error by setting the derivatives of E to zero.

The new feature is this: E depends on two variables x and y. *Therefore E has two derivatives*. They both have to be zero at the minimum. That gives two equations for x and y:

$$x \text{ derivative of } E \text{ is zero: } 2(x + y) + 2(x + 2y - 5) \quad + 2(x + 3y - 4) \quad = 0$$
$$y \text{ derivative of } E \text{ is zero: } 2(x + y) + 2(x + 2y - 5)(2) + 2(x + 3y - 4)(3) = 0.$$

When we divide by 2, those are the normal equations $3x + 6y = 9$ and $6x + 14y = 22$. The minimizing x and y from calculus are the same numbers -1 and 2.

The x derivative treats y as a constant. The y derivative treats x as a constant. These are ***partial derivatives***. This calculus approach to least squares is in Chapter 13, as an important application of partial derivatives.

We now summarize the *least squares problem*—to find the closest line to n data points. In practice n may be 1000 instead of 3. The points have horizontal coordinates $b_1, b_2, \ldots, b_n$. The vertical coordinates are $d_1, d_2, \ldots, d_n$. These vectors **b** and **d**, together with $\mathbf{a} = (1, 1, \ldots, 1)$, determine a projection—the combination $\mathbf{p} = x\mathbf{a} + y\mathbf{b}$
that is closest to **d**. This problem is the same in n dimensions—the error $\mathbf{d} - \mathbf{p}$ is perpendicular to **a** and **b**. That is still tested by dot products, $\mathbf{p} \cdot \mathbf{a} = \mathbf{d} \cdot \mathbf{a}$ and $\mathbf{p} \cdot \mathbf{b} = \mathbf{d} \cdot \mathbf{b}$, which give the normal equations for x and y:

$$\begin{matrix} (\mathbf{a}\cdot\mathbf{a})x + (\mathbf{a}\cdot\mathbf{b})y = \mathbf{a}\cdot\mathbf{d} \\ (\mathbf{b}\cdot\mathbf{a})x + (\mathbf{b}\cdot\mathbf{b})y = \mathbf{b}\cdot\mathbf{d} \end{matrix} \quad \text{or} \quad \begin{matrix} (n)\ \ x + (\Sigma\, b_i)y = \Sigma\, d_i \\ (\Sigma\, b_i)x + (\Sigma\, b_i^2)y = \Sigma\, b_i d_i\,. \end{matrix} \tag{17}$$

11K The least squares problem projects **d** onto the plane of **a** and **b**. The projection is $\mathbf{p} = x\mathbf{a} + y\mathbf{b}$, in n dimensions. The closest line $f = x + yt$, in two dimensions. The normal equations (17) give the best x and y.

11.4 EXERCISES

Read-through questions

The equations $3x + y = 8$ and $x + y = 6$ combine into the vector equation x__a__$+ y$__b__ $=$ __c__ $= \mathbf{d}$. The left side is $A\mathbf{u}$, with coefficient matrix $A =$ __d__ and unknown vector $\mathbf{u} =$ __e__. The determinant of A is __f__, so this problem is not __g__. The row picture shows two intersecting __h__. The column picture shows $x\mathbf{a} + y\mathbf{b} = \mathbf{d}$, where $\mathbf{a} =$ __i__ and $\mathbf{b} =$ __j__. The inverse matrix is $A^{-1} =$ __k__ . The solution is $\mathbf{u} = A^{-1}\mathbf{d} =$ __l__.

A matrix-vector multiplication produces a vector of dot __m__ from the rows, and also a combination of the __n__:

$$\begin{bmatrix} \mathbf{A} \\ \mathbf{B} \end{bmatrix}\begin{bmatrix} \mathbf{u} \end{bmatrix} = \begin{bmatrix} - \\ - \end{bmatrix}, \quad \begin{bmatrix} \mathbf{a} & \mathbf{b} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \\ - \end{bmatrix}, \quad \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} - \\ - \end{bmatrix}.$$

If the entries are a, b, c, d, the determinant is $D =$ __o__. A^{-1} is [__p__] divided by D. Cramer's Rule shows components of $\mathbf{u} = A^{-1}\mathbf{d}$ as ratios of determinants: $x =$ __q__$/D$ and $y =$ __r__$/D$.

A matrix-matrix multiplication MV yields a matrix of dot products, from the rows of __s__ and the columns of __t__:

$$\begin{bmatrix} \mathbf{A} \\ \mathbf{B} \end{bmatrix}\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} - & - \\ - & - \end{bmatrix} \qquad \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} - & - \\ - & - \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 3/2 \end{bmatrix} = \begin{bmatrix} - & - \\ - & - \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} - & - \\ - & - \end{bmatrix}.$$

The last line contains the __u__ matrix, denoted by I. It has the property that $IA = AI =$ __v__ for every matrix A, and $I\mathbf{u} =$ __w__ for every vector **u**. The inverse matrix satisfies $A^{-1}A =$ __x__. Then $A\mathbf{u} = \mathbf{d}$ is solved by multiplying both sides by __y__, to give $\mathbf{u} =$ __z__. There is no inverse matrix when __A__.

The combination $x\mathbf{a} + y\mathbf{b}$ is the projection of **d** when the error __B__ is perpendicular to __C__ and __D__. If $\mathbf{a} = (1, 1, 1)$, $\mathbf{b} = (1, 2, 3)$, and $\mathbf{d} = (0, 8, 4)$, the equations for x and y are __E__. Solving them also gives the closest __F__ to the data points $(1, 0)$, __G__, and $(3, 4)$. The solution is $x = 0, y = 2$, which means the best line is __H__. The projection is $0\mathbf{a} + 2\mathbf{b} =$ __I__. The three error components are __J__ . Check perpendicularity: __K__ $= 0$ and __L__ $= 0$. Applying calculus to this problem, x and y minimize the sum of squares $E =$ __M__.

In 1–8 find the point (x, y) where the two lines intersect (if they do). Also show how the right side is a combination of the columns on the left side (if it is). Also find the determinant D.

1 $x + y = 7$
$x - y = 3$

2 $2x + y = 11$
$x + y = 6$

3 $3x - y = 8$, $x - 3y = 0$

4 $x + 2y = 3$, $2x + 4y = 7$

5 $2x - 4y = 0$, $x - 2y = 0$

6 $10x + y = 1$, $x + y = 1$

7 $ax + by = 0$, $2ax + 2by = 2$

8 $ax + by = 1$, $cx + dy = 1$

9 Solve Problem 3 by Cramer's Rule.

10 Try to solve Problem 4 by Cramer's Rule.

11 What are the ratios for Cramer's Rule in Problem 5?

12 If $A = I$ show how Cramer's Rule solves $A\mathbf{u} = \mathbf{d}$.

13 Draw the row picture and column picture for Problem 1.

14 Draw the row and column pictures for Problem 6.

15 Find A^{-1} in Problem 1.

16 Find A^{-1} in Problem 8 if $ad - bc = 1$.

17 A 2 by 2 system is *singular* when the two lines in the row picture ______. This system is still solvable if one equation is a ______ of the other equation. In that case the two lines are ______ and the number of solution is ______.

18 Try Cramer's Rule when there is no solution or infinitely many:

$$\begin{aligned} 3x + y &= 0 \\ 6x + 2y &= 2 \end{aligned} \quad \text{or} \quad \begin{aligned} 3x + y &= 1 \\ 6x + 2y &= 2. \end{aligned}$$

19 $A\mathbf{u} = \mathbf{d}$ is singular when the columns of A are ______. A solution exists if the right side $\mathbf{d}$ is ______. In this solvable case the number of solutions is ______.

20 The equations $x - y = d_1$ and $9x - 9y = d_2$ can be solved if ______.

21 Suppose $x = \frac{1}{4}$ billion people live in the U.S. and $y = 5$ billion live outside. If 4 per cent of those inside move out and 2 per cent of those outside move in, find the populations d_1 inside and d_2 outside after the move. Express this as a matrix multiplication $A\mathbf{u} = \mathbf{d}$ (and find the matrix).

22 In Problem 21 what is special about $a_1 + a_2$ and $b_1 + b_2$ (the sums down the columns of A)? Explain why $d_1 + d_2$ equals $x + y$.

23 With the same percentages moving, suppose $d_1 = 0.58$ billion are inside and $d_2 = 4.92$ billion are outside *at the end*. Set up and solve two equations for the original populations x and y.

24 What is the determinant of A in Problems 21–23? What is A^{-1}? Check that $A^{-1}A = I$.

25 The equations $ax + y = 0$, $x + ay = 0$ have the solution $x = y = 0$. For which two values of a are there other solutions (and what are the other solutions)?

26 The equations $ax + by = 0$, $cx + dy = 0$ have the solution $x = y = 0$. There are other solutions if the two lines are ______. This happens if a, b, c, d satisfy ______.

27 Find the determinant and inverse of $A = \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix}$. Do the same for $2A, A^{-1}, -A$, and I.

28 Show that the determinant of A^{-1} is $1/\det A$:

$$A^{-1} = \begin{bmatrix} d/(ad-bc) & -b/(ad-bc) \\ -c/(ad-bc) & a/(ad-bc) \end{bmatrix}$$

29 Compute AB and BA and also BC and CB:

$$A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}.$$

Verify the *associative law*: AB times C equals A times BC.

30 (a) Find the determinants of A, B, AB, and BA above.

(b) Propose a law for the determinant of BC and test it.

31 For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$ write out AB and factor its determinant into $(ad - bc)(eh - fg)$. Therefore $\det(AB) = (\det A)(\det B)$.

32 Usually $\det(A + B)$ does *not* equal $\det A + \det B$. Find examples of inequality and equality.

33 Find the inverses, and check $A^{-1}A = I$ and $BB^{-1} = I$, for

$$A = \begin{bmatrix} 1 & 4 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix}.$$

34 In Problem 33 compute AB and the inverse of AB. Check that this inverse equals B^{-1} times A^{-1}.

35 The matrix product $ABB^{-1}A^{-1}$ equals the ______ matrix. Therefore the inverse of AB is ______. *Important*: The associative law in Problem 29 allows you to multiply BB^{-1} first.

36 The matrix multiplication $C^{-1}B^{-1}A^{-1}ABC$ yields the ______ matrix. Therefore the inverse of ABC is ______.

37 The equations $x + 2y + 3z$ and $4x + 5y + cz = 0$ always have a nonzero solution. The vector $\mathbf{u} = (x, y, z)$ is required to be ______ to $\mathbf{v} = (1, 2, 3)$ and $\mathbf{w} = (4, 5, c)$. So choose $\mathbf{u} =$ ______.

38 Find the combination $\mathbf{p} = x\mathbf{a} + y\mathbf{b}$ of the vectors $\mathbf{a} = (1, 1, 1)$ and $\mathbf{b} = (-1, 0, 1)$ that comes closest to $\mathbf{d} = (2, 6, 4)$. (a) Solve the normal equations (14) for x and y. (b) Check that the error $\mathbf{d} - \mathbf{p}$ is perpendicular to $\mathbf{a}$ and $\mathbf{b}$.

39 Plot the three data points $(-1, 2), (0, 6), (1, 4)$ in a plane. Draw the straight line $x + yt$ with the same x and y as in Problem 38. Locate the three errors up or down from the data points and compare with Problem 38.

40 Solve equation (14) to find the combination $x\mathbf{a} + y\mathbf{b}$ of $\mathbf{a} = (1, 1, 1)$ and $\mathbf{b} = (-1, 1, 2)$ that is closest to $\mathbf{d} = (1, 1, 3)$. Draw the corresponding straight line for the data points $(-1, 1), (1, 1)$, and $(2, 3)$. What is the vector of three errors and what is it perpendicular to?

41 Under what condition on d_1, d_2, d_3 do the three points $(0, d_1), (1, d_2), (2, d_3)$ lie on a line?

42 Find the matrices that reverse x and y and project:

$$M\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix} \quad \text{and} \quad P\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}.$$

43 Multiplying by $P = \begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix}$ projects $\mathbf{u}$ onto the 45° line.

(a) Find the projection $P\mathbf{u}$ of $\mathbf{u} = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$.

(b) Why does P times P equal P?

(c) Does P^{-1} exist? What vectors give $P\mathbf{u} = \mathbf{0}$?

44 Suppose $\mathbf{u}$ is not the zero vector but $A\mathbf{u} = \mathbf{0}$. Then A^{-1} can't exist: It would multiply ______ and produce $\mathbf{u}$.

11.5 Linear Algebra

This section moves from two to three dimensions. There are three unknowns x, y, z and also three equations. This is at the crossover point between formulas and algorithms—it is real linear algebra. The formulas give a direct solution using determinants. The algorithms use elimination and the numbers x, y, z appear at the end. In practice that end result comes quickly. *Computers solve linear equations by elimination*.

The situation for a nonlinear equation is similar. Quadratic equations $ax^2+bx+c=0$ are solved by a formula. Cubic equations are solved by Newton's method (even though a formula exists). For equations involving x^5 or x^{10}, algorithms take over completely.

Since we are at the crossover point, we look both ways. This section has a lot to do, in mixing geometry, determinants, and 3 by 3 matrices:

1. The row picture: three planes intersect at the solution
2. The column picture: a vector equation combines the columns
3. The formulas: determinants and Cramer's Rule
4. Matrix multiplication and A^{-1}
5. The algorithm: Gaussian elimination.

Part of our goal is three-dimensional calculus. Another part is n-dimensional algebra. And a third possibility is that you may not take mathematics next year. If that happens, I hope you will *use* mathematics. Linear equations are so basic and important, in such a variety of applications, that the effort in this section is worth making.

An example is needed. It is convenient and realistic if the matrix contains zeros. Most equations in practice are fairly simple—a thousand equations each with 990 zeros would be very reasonable. Here are three equations in three unknowns:

$$\begin{aligned} x + y \qquad &= 1 \\ x \qquad + 2z &= 0 \\ -2y + 2z &= -4. \end{aligned} \tag{1}$$

In matrix-vector form, the unknown $\mathbf{u}$ has components x, y, z. The right sides $1, 0, -4$ go into $\mathbf{d}$. The nine coefficients, including three zeros, enter the matrix A:

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & -2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix} \quad \text{or} \quad A\mathbf{u} = \mathbf{d}. \tag{2}$$

The goal is to understand that system geometrically, and then solve it.

THE ROW PICTURE: INTERSECTING PLANES

Start with the first equation $x+y=1$. In the xy plane that produces a line. In three dimensions it is a *plane*. It has the usual form $ax+by+cz=d$, except that c happens to be zero. The plane is easy to visualize (Figure 11.20a), because it cuts straight down through the line. The equation $x+y=1$ allows z to have any value, so the graph includes all points above and below the line.

The second equation $x+2z=0$ gives a second plane, which goes through the origin. *When the right side is zero, the point* $(0,0,0)$ *satisfies the equation*. This time y is absent from the equation, so the plane contains the whole y axis. All points

$(0, y, 0)$ meet the requirement $x + 2z = 0$. *The normal vector to the plane is* $\mathbf{N} = \mathbf{i} + 2\mathbf{k}$. The plane cuts across, rather than down, in 11.20b.

Before the third equation we combine the first two. ***The intersection of two planes is a line***. In three-dimensional space, two equations (not one) describe a line. The points on the line have to satisfy $x + y = 1$ and also $x + 2z = 0$. A convenient point is $P = (0, 1, 0)$. Another point is $Q = (-1, 2, \frac{1}{2})$. The line through P and Q extends out in both directions.

The solution is on that line. The third plane decides where.

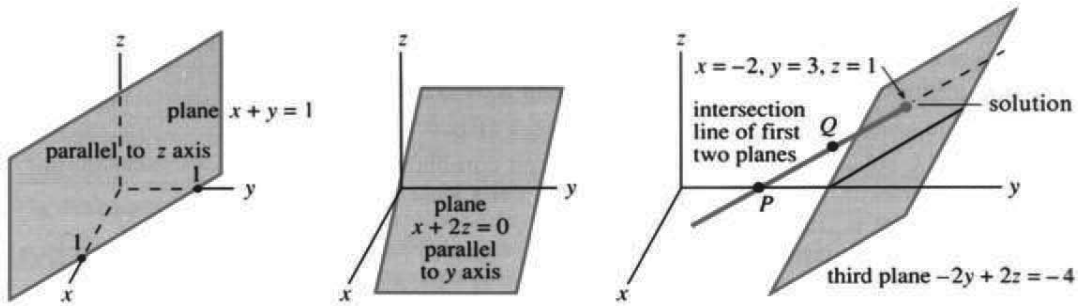


Fig. 11.20 First plane, second plane, interesection line meets third plane at solution.

The third equation $-2y + 2z = -4$ gives the third plane—which misses the origin because the right side is not zero. What is important is *the point where the three planes meet*. The intersection line of the first two planes crosses the third plane. We used determinants (but elimination is better) to find $x = -2,\ y = 3,\ z = 1$. This solution satisfies the three equations and lies on the three planes.

A brief comment on 4 by 4 systems. The first equation might be $x + y + z - t = 0$. It represents a three-dimensional "hyperplane" in four-dimensional space. (In physics this is space-time.) The second equation gives a second hyperplane, and its intersection with the first one is two-dimensional. The third equation (third hyperplane) reduces the intersection to a line. The fourth hyperplane meets that line at a point, which is the solution. It satisfies the four equations and lies on the four hyperplanes. In this course three dimensions are enough.

COLUMN PICTURE: COMBINATION OF COLUMN VECTORS

There is an extremely important way to rewrite our three equations. In (1) they were separate, in (2) they went into a matrix. Now they become a vector equation:

$$x \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + z \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix}. \qquad (3)$$

The columns of the matrix are multiplied by x, y, z. That is a special way to see matrix-vector multiplication: $A\mathbf{u}$ ***is a combination of the columns of*** A. We are looking for the numbers x, y, z so that the combination produces the right side $\mathbf{d}$.

The column vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are shown in Figure 11.21a. The vector equation is $x\mathbf{a} + y\mathbf{b} + z\mathbf{c} = \mathbf{d}$. The combination that solves this equation must again be $x = -2$, $y = 3, z = 1$. That agrees with the intersection point of the three planes in the row picture.

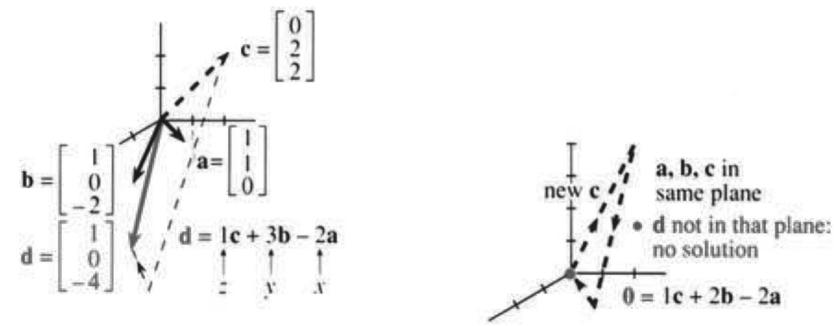


Fig. 11.21 Columns combine to give **d**. Columns combine to give **zero** (singular case).

THE DETERMINANT AND THE INVERSE MATRIX

For a 3 by 3 determinant, the section on cross products gave two formulas. One was the triple product $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$. The other wrote out the six terms:

$$\det A = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1).$$

Geometrically this is *the volume of a box*. The columns $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are the edges going out from the origin. In our example the determinant and volume are 2:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & -2 & 2 \end{vmatrix} = \begin{matrix} (1)(0)(2) - (1)(-2)(2) + (1)(-2)(0) \\ -(1)(1)(2) + (0)(1)(2) - (0)(0)(0) \end{matrix} = 2.$$

A slight dishonesty is present in that calculation, and will be admitted now. In Section 11.3 the vectors $\mathbf{A}, \mathbf{B}, \mathbf{C}$ were *rows*. In this section $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are *columns*. It doesn't matter, because the determinant is the same either way. Any matrix can be "transposed" —exchanging rows for columns—without altering the determinant. The six terms ($a_1b_2c_3$ is the first) may come in a different order, but they are the same six terms. Here four of those terms are zero, because of the zeros in the matrix. The sum of all six terms is $D = \det A = 2$.

Since D is not zero, the equations can be solved. The three planes meet at a point. The column vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ produce a genuine box, and are not flattened into the same plane (with zero volume). The solution involves ***dividing by*** D—which is only possible if $D = \det A$ is not zero.

> **11L** When the determinant D is not zero, A bas an inverse: $AA^{-1} = A^{-1}A = I$. Then the equations $A\mathbf{u} = \mathbf{d}$ have one and only one solution $\mathbf{u} = A^{-1}\mathbf{d}$.

The 3 by 3 identity matrix I is at the end of equation (5). Always $I\mathbf{u} = \mathbf{u}$.

We now compute A^{-1}, first with letters and then with numbers. The neatest formula uses cross products of the columns of A—it is special for 3 by 3 matrices.

$$\textit{Every entry is divided by } D\textit{: The inverse matrix is } A^{-1} = \frac{1}{D}\begin{bmatrix} \mathbf{b}\times\mathbf{c} \\ \mathbf{c}\times\mathbf{a} \\ \mathbf{a}\times\mathbf{b} \end{bmatrix}. \quad (4)$$

To test this formula, multiply by A. ***Matrix multiplication produces a matrix of dot products***—from the rows of the first matrix and the columns of the second, $A^{-1}A =$

I:

$$\frac{1}{D}\begin{bmatrix}\mathbf{b}\times\mathbf{c}\\ \mathbf{c}\times\mathbf{a}\\ \mathbf{a}\times\mathbf{b}\end{bmatrix}\begin{bmatrix}\mathbf{a} & \mathbf{b} & \mathbf{c}\end{bmatrix}=\frac{1}{D}\begin{bmatrix}\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) & \mathbf{b}\cdot(\mathbf{b}\times\mathbf{c}) & \mathbf{c}\cdot(\mathbf{b}\times\mathbf{c})\\ \mathbf{a}\cdot(\mathbf{c}\times\mathbf{a}) & \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) & \mathbf{c}\cdot(\mathbf{c}\times\mathbf{a})\\ \mathbf{a}\cdot(\mathbf{a}\times\mathbf{b}) & \mathbf{b}\cdot(\mathbf{a}\times\mathbf{b}) & \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}. \tag{5}$$

On the right side, six of the triple products are zero. They are the off-diagonals like $\mathbf{b}\cdot(\mathbf{b}\times\mathbf{c})$, which contain the same vector twice. Since $\mathbf{b}\times\mathbf{c}$ is perpendicular to $\mathbf{b}$, this triple product is zero. The same is true of the others, like $\mathbf{a}\cdot(\mathbf{a}\times\mathbf{b})=0$. That is the volume of a box with two identical sides. The six off-diagonal zeros are the volumes of completely flattened boxes.

On the main diagonal the triple products equal D. The order of vectors can be **abc** or **bca** or **cab**, and the volume of the box stays the same. Dividing by this number D, which is placed outside for that purpose, gives the 1's in the identity matrix I.

Now we change to numbers. The goal is to find A^{-1} and to test it.

EXAMPLE 1 *The inverse of* $A=\begin{bmatrix}1&1&0\\1&0&2\\0&-2&2\end{bmatrix}$ *is* $A^{-1}=\frac{1}{2}\begin{bmatrix}4&-2&2\\-2&2&-2\\-2&2&-1\end{bmatrix}$.

That comes from the formula, and it absolutely has to be checked. Do not fail to multiply A^{-1} times A (or A times A^{-1}). Matrix multiplication is much easier than the formula for A^{-1}. We highlight row 3 times column 1, with dot product zero:

$$\frac{1}{2}\begin{bmatrix}4&-2&2\\-2&2&-2\\-2&2&-1\end{bmatrix}\begin{bmatrix}1&1&0\\1&0&2\\0&-2&2\end{bmatrix}=\frac{1}{2}\begin{bmatrix}4-2&4-4&-4+4\\-2+2&-2+4&4-4\\-2+2&-2+2&4-2\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}.$$

Remark on A^{-1} Inverting a matrix requires $D\neq 0$. We divide by $D=\det A$. The cross products $\mathbf{b}\times\mathbf{c}$ and $\mathbf{c}\times\mathbf{a}$ and $\mathbf{a}\times\mathbf{b}$ give A^{-1} in a neat form, but errors are easy. We prefer to avoid writing $\mathbf{i},\mathbf{j},\mathbf{k}$. There are nine 2 by 2 determinants to be calculated, and here is A^{-1} in full—containing the nine "***cofactors***" divided by D:

$$A^{-1}=\frac{1}{D}\begin{bmatrix}b_2c_3-b_3c_2 & b_3c_1-b_1c_3 & b_1c_2-b_2c_1\\ c_2a_3-c_3a_2 & c_3a_1-c_1a_3 & c_1a_2-c_2a_1\\ a_2b_3-a_3b_2 & a_3b_1-a_1b_3 & a_1b_2-a_2b_1\end{bmatrix}. \tag{6}$$

Important: The first row of A^{-1} does not use the first column of A, except in $1/D$. In other words, $\mathbf{b}\times\mathbf{c}$ does not involve $\mathbf{a}$. Here are the 2 by 2 determinants that produce $4,-2,2$—which is divided by $D=2$ in the top row of A^{-1}:

$$\begin{bmatrix}1&1&0\\1&0&2\\0&-2&2\end{bmatrix}\begin{bmatrix}1&1&0\\1&0&2\\0&-2&2\end{bmatrix}\begin{bmatrix}1&1&0\\1&0&2\\0&-2&2\end{bmatrix}\qquad\begin{bmatrix}+&-&+\\-&+&-\\+&-&+\end{bmatrix}. \tag{7}$$

The second highlighted determinant looks like $+2$ not -2. But the ***sign matrix*** on the right assigns a minus to that position in A^{-1}. We reverse the sign of $b_1c_3-b_3c_1$, to find the cofactor $b_3c_1-b_1c_3$ in the top row of (6).

To repeat: ***For a row of A^{-1}, cross out the corresponding column of A. Find the three 2 by 2 determinants, use the sign matrix, and divide by D.***

EXAMPLE 2
$$B=\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix} \text{ has } D=1 \text{ and } B^{-1}=\begin{bmatrix}1&-1&0\\0&1&-1\\0&0&1\end{bmatrix}. \qquad (8)$$

The multiplication $BB^{-1}=I$ checks the arithmetic. Notice how $\begin{smallmatrix}1&1\\1&1\end{smallmatrix}$ in B leads to a zero in the top row of B^{-1}. To find row 1, column 3 of B^{-1} we ignore column 1 and row 3 of B. (Also: the inverse of a triangular matrix is triangular.) The minus signs come from the sign matrix.

THE SOLUTION $\mathbf{u}=A^{-1}\mathbf{d}$

The purpose of A^{-1} is to solve the equation $A\mathbf{u}=\mathbf{d}$. Multiplying by A^{-1} produces $I\mathbf{u}=A^{-1}\mathbf{d}$. The matrix becomes the identity, $I\mathbf{u}$ equals $\mathbf{u}$, and the solution is immediate:

$$\mathbf{u}=A^{-1}\mathbf{d}=\frac{1}{D}\begin{bmatrix}\mathbf{b}\times\mathbf{c}\\\mathbf{c}\times\mathbf{a}\\\mathbf{a}\times\mathbf{b}\end{bmatrix}\begin{bmatrix}\\\mathbf{d}\\\\\end{bmatrix}=\frac{1}{D}\begin{bmatrix}\mathbf{d}\cdot(\mathbf{b}\times\mathbf{c})\\\mathbf{d}\cdot(\mathbf{c}\times\mathbf{a})\\\mathbf{d}\cdot(\mathbf{a}\times\mathbf{b})\end{bmatrix}. \qquad (9)$$

By writing those components x, y, z as *ratios of determinants*, we have Cramer's Rule:

11M (*Cramer's Rule*)

The solution is $x=\dfrac{|\mathbf{d}\ \mathbf{b}\ \mathbf{c}|}{|\mathbf{a}\ \mathbf{b}\ \mathbf{c}|},\quad y=\dfrac{|\mathbf{a}\ \mathbf{d}\ \mathbf{c}|}{|\mathbf{a}\ \mathbf{b}\ \mathbf{c}|},\quad z=\dfrac{|\mathbf{a}\ \mathbf{b}\ \mathbf{d}|}{|\mathbf{a}\ \mathbf{b}\ \mathbf{c}|}.$ (10)

The right side $\mathbf{d}$ replaces, in turn, columns $\mathbf{a}$ and $\mathbf{b}$ and $\mathbf{c}$. All denominators are $D=\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$. The numerator of x is the determinant $\mathbf{d}\cdot(\mathbf{b}\times\mathbf{c})$ in (9). The second numerator agrees with the second component $\mathbf{d}\cdot(\mathbf{c}\times\mathbf{a})$, because the cyclic order is correct. The third determinant with columns **abd** equals the triple product $\mathbf{d}\cdot(\mathbf{a}\times\mathbf{b})$ in $A^{-1}\mathbf{u}$. Thus (10) is the same as (9).

EXAMPLE A: Multiply by A^{-1} to find the known solution $x=-2$, $y=3$, $z=1$:

$$\mathbf{u}=A^{-1}\mathbf{d}=\frac{1}{2}\begin{bmatrix}4&-2&2\\-2&2&-2\\-2&2&-1\end{bmatrix}\begin{bmatrix}1\\0\\-4\end{bmatrix}=\frac{1}{2}\begin{bmatrix}4-8\\-2+8\\-2+4\end{bmatrix}=\begin{bmatrix}-2\\3\\1\end{bmatrix}.$$

EXAMPLE B: Multiply by B^{-1} to solve $B\mathbf{u}=\mathbf{d}$ when $\mathbf{d}$ is the column $(6,5,4)$:

$$\mathbf{u}=B^{-1}\mathbf{d}=\begin{bmatrix}1&-1&0\\0&1&-1\\0&0&1\end{bmatrix}\begin{bmatrix}6\\5\\4\end{bmatrix}=\begin{bmatrix}1\\1\\4\end{bmatrix}. \quad \textit{Check } B\mathbf{u}=\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}\begin{bmatrix}1\\1\\4\end{bmatrix}=\begin{bmatrix}6\\5\\4\end{bmatrix}.$$

EXAMPLE C: Put $\mathbf{d}=(6,5,4)$ in each column of B. Cramer's Rule gives $\mathbf{u}=(1,1,4)$:

$$\begin{vmatrix}6&1&1\\5&1&1\\4&0&1\end{vmatrix}=1\quad\begin{vmatrix}1&6&1\\0&5&1\\0&4&1\end{vmatrix}=1\quad\begin{vmatrix}1&1&6\\0&1&5\\0&0&4\end{vmatrix}=4\quad\text{all divided by } D=\begin{vmatrix}1&1&1\\0&1&1\\0&0&1\end{vmatrix}=1.$$

This rule fills the page with determinants. Those are good ones to check by eye, without writing down the six terms (three $+$ and three $-$).

The formulas for A^{-1} are honored chiefly in their absence. They are not used by the computer, even though the algebra is in some ways beautiful. In big calculations, the computer never finds A^{-1}—just the solution.

We now look at the singular case $D=0$. Geometry-algebra-algorithm must all break down. After that is the algorithm: Gaussian elimination.

THE SINGULAR CASE

Changing one entry of a matrix can make the determinant zero. The triple product $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$, which is also the volume, becomes $D=0$. The box is flattened and the matrix is singular. That happens in our example when the lower right entry is changed from 2 to 4:

$$S=\begin{bmatrix}1&1&0\\1&0&2\\0&-2&4\end{bmatrix}\text{ has determinant } D=0.$$

This does more than change the inverse. It *destroys* the inverse. We can no longer divide by D. There is no S^{-1}.

What happens to the row picture and column picture? For 2 by 2 systems, the singular case had two parallel lines. Now the row picture has three planes, which need not be parallel. Here the planes are *not parallel*. Their normal vectors are the rows of S, which go in different directions. But somehow the planes fail to go through a common point.

What happens is more subtle. The intersection line from two planes misses the third plane. The line is parallel to the plane and stays above it (Figure 11.22)a. When all three planes are drawn, they form an open tunnel. The picture tells more than the numbers, about how three planes can fail to meet. The third figure shows an end view, where the planes go directly into the page. Each pair meets in a line, but those lines don't meet in a point.

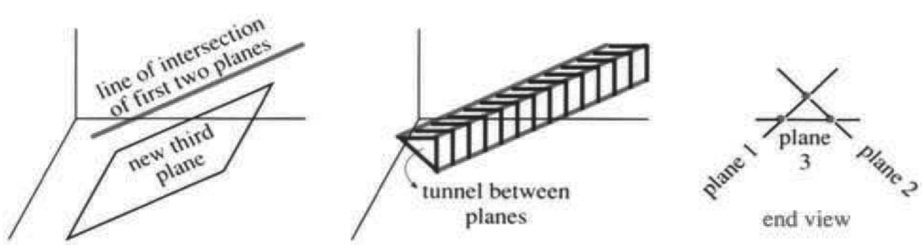


Fig. 11.22 The row picture in the singular case: no interestion point, no solutions.

When two planes are parallel, the determinant is again zero. One row of the matrix is a multiple of another row. The extreme case has all three planes parallel—as in a matrix with nine 1's.

The column picture must also break down. In the 2 by 2 failure (previous section), the columns were on the same line. *Now the three columns are in the same plane*. The combinations of those columns produce **d** only if it happens to lie in that particular plane. Most vectors **d** will be outside the plane, so most singular systems have no solution.

When the determinant is zero, $A\mathbf{u}=\mathbf{d}$ ***has no solution or infinitely many.***

THE ELIMINATION ALGORITHM

Go back to the 3 by 3 example $A\mathbf{u}=\mathbf{d}$. If you were given those equations, you would never think of determinants. You would—*quite correctly*—start with the first equation. It gives $x=1-y$, which goes into the next equation to eliminate x:

$$\begin{aligned} x+\ \ y \quad\ \ &= \ \ 1 \\ x \qquad +2z &= \ \ 0 \\ -2y+2z &= -4 \end{aligned} \quad \xrightarrow{x=1-y} \quad \begin{aligned} 1-\ \ y+2z &= \ \ 0 \\ -2y+2z &= -4. \end{aligned}$$

Stop there for a minute. On the right is a 2 by 2 system for y and z. The first equation and first unknown are eliminated—exactly what we want. But that step was not organized in the best way, because a "1" ended up on the left side. Constants should stay on the right side—the pattern should be preserved. It is better to take the same step by ***subtracting the first equation from the second***:

$$\begin{aligned} x+\ \ y \quad\ \ &= \ \ 1 \\ x \qquad +2z &= \ \ 0 \\ -2y+2z &= -4 \end{aligned} \quad \longrightarrow \quad \begin{aligned} -\ \ y+2z &= -1 \\ -2y+2z &= -4. \end{aligned} \tag{11}$$

Same equations, better organization. Now look at the corner term $-y$. Its coefficient -1 is the ***second pivot***. (The first pivot was $+1$, the coefficient of x in the first corner.) We are ready for the next elimination step:

Plan: Subtract a multiple of the "pivot equation" from the equation below it.
Goal: To produce a zero below the pivot, so y is eliminated.
Method: Subtract 2 times the pivot equation to cancel $-2y$.

$$\begin{aligned} -\ \ y+2z &= -1 \\ -2y+2z &= -4 \end{aligned} \quad \rightarrow \quad -2z=-2. \tag{12}$$

The answer comes by ***back substitution***. Equation (12) gives $z=1$. Then equation (11) gives $y=3$. Then the first equation gives $x=-2$. This is much quicker than determinants. You may ask: *Why use Cramer's Rule*? Good question.

With numbers elimination is better. It is faster and also safer. (To check against error, substitute $-2,3,1$ into the original equations.) The algorithm reaches the answer *without the determinant and without the inverse*. Calculations with letters use $\det A$ and A^{-1}.

Here are the steps in a definite order (top to bottom):

Subtract a multiple of equation 1 to produce $0x$ in equation 2
Subtract a multiple of equation 1 to produce $0x$ in equation 3
Subtract a multiple of equation 2 (new) to produce $0y$ in equation 3.

EXAMPLE (notice the zeros appearing under the pivots):

$$\begin{array}{rcl} x+\ y+\ z=\ 1 & & x+\ y+\ z=1 & & x+\ y+z=1 \\ 2x+5y+3z=\ 7 & \to & 3y+\ z=5 & \to & 3y+z=5 \\ 4x+7y+6z=11 & & 3y+2z=7 & & z=2. \end{array}$$

Elimination leads to a ***triangular system***. The coefficients below the diagonal are zero.
First $z=2$, then $y=1$, then $x=-2$. *Back substitution solves triangular systems* (fast).

As a final example, try the singular case $S\mathbf{u}=\mathbf{d}$ when the corner entry is changed from 2 to 4. With $D=0$, there is no inverse matrix S^{-1}. Elimination also fails, by reaching an impossible equation $0=-2$:

$$\begin{array}{rcl} x+\ y \qquad =\ 1 & & x+\ y \qquad =\ 1 & & x+y \qquad =\ 1 \\ x \qquad +2z=\ 0 & \to & -\ y+2z=-1 & \to & -\ y+\ 2z=-1 \\ -2y+\underline{4}z=-4 & & -2y+\underline{4}z=-4 & & \underline{0=-2} \end{array}$$

The three planes do not meet at a point—a fact that was not obvious at the start. Algebra discovers this fact from $D=0$. Elimination discovers it from $0=-2$. The chapter is ending at the point where my linear algebra book begins.

One final comment. In actual computing, you will use a code written by professionals. The steps will be the same as above. A multiple of equation 1 is subtracted from each equation below it, to eliminate the first unknown x. With one fewer unknown and equation, elimination starts again. (A parallel computer executes many steps at once.) Extra instructions are included to reduce roundoff error. You only see the result! But it is more satisfying to know what the computer is doing.

In the end, solving linear equations is the key step in solving nonlinear equations. The central idea of differential calculus is to *linearize* near a point.

11.5 EXERCISES

Read-through questions

Three equations in three unknowns can be written as $A\mathbf{u}=\mathbf{d}$. The ___a___ $\mathbf{u}$ has components x,y,z and A is a ___b___. The row picture has a ___c___ for each equation. The first two planes intersect in a ___d___, and all three planes intersect in a ___e___, which is ___f___. The column picture starts with vectors $\mathbf{a},\mathbf{b},\mathbf{c}$ from the columns of ___g___ and combines them to produce ___h___. The vector equation is ___i___ $=\mathbf{d}$.

The determinant of A is the triple product ___j___. This is the volume of a box, whose edges from the origin are ___k___. If $\det A=$ ___l___ then the system is ___m___. Otherwise there is an ___n___ matrix such that $A^{-1}A=$ ___o___ (the ___p___ matrix). In this case the solution to $A\mathbf{u}=\mathbf{d}$ is $\mathbf{u}=$ ___q___.

The rows of A^{-1} are the cross products $\mathbf{b}\times\mathbf{c}$, ___r___, ___s___, divided by D. The entries of A^{-1} are 2 by 2 ___t___, divided by D. The upper left entry equals ___u___. The 2 by 2 determinants needed for a row of A^{-1} do not use the corresponding ___v___ of A.

The solution is $\mathbf{u}=A^{-1}\mathbf{d}$. Its first component x is a ratio of determinants, $|\mathbf{d}\,\mathbf{b}\,\mathbf{c}|$ divided by ___w___. Cramer's Rule breaks down when $\det A=$ ___x___. Then the columns $\mathbf{a},\mathbf{b},\mathbf{c}$ lie in the same ___y___. There is no solution to $x\mathbf{a}+y\mathbf{b}+z\mathbf{c}=\mathbf{d}$, if $\mathbf{d}$ is not on that ___z___. In a singular row picture, the intersection of planes 1 and 2 is ___A___ to the third plane.

In practice $\mathbf{u}$ is computed by ___B___. The algorithm starts by subtracting a multiple of row 1 to eliminate x from ___C___. If the first two equations are $x-y=1$ and $3x+z=7$, this elimination step leaves __D__. Similarly x is eliminated from the third equation, and then __E__ is eliminated. The equations are solved by back __F__. When the system has no solution, we reach an impossible equation like __G__. The example $x-y=1, 3x+z=7$ has no solution if the third equation is __H__.

Rewrite 1–4 as matrix equations $A\mathbf{u}=\mathbf{d}$ (do not solve).

1 $\mathbf{d}=(0,0,8)$ is a combination of $\mathbf{a}=(1,2,0)$ and $\mathbf{b}=(2,3,2)$ and $\mathbf{c}=(2,5,2)$.

2 The planes $x+y=0, x+y+z=1$, and $y+z=0$ intersect at $\mathbf{u}=(x,y,z)$.

3 The point $\mathbf{u}=(x,y,z)$ is on the planes $x=y$, $y=z$, $x-z=1$.

4 A combination of $\mathbf{a}=(1,0,0)$ and $\mathbf{b}=(0,2,0)$ and $\mathbf{c}=(0,0,3)$ equals $\mathbf{d}=(5,2,0)$.

5 Show that Problem 3 has no solution in two ways: find the determinant of A, and combine the equations to produce $0=1$.

6 Solve Problem 2 in two ways: by inspiration and Cramer's Rule.

7 Solve Problem 4 in two ways: by inspection and by computing the determinant and inverse of the *diagonal matrix*

$$A=\begin{bmatrix}1&0&0\\0&2&0\\0&0&3\end{bmatrix}.$$

8 Solve the three equations of Problem 1 by elimination.

9 The vectors $\mathbf{b}$ and $\mathbf{c}$ lie in a plane which is perpendicular to the vector ______. In case the vector $\mathbf{a}$ also lies in that plane, it is also perpendicular and $\mathbf{a}\cdot$ ______ $=0$. The ______ of the matrix with columns in a plane is ______.

10 The plane $a_1x+b_1y+c_1z=d_1$ is perpendicular to its normal vector $\mathbf{N}_1=$ ______ . The plane $a_2x+b_2y+c_2z=d_2$ is perpendicular to $\mathbf{N}_2=$ ______. The planes meet in a line that is perpendicular to both vectors, so the line is parallel to their ______ product. If this line is also parallel to the third plane and perpendicular to $\mathbf{N}_3$, the system is ______. The matrix has no ______, which happens when $(\mathbf{N}_1\times\mathbf{N}_2)\cdot\mathbf{N}_3=0$.

Problems 11–24 use the matrices A, B, C.

$$A=\begin{bmatrix}1&4&0\\0&2&6\\0&0&3\end{bmatrix}\quad B=\begin{bmatrix}0&0&1\\2&1&0\\6&4&0\end{bmatrix}\quad C=\begin{bmatrix}1&-1&-3\\-1&2&0\\0&-1&3\end{bmatrix}.$$

11 Find the determinants $|A|, |B|, |C|$. Since A is triangular, its determinant is the product ______.

12 Compute the cross products of each pair of columns in B (three cross products).

13 Compute the inverses of A and B above. Check that $A^{-1}A=I$ and $B^{-1}B=I$.

14 Solve $A\mathbf{u}=\begin{bmatrix}1\\0\\0\end{bmatrix}$ and $B\mathbf{u}=\begin{bmatrix}1\\0\\0\end{bmatrix}$. With this right side $\mathbf{d}$, why is $\mathbf{u}$ the first column of the inverse?

15 Suppose all three columns of a matrix add to zero, as in C above. The dot product of each column with $\mathbf{v}=(1,1,1)$ is ______. All three columns lie in the same ______. The determinant of C must be ______.

16 Find a nonzero solution to $C\mathbf{u}=\mathbf{0}$. Find all solutions to $C\mathbf{u}=\mathbf{0}$.

17 Choose any right side $\mathbf{d}$ that is perpendicular to $\mathbf{v}=(1,1,1)$ and solve $C\mathbf{u}=\mathbf{d}$. Then find a second solution.

18 Choose any right side $\mathbf{d}$ that is not perpendicular to $\mathbf{v}=(1,1,1)$. Show by elimination (reach an impossible equation) that $C\mathbf{u}=\mathbf{d}$ has no solution.

19 Compute the matrix product AB and then its determinant. How is $\det AB$ related to $\det A$ and $\det B$?

20 Compute the matrix products BC and CB. All columns of CB add to ______, and its determinant is ______.

21 Add A and C by adding each entry of A to the corresponding entry of C. Check whether the determinant of $A+C$ equals $\det A+\det C$.

22 Compute $2A$ by multiplying each entry of A by 2. The determinant of $2A$ equals ______ times the determinant of A.

23 Which four entries of A give the upper left corner entry p of A^{-1}, after dividing by $D=\det A$? Which four entries of A give the entry q in row 1, column 2 of A^{-1}? Find p and q.

24 The 2 by 2 determinants from the first two rows of B are -1 (from columns $2,3$) and -2 (from columns $1,3$) and ______ (from columns $1,2$). These numbers go into the third ______ of B^{-1}, after dividing by ______ and changing the sign of ______.

25 Why does every inverse matrix A^{-1} have an inverse?

26 From the multiplication $ABB^{-1}A^{-1}=I$ it follows that the inverse of AB is ______. The separate inverses come in ______ order. If you put on socks and then shoes, the inverse begins by taking off ______.

27 Find the determinants of these four *permutation matrices*:

$$P=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}\quad Q=\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}\quad PQ=\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}$$

and $QP=$ ______. Multiply $\mathbf{u}=(x,y,z)$ by each permutation to find $P\mathbf{u}$, $Q\mathbf{u}$, $PQ\mathbf{u}$, and $QP\mathbf{u}$.

28 Find all six of the 3 by 3 permutation matrices (including I), with a single 1 in each row and column. Which of them are "even" (determinant 1) and which are "odd" (determinant -1)?

29 How many 2 by 2 permutation matrices are there, including I? How many 4 by 4?

30 Multiply any matrix A by the permutation matrix P and explain how PA is related to A. In the opposite order explain how AP is related to A.

31 Eliminate x from the last two equations by subtracting the first equation. Then eliminate y from the new third equation by using the new second equation:

$$\text{(a)}\quad \begin{aligned} x+\ y+\ z&=2\\ x+3y+3z&=0\\ x+3y+7z&=2 \end{aligned} \qquad\qquad \text{(b)}\quad \begin{aligned} x+y\qquad&=1\\ x+\qquad z&=3\\ y+z&=5. \end{aligned}$$

After elimination solve for z, y, x (back substitution).

32 By elimination and back substitution solve

$$\text{(a)}\quad \begin{aligned} x\ +2y+2z&=0\\ 2x+3y+5z&=0\\ 2y+2z&=8 \end{aligned} \qquad\qquad \text{(b)}\quad \begin{aligned} x-y\qquad&=1\\ x\qquad -z&=4\\ y-z&=7. \end{aligned}$$

33 Eliminate x from equation 2 by using equation 1:

$$\begin{aligned} x+2y+2z&=0\\ 2x+4y+5z&=0\\ 2y+2z&=8. \end{aligned}$$

Why can't the new second equation eliminate y from the third equation? Is there a solution or is the system singular?

Note: If elimination creates a zero in the "pivot position," try to exchange that pivot equation with an equation below it. Elimination succeeds when there is a full set of pivots.

34 The pivots in Problem 32a are $1, -1$, and 4. Circle those as they appear along the diagonal in elimination. Check that the product of the pivots equals the determinant. (This is how determinants are computed.)

35 Find the pivots and determinants in Problem 31.

36 Find the inverse of $A = \begin{bmatrix} 1 & 1 & 0\\ 0 & 1 & 1\\ 0 & 0 & 1 \end{bmatrix}$ and also of $B = A^2$.

37 The symbol a_{ij} stands for the entry in row i, column j. Find a_{12} and a_{21} in Problem 36. The formula $\Sigma a_{ij} b_{jk}$ gives the entry in which row and column of the matrix product AB?

38 Write down a 3 by 3 singular matrix S in which no two rows are parallel. Find a combination of rows 1 and 2 that is parallel to row 3. Find a combination of columns 1 and 2 that is parallel to column 3. Find a nonzero solution to $S\mathbf{u} = 0$.

39 Compute these determinants. The 2 by 2 matrix is invertible if ______ . The 3 by 3 matrix (is)(is not) invertible.