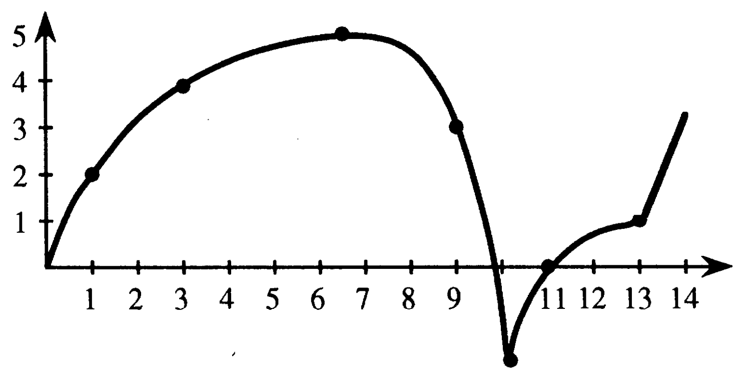
CHAPTER 2 DERIVATIVES

2.1 The Derivative of a Function (page 49)

In this section you are mainly concerned with learning the meaning of the derivative, and also the notation. The list of functions with known derivatives includes $f(t)$ = constant, Vt, $\frac{1}{2}at^2$, and $1/t$. Those functions have $f'(t) = 0$, V, at, and $-1/t^2$. We also establish the "square rule", that the derivative of $(f(t))^2$ is $2f(t)f'(t)$. Soon you will see other quick techniques for finding derivatives. But learn the basics first.

The derivative is the slope of the tangent line. Mathematicians often say the "slope" of a function when they mean the "derivative." Questions 1–7 refer to this figure:



1. If the slope of the tangent line at (1,2) is $\frac{3}{2}$, this means that $f'(\_\_) = \_\_\_\_$.
   - It means that $f'(1) = \frac{3}{2}$. The first coordinate of (1,2) goes into $f'(\_\_)$.
2. If $f'(3) = \frac{1}{2}$, then the slope of the tangent line through (___, ___) is $\frac{1}{2}$.
   - The point is (3,4).
3. The graph indicates that $f'(\_\_) = 0$. This zero derivative means ____ .
   - The graph has $f'(6\frac{1}{2}) = 0$. The tangent is horizontal at $x = 6\frac{1}{2}$. Later we learn: This happens at a maximum or minimum.
4. $\frac{dy}{dx}$ is negative when ___ $< x <$ ___ . In this interval the function is ____ .
   - The slope is negative when $6\frac{1}{2} < x < 10$. The function is decreasing (left to right).
5. The derivative $\frac{dy}{dx}$ is not defined when $x =$ ____ and ____ .
   - $\frac{dy}{dx}$ is undefined at $x = 10$ because the tangent line is vertical. You might say "infinite slope." Also there is no derivative at $x = 13$. The graph has a *corner*. The slope is different on the left side and right side of the corner.
6. $f(x)$ is (positive or negative?) at (9,3) while $f'(x)$ is (positive or negative?).
   - $f(9) = 3$ is positive. The graph is above the y axis. The slope $f'(9)$ is negative.
7. $f(10.5)$ is (positive or negative?) while $f'(10.5)$ is ____ .
   - At $(10.5, -1)$ the graph is below the y axis and rising. $f(x)$ is negative and $f'(x)$ is positive.

The derivative is the rate of change. When the function is the distance $f(t)$, its derivative is (instantaneous) velocity. When the function is the velocity $v(t)$, its derivative is (instantaneous) acceleration.

8. Suppose a vehicle travels according to the rule $f(t) = 3t^3 - t$. Find its velocity at time $t = 4$.

   - Even though you have in mind a special time $t = 4$, you cannot substitute $t = 4$ until the end. (You don't want $f(t) = 44$. A constant function has velocity zero! What you will use is $f(4) = 44$.) First find the average velocity $\frac{\Delta f}{\Delta t}$ using Δt or h:

     $$\frac{\Delta f}{\Delta t} = \frac{f(t+\Delta t) - f(t)}{\Delta t} = \frac{3(t^2 + 2t\ \Delta t + (\Delta t)^2) - (t + \Delta t) - 3t^2 + t}{\Delta t}.$$

     Notice that $3t^2$ cancels with $-3t^2$ and also t cancels $-t$. This produces

     $$\frac{\Delta f}{\Delta t} = \frac{6t\Delta t + 3(\Delta t)^2 - \Delta t}{\Delta t} = 6t + 3\Delta t - 1.$$

     The division removed Δt from the denominator. *Now let* Δt *go to* 0. The limit is $f'(t) = 6t - 1 = v(t)$. At $t = 4$, the velocity is $v(4) = 6 \times 4 - 1 = 23$. To find acceleration, go back to the formula for velocity (*before* you plugged in $t = 4$). Take the derivative of $v(t) = 6t - 1$:

     $$\frac{\Delta v}{\Delta t} = \frac{v(t+\Delta t) - v(t)}{\Delta t} = \frac{[6(t+\Delta t) - 1] - (6t - 1)}{\Delta t} = \frac{6\Delta t}{\Delta t} = 6.$$

     As $\Delta t \to 0$, the average $\frac{\Delta v}{\Delta t}$ stays at 6. So the acceleration is $\frac{dv}{dt} = 6$. The graph of $v(t) = 6t - 1$ is a line with constant slope 6.

   - A simple but important point is that if the graph is shifted up or down, its slope does not change. Another way to say this is that if $g(x) = f(x) + c$, where c is a constant number, then $g'(x) = f'(x)$. Question: *Does* $f'(x)$ *change if the graph is shifted right or left?* Yes it does. The slope graph shifts too. The slope of $f(x) + 1$ is $f'(x)$, but the slope of $f(x+1)$ is $f'(x+1)$.

9. Find functions that have the same derivative as $f(x) = \frac{1}{x}$.

   - $g(x) = \frac{1}{x} - 5$ and $h(x) = \frac{1}{x} + 2$ and any $\frac{1}{x} + C$. No other possibilities!

Read-throughs and selected even-numbered solutions :

The derivative is the **limit** of $\Delta f/\Delta t$ as Δt approaches **zero.** Here Δf equals $\mathbf{f(t + \Delta\ t) - f(t)}$. The step Δt can be positive or **negative.** The derivative is written v or **df/dt** or $\mathbf{f'(t)}$. If $f(x) = 2x + 3$ and $\Delta x = 4$ then $\Delta f = \mathbf{8}$. If $\Delta x = -1$ then $\Delta f = \mathbf{-2}$. If $\Delta x = 0$ then $\Delta f = \mathbf{0}$. The slope is not 0/0 but $df/dx = \mathbf{2}$.

The derivative does not exist where $f(t)$ has a **corner** and $v(t)$ has a **jump.** For $f(t) = 1/t$ the derivative is $\mathbf{-1/t^2}$. The slope of $y = 4/x$ is $dy/dx = \mathbf{-4/x^2}$. A decreasing function has a **negative** derivative. The **independent** variable is t or x and the **dependent** variable is f or y. The slope of y^2 **(is not)** $(dy/dx)^2$. The slope of $(u(x))^2$ is $\mathbf{2u(x)\ du/dx}$ by the square rule. The slope of $(2x+3)^2$ is $\mathbf{2(2x+3)2 = 8x + 12}$.

2 (a) $\frac{\Delta f}{h} = \frac{2hx+h^2}{h}$ becomes **2x** at $h = 0$ (b) $\frac{(x+5h)^2 - x^2}{5h} = \frac{10hx + 25h^2}{5h} = 2x + 5h$ becomes **2x** at $h = 0$ (c) $\frac{(x+h)^2-(x-h)^2}{2h} = \frac{4xh}{2h} =$ **2x** *always* (d) $\frac{(x+1)^2 - x^2}{h} = \frac{2x+1}{h} \to \infty$ as $h \to 0$ **4** x^2+1, x^2+10, x^2-100

6 The line and parabola have slopes 1 and $2x$. So the touching point must have $x = \frac{1}{2}$. There $y = \frac{1}{2}$ for the line, $y = (\frac{1}{2})^2 + c$ for the parabola so $c = \mathbf{\frac{1}{4}}$.

22 The graph of $f(t)$ has slope -2 until it reaches $t = 2$ where $f(2)$ equals -1; after that it has slope zero.

36 (a) **False** First draw a curve that stays below $y = x$ but comes upward steeply for negative x. Then create a formula like $\mathbf{y = -x^2 - 10}$. (b) **False** $f(x)$ could be any constant, for example $f(x) = 10$. Note what is *true*: If $\frac{df}{dx} \le 1$ and $f(x) \le x$ at some point then $f(x) \le x$ everywhere **beyond that point.**

2.2 Powers and Polynomials (page 56)

The derivatives of x^5 and $x^5+x^{-\sqrt{2}}$ and $3x^5$ come from **2B, 2C, 2D**. Practice until $5x^4$ and $5x^4-\sqrt{2}x^{-\sqrt{2}-1}$ and $15x^4$ are automatic. It's not the end of the world if the binomial formula escapes you, but you have to find derivatives of powers and polynomials. Note that the derivative of x^n is nx^{n-1} for *all* numbers n (fractions, negative numbers, $\cdots$). So far this has only been proved when n is a positive integer. *Find* $\frac{dy}{dx}$ *in Problems* 1–6:

1. $y = 4x+2$. This line has slope $\frac{dy}{dx} = 4(1)+0 = 4$. Use **2B** (for x) and **2D** (for $4x$) and **2C** (for $4x+2$).
2. $y = \frac{1}{x^2} = x^{-2}$. This has $n=-2$ so $\frac{dy}{dx} = -2x^{-3} = \frac{-2}{x^3}$.
3. $y = 5x^{-\frac{2}{5}} + 10x^{\frac{1}{5}}$. Fractional powers give $\frac{dy}{dx} = 5(-\frac{2}{5})x^{-\frac{7}{5}} + 10(\frac{1}{5})x^{-\frac{4}{5}} = -2x^{-\frac{7}{5}} + 2x^{-\frac{4}{5}}$. *Why* $-\frac{7}{5}$?
4. $y = 4\sqrt{x} = 4x^{\frac{1}{2}}$. Here $n=\frac{1}{2}$. Then $\frac{dy}{dx} = 4(\frac{1}{2})x^{-\frac{1}{2}} = \frac{2}{\sqrt{x}}$.
5. $y=(3x-5)^2$. The answer is *not* $\frac{dy}{dx} = 2(3x-5)$! Use the square rule to get $2(3x-5)(3) = 18x-30$. *The extra* 3 *comes from the derivative of* $3x-5$. You can expand $(3x-5)^2$ and take the derivative of each term:

$$y = (3x-5)(3x-5) = 9x^2-15x-15x+25 \quad \text{so} \quad \frac{dy}{dx} = 9(2x)-15-15 = 18x-30.$$

6. $y = 4x^{\sqrt{3}}$. Don't let $\sqrt{3}$ throw you! Just forge ahead to find $\frac{dy}{dx} = 4\sqrt{3}x^{\sqrt{3}-1}$.
7. Find the second derivative of $y = 4x^3-2x+\frac{6}{x}$.
   - The first derivative is $\frac{dy}{dx} = 12x^2-2-6x^{-2}$. Then the second derivative is $24x+0+12x^{-3}$.
8. Find a function that has $\frac{dy}{dx} = -y^2$. This is a *differential equation*, with y on both sides.
   - Solving means finding $y(x)$ from information about $\frac{dy}{dx}$. At this point we don't know the derivatives of too many functions. We can guess $y=cx^n$ and work backward. Then $\frac{dy}{dx} = cnx^{n-1}$ and $-y^2 = -c^2x^{2n}$. If we can choose c and n so that $cnx^{n-1} = -c^2x^{2n}$, we have solved the differential equation. Match the powers $n-1=2n$ to get $n=-1$. Now we want cn to equal $-c^2$, so $c=1$. The answer is $y=x^{-1}$. Check that $\frac{dy}{dx} = -y^2$. Both sides equal $-x^{-2}$.

Read-throughs and selected even-numbered solutions :

The derivative of $f = x^4$ is $f' = \mathbf{4x^3}$. That comes from expanding $(x+h)^4$ into the five terms $\mathbf{x^4+4x^3h+6x^2h^2+4xh^3+h^4}$. Subtracting x^4 and dividing by h leaves the four terms, $\mathbf{4x^3+6x^2h+4xh^2+h^3}$. This is $\Delta f/h$, and its limit is $\mathbf{4x^3}$.

The derivative of $f=x^n$ is $f' = \mathbf{nx^{n-1}}$. Now $(x+h)^n$ comes from the **binomial** theorem. The terms to look for are $x^{n-1}h$, containing only one **h**. There are **n** of those terms, so $(x+h)^n = x^n + \mathbf{nx^{n-1}h} + \cdots$. After subtracting $\mathbf{x^n}$ and dividing by h, the limit of $\Delta f/h$ is $\mathbf{nx^{n-1}}$. The coefficient of $x^{n-j}h^j$, not needed here, is "n choose j" $=$ **n!/j!(n-j)!**, where $n!$ means **n(n-1)** $\cdots$ **(1)**.

The derivative of x^{-2} is $\mathbf{-2x^{-3}}$. The derivative of $x^{1/2}$ is $\mathbf{\frac{1}{2}x^{-1/2}}$. The derivative of $3x+(1/x)$ is $\mathbf{3-1/x^2}$, which uses the following rules: the derivative of $3f(x)$ is $\mathbf{3f'(x)}$ and the derivative of $f(x)+g(x)$ is $\mathbf{f'(x)+g'(x)}$.

Integral calculus recovers **y** from dy/dx. If $dy/dx = x^4$ then $y(x) = \mathbf{x^5/5 + C}$.

2 $f(x) = \mathbf{\frac{1}{7}x^7}$ (or $\frac{1}{7}x^7 + C$)
8 $\frac{df}{dx} = \frac{1}{n!}(nx^{n-1}) = \mathbf{\frac{x^{n-1}}{(n-1)!}}$. Note the step $\frac{n}{n(n-1)\cdots(1)} = \frac{1}{(n-1)\cdots(1)} = \frac{1}{(n-1)!}$
10 $f'(x) = \frac{2}{3}(\frac{3}{2}x^{1/2}) + \frac{2}{5}(\frac{5}{2}x^{3/2}) = x^{1/2} + x^{3/2}$.
14 The slope of $x + \frac{1}{x}$ is $1 - \frac{1}{x^2}$ which is zero at $x = 1$. At that point the graph of $x + \frac{1}{x}$ levels off. (The function reaches its minimum, which is 2. For any other positive x, the combination $x + \frac{1}{x}$ is larger than 2.)
22 If $y = \frac{1}{\sqrt{x}}$ then $\Delta y = \frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}} = \frac{\sqrt{x}-\sqrt{x+h}}{\sqrt{x+h}\sqrt{x}}$ = (multiply top and bottom by $\sqrt{x} + \sqrt{x+h}$) = $\frac{\mathbf{x-(x+h)}}{\mathbf{\sqrt{x+h}\sqrt{x}(\sqrt{x}+\sqrt{x+h})}}$. Cancel $x - x$ in the numerator and divide by h : $\frac{\Delta y}{h} = \frac{-1}{\sqrt{x+h}\sqrt{x}(\sqrt{x}+\sqrt{x+h})}$. Now let $h \to 0$ to find $\frac{dy}{dx} = \frac{-1}{2x^{3/2}} = -\frac{1}{2}x^{-3/2}$ (which is nx^{n-1}).
38 If $y = y_0 + cx$ then $E(x) = \frac{dy/dx}{y/x} = \frac{c}{\frac{y_0}{x}+c}$ which approaches **1** as $x \to \infty$.
42 $y = x^n$ has $E = \frac{dy/dx}{y/x} = \mathbf{n}$. The revenue $xy = x^{n+1}$ has $E = \mathbf{n+1}$.
44 Marginal propensity to save is $\mathbf{\frac{dS}{dI}}$. Elasticity is not needed because S and I have the same **units**. Applied to the whole economy this is **macroeconomics**.

2.3 The Slope and the Tangent Line (page 63)

Questions 1–5 refer to the curve $y = x^3 - 2x^2$. The derivative is $\frac{dy}{dx} = 3x^2 - 4x$.

1. Find the slope at $x = 3$.
   - Substitute $x = 3$ in $\frac{dy}{dx}$ (not in y!). The slope is $27 - 12 = 15$.
2. Find the equation of the tangent line at $x = 3$.
   - From question 1, the slope at $x = 3$ is 15. The function itself is $y = 27 - 18 = 9$. Use the point-slope form to get the equation of the tangent line: $y - 9 = 15(x - 3)$. You can rewrite this as $y = 15x - 36$.
3. Find the equation of the normal line at $x = 3$. This line is perpendicular to the curve.
   - We know that the slope of the curve at $(3, 9)$ is 15. Therefore the slope of the normal is $-\frac{1}{15}$. *When two lines are perpendicular, the second slope is* $\frac{-1}{\text{first slope}}$. The normal line is $y - 9 = -\frac{1}{15}(x - 3)$.
4. Find the secant line from (0,0) to (3,9). A *secant* connects two points on the curve.
   - The slope of the secant is $\frac{9-0}{3-0} = 3$. The secant line is $y - 0 = 3(x - 0)$, or $y = 3x$.
5. Where does the curve $y = x^3 - 2x^2$ have a horizontal tangent line? This means $\frac{dy}{dx} = 0$.
   - The slope is $\frac{dy}{dx} = 3x^2 - 4x = x(3x - 4)$. Then $\frac{dy}{dx} = 0$ when $x = 0$ or $x = \frac{4}{3}$. Horizontal (flat) tangent lines are found at (0,0) and $(\frac{4}{3}, \frac{-32}{27})$.
6. (This is Problem 2.3.13) At $x = a$ compute (a) the equation of the tangent line to the curve $y = \frac{1}{x}$ and (b) the points where that line crosses the axes. (c) The triangle between the tangent line and the axes always has area ________ .
   - (a) $y = x^{-1}$ has slope $\frac{dy}{dx} = -x^{-2} = -\frac{1}{x^2}$. At $(a, \frac{1}{a})$ this slope is $-\frac{1}{a^2}$. The tangent line has equation $y - \frac{1}{a} = -\frac{1}{a^2}(x - a)$.

- (b) This line crosses the x axis when $y = 0$ so $0 - \frac{1}{a} = -\frac{1}{a^2}(x - a)$. Multiply by $-a^2$ to find $a = x - a$ and $x = 2a$. The line crosses the y axis when $x = 0: y - \frac{1}{a} = -\frac{1}{a^2}(0 - a)$ which gives $y = \frac{2}{a}$.
- (c) The area of the triangle is $\frac{1}{2}(\text{base} \times \text{height}) = \frac{1}{2}(2a)(\frac{2}{a})$. This area is always 2.

4. Turn off your calculator and use the methods of this section to estimate $\sqrt[3]{66} = 66^{1/3}$.

   - Let $f(x) = x^{1/3}$. We know that $64^{1/3} = 4$. (64 is chosen because it is the closest perfect cube to 66.) The plan is to find the tangent line through the point $x = 64$, $y = 4$. Use this tangent line to approximate the cube root function:

$$x^{1/3} \text{ has derivative } \frac{1}{3}x^{-2/3} \text{ and } f'(64) = \frac{1}{3}(64)^{-2/3} = \frac{1}{3}(4^{-2}) = \frac{1}{3(16)} = \frac{1}{48}.$$

The tangent line is $y - 4 = \frac{1}{48}(x - 64)$. If $x = 66$ then $y = 4 + \frac{1}{48}(2) = 4\frac{1}{24}$. The cube root curve goes up approximately $\frac{1}{24}$ to $66^{1/3} \approx 4\frac{1}{24}$.

Read-throughs and selected even-numbered solutions :

A straight line is determined by **2** points, or one point and the **slope.** The slope of the tangent line equals the slope of the **curve.** The point-slope form of the tangent equation is $y - f(a) = \mathbf{f'(a)(x - a)}$.

The tangent line to $y = x^3 + x$ at $x = 1$ has slope **4.** Its equation is $\mathbf{y - 2 = 4(x - 1)}$. It crosses the y axis at $\mathbf{y = -2}$ and the x axis at $\mathbf{x = \frac{1}{2}}$. The normal line at this point (1,2) has slope $-\frac{1}{4}$. Its equation is $y - 2 = -\frac{1}{4}\mathbf{(x-1)}$. The secant line from (1,2) to (2, **10**) has slope **8.** Its equation is $y - 2 = \mathbf{8(x-1)}$.

The point $(c, f(c))$ is on the line $y - f(a) = m(x - a)$ provided $m = \mathbf{\frac{f(c)-f(a)}{c-a}}$. As c approaches a, the slope m approaches $\mathbf{f'(a)}$. The secant line approaches the **tangent** line.

2 $y = x^2 + x$ has $\frac{dy}{dx} = 2x + 1 = 3$ at $x = 1, y = 2$. The tangent line is $\mathbf{y - 2 = 3(x - 1)}$ or $y = 3x - 1$. The normal line is $\mathbf{y - 2 = -\frac{1}{3}(x - 1)}$ or $y = -\frac{x}{3} + \frac{7}{3}$. The secant line is $\mathbf{y - 2 = m(x - 1)}$ with $m = \frac{(1+h)^2+(1+h)-2}{(1+h)-1} = 3 + h$.

8 $(x - 1)(x - 2)$ is zero at $x = 1$ and $x = 2$. If this is the slope (it is $x^2 - 3x + 2$) then the function can be $\mathbf{\frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x}$. We can add any $Cx + D$ to this answer, and the slopes at $x = 1$ and 2 are still equal. $y = x^4 - 2x^2$ has $\frac{dy}{dx} = 4x^3 - 4x$. At $\mathbf{x = 1}$ **and** $\mathbf{x = -1}$ the slopes are zero and the y's are equal. The tangent line (horizontal) is the same.

18 Tangency requires $4x = cx^2$ and also (slopes) $4 = 2cx$ at the same x. The second equation gives $x = \frac{2}{c}$ and then the first is $\frac{8}{c} = \frac{4}{c}$ which has no solution.

30 The tangent line is $y - f(a) = f'(a)(x - a)$. This goes through $y = g(b)$ at $x = b$ if $\mathbf{g(b) - f(a) = f'(a)(b - a)}$. The slopes are the same if $\mathbf{g'(b) = f'(a)}$.

46 To *just* pass the baton, the runners reach the same point at the same time ($vt = -8 + 6t - \frac{1}{2}t^2$) and with the same speed ($v = 6 - t$). Then $(6 - t)t = -8 + 6t - \frac{1}{2}t^2$ and $\frac{1}{2}t^2 - 8 = 0$. Then $\mathbf{t = 4}$ and $\mathbf{v = 2}$.

2.4 The Derivative of the Sine and Cosine (page 70)

This section proves that $\frac{\sin h}{h} \to 1$ as $h \to 0$. In other words $\lim_{h\to 0} \frac{\sin h}{h} = 1$. The separate limits of $\sin h$ and h lead to $\frac{0}{0}$ which is undefined. The key to differential calculus is that the ratio approaches a definite limit.

Once that limit is established, algebraic substitution lets us conclude that $\lim_{5h\to 0} \frac{\sin 5h}{5h} = 1$. Similarly $\frac{\sin(-2z)}{(-2z)}$ approaches 1. The limit of $\frac{\sin\square}{\square}$ is 1 as long as both boxes are the same and approach zero. However $\frac{\sin 4h}{h}$ approaches 4. We show this by multiplying by $\frac{4}{4}$ to get $\frac{4\sin 4h}{4h}$. The limit is 4 times the limit of $\frac{\sin 4h}{4h}$, or $4 \times 1 = 4$.

You can factor out 4 or any constant from inside the limit. Questions 1–3 use this trick.

1. Find $\lim_{\theta\to 0} \frac{\theta}{\sin 6\theta}$.

   - Multiply and divide by 6, to get $\frac{1}{6} \cdot \frac{6\theta}{\sin 6\theta}$. As $\theta \to 0$ the limit is $\frac{1}{6} \times 1 = \frac{1}{6}$.

2. Find $\lim_{h\to 0} \frac{\sin 4h}{\sin 2h}$.

   - Write $\frac{\sin 4h}{\sin 2h}$ as $\frac{\sin 4h}{4h} \cdot \frac{2h}{\sin 2h} \cdot 2$. (Check this to satisfy yourself that we have just multiplied by 1 in a good way.) As $h \to 0$, we have $\frac{\sin 4h}{4h} \to 1$ and $\frac{2h}{\sin 2h} \to 1$. The limit of $\frac{\sin 4h}{\sin 2h}$ is $1 \cdot 1 \cdot 2 = 2$.

3. Find $\lim_{z\to 0} \frac{\tan z}{z}$.

   - The ratio is $\frac{\tan z}{z} = \frac{\sin z}{z \cos z}$. We know that $\frac{\sin z}{z} \to 1$ and $\cos z \to 1$ as $z \to 0$. Divide to find $\frac{\tan z}{z} \to 1$. What does this mean about the graph of $y = \tan z$? It means that the slope of $\tan z$ at $z = 0$ is $\frac{dy}{dz} = 1$. Reason: $\frac{\tan z}{z}$ is the average slope. Its limit is the exact slope $\frac{dy}{dz} = 1$.

4. Find $\frac{dy}{dx}$ for $y = \sin 2x$, first at $x = 0$ and then at every x.

   - The slope at $x = 0$ is $\lim \frac{\sin 2x}{x}$. (Maybe I should say $\lim \frac{\sin 2h}{h}$ – the same.) This is $2 \lim \frac{\sin 2x}{2x} = 2$.
   - The slope at any x is $\lim \frac{\sin 2(x+h) - \sin 2x}{h}$. We need a formula for $\sin 2(x+h)$. Equation (9) on page 32 gives the addition formula $\sin 2x \cos 2h + \cos 2x \sin 2h$. Group the $\sin 2x$ terms separately from the $\cos 2x$ term to get $\frac{\sin 2x(\cos 2h - 1) + \cos 2x \sin 2h}{h}$. Now take limits of those two parts:

$$\begin{aligned}\frac{dy}{dx} &= (\sin 2x)\lim_{h\to 0}\left(\frac{\cos 2h - 1}{h}\right) + (\cos 2x)\lim_{h\to 0}\left(\frac{\sin 2h}{h}\right) \\ &= (\sin 2x)(0) + (\cos 2x)(2) = 2\cos 2x.\end{aligned}$$

   The same methods show that $y = \sin(nx)$ has $\frac{dy}{dx} = n\cos(nx)$. Similarly $y = \cos(nx)$ has $\frac{dy}{dx} = -n\sin(nx)$. These are important. *Notice the extra factor n – later it comes from the chain rule.*

5. Write the equation of the tangent line to $f(x) = \cos x$ at $x = \frac{\pi}{4}$. Where does this line cross the y axis?

   - At $x = \frac{\pi}{4}$ (which is 45°) the cosine is $y = \frac{\sqrt{2}}{2}$ and the slope is $y' = -\sin\frac{\pi}{4} = -\frac{\sqrt{2}}{2}$. The tangent line is $y - \frac{\sqrt{2}}{2} = -\frac{\sqrt{2}}{2}(x - \frac{\pi}{4})$. Set $x = 0$ to find the y-intercept $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}\pi}{8} \approx 1.26$.

Read-throughs and selected even-numbered solutions :

The derivative of $y = \sin x$ is $y' = \mathbf{\cos x}$. The second derivative (the **derivative** of the derivative) is $y'' = \mathbf{-\sin x}$. The fourth derivative is $y'''' = \mathbf{\sin x}$. Thus $y = \sin x$ satisfies the differential equations $y'' = \mathbf{-y}$

and $y'''' = \mathbf{y}$. So does $y = \cos x$, whose second derivative is $-\mathbf{cos\ x}$.

All these derivatives come from one basic limit: $(\sin h)/h$ approaches **1**. The sine of .01 radians is very close to **.01**. So is the **tangent** of .01. The cosine of .01 is not .99, because $1 - \cos h$ is much **smaller** than h. The ratio $(1 - \cos h)/h^2$ approaches $\mathbf{\frac{1}{2}}$. Therefore $\cos h$ is close to $1 - \frac{1}{2}h^2$ and $\cos .01 \approx$ **.99995**. We can replace h by x.

The differential equation $y'' = -y$ leads to **oscillation**. When y is positive, y'' is **negative**. Therefore y' is **decreasing**. Eventually y goes below zero and y'' becomes **positive**. Then y' is **increasing**. Examples of oscillation in real life are **springs** and **heartbeats**.

4 $\tan h = 1.01h$ at $h = 0$ and $h = \pm.17$; $\tan h = h$ at $h = 0$.

10 (a) $\frac{1-\cos h}{h^2} = \frac{1-\cos^2 h}{(1+\cos h)h^2} = \frac{1}{1+\cos h}(\frac{\sin h}{h})^2 \to \mathbf{\frac{1}{2}}$ (b) $\frac{1-\cos^2 h}{h^2} = (\frac{\sin h}{h})^2 \to \mathbf{1}$ (c) $\frac{1-\cos^2 h}{\sin^2 h} = \mathbf{1}$
(d) $\frac{1-\cos 2h}{h} = 2\frac{1-\cos 2h}{2h} \to 2(0) = \mathbf{0}$.

24 The maximum of $y = \sin x + \sqrt{3}\cos x$ is at $\mathbf{x} = \mathbf{\frac{\pi}{6}}$ (or 30°) where $\mathbf{y} = \mathbf{\frac{1}{2}} + \sqrt{\mathbf{3}}\mathbf{\frac{\sqrt{3}}{2}} = \mathbf{2}$. The slope at that point is $\cos x - \sqrt{3}\sin x = \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = \mathbf{0}$. Note that y is the same as $2\cos x$ shifted to the right by $\frac{\pi}{6}$.

26 (a) **False** (use the square rule) (b) **True** (because $\cos(-x) = \cos x$) (c) **False** for $y = x^2$ (happens to be true for $y = \sin x$) (d) **True** (y'' = slope of y' = positive when y' increases)

2.5 The Product and Quotient and Power Rules (page 77)

You have to learn the rules that are boxed on page 76. Many people memorize the product rule $uv' + vu'$ this way: *The derivative of a product is the first times the derivative of the second plus the second times the derivative of the first.* The derivative of a quotient is : "The bottom times the derivative of the top minus the top times the derivative of the bottom, all divided by the bottom squared." I chant these to myself as I use them. (This is from Jennifer Carmody. Professor Strang just mumbles them.) Questions 1–6 ask for $\frac{dy}{dx}$ from the rules for derivatives.

1. $y = (4x^3 - 2x + 7)^5$.

   - This has the form u^5 where $u = 4x^3 - 2x + 7$. Use the power rule $5u^4\frac{du}{dx}$ noting that $\frac{du}{dx} = 12x^2 - 2$. Then $y' = 5(4x^3 - 2x + 7)^4(12x^2 - 2)$.

2. $y = \sqrt{\cos x + \sin x}$.

   - Here $y = u^{1/2}$ where $u = \cos x + \sin x$ and $u' = -\sin x + \cos x$. Use the power rule with $n = \frac{1}{2}$:

$$y' = \frac{1}{2}u^{-\frac{1}{2}}\frac{du}{dx} = \frac{1}{2}(\cos x + \sin x)^{-\frac{1}{2}}(-\sin x + \cos x).$$

3. $y = (4x - 7)^3(2x + 3)^9$.

   - Use the product rule for $y = uv$, with $u = (4x - 7)^3$ and $v = (2x + 3)^9$. But we need the power rule to find $u' = 3(4x - 7)^2(4)$ and $v' = 9(2x + 3)^8(2)$. *Where did the 4 and 2 come from?*

Putting it all together, y' is

$$\begin{aligned} uv' + vu' &= (4x-7)^3(9)(2x+3)^8(2) + (2x+3)^9(3)(4x-7)^2(4) \\ &= 18(4x-7)^3(2x+3)^8 + 12(2x+3)^9(4x-7)^2. \end{aligned}$$

4. Verify that $(\sec x)' = \sec x \tan x$. Since $\sec x = \frac{1}{\cos x}$, we can use the reciprocal rule.

   - *This is the quotient rule with* $u = 1$ *on top.* The bottom is $v = \cos x$. We want $\frac{-v'}{v^2}$:

$$(\sec x)' = (\frac{1}{\cos x})' = \frac{-(-\sin x)}{\cos^2 x} = \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} = \sec x \tan x.$$

5. $y = \frac{\sin x}{1+x^2}$.

   - Use the quotient rule with $u = \sin x$ and $v = 1 + x^2$. Find $u' = \cos x$ and $v' = 2x$:

$$y' = \frac{vu' - uv'}{v^2} = \frac{(\text{bottom})(\text{top})' - (\text{top})(\text{bottom})'}{(\text{bottom})^2} = \frac{(1+x^2)\cos x - \sin x(2x)}{(1+x^2)^2}.$$

6. $y = \cos x(1 + \tan x)(1 + \sin^2 x)$. Use the triple product rule from Example 5 on page 72.

   - The three factors are $u = \cos x$ with $u' = -\sin x$; $v = 1 + \tan x$ with $v' = \sec^2 x$; $w = 1 + \sin^2 x$ with $w' = 2 \sin x \cos x$ (by the power rule). The triple product rule is $uvw' + uv'w + u'vw$:

$$(uvw)' = \cos x(1+\tan x)(2\sin x \cos x) + \cos x(\sec^2 x)(1 + \sin^2 x) - \sin x(1 + \tan x)(1 + \sin^2 x).$$

Read-throughs and selected even-numbered solutions :

The derivatives of $\sin x \cos x$ and $1/\cos x$ and $\sin x/\cos x$ and $\tan^3 x$ come from the **product** rule, **reciprocal** rule, **quotient** rule, and **power** rule. The product of $\sin x$ times $\cos x$ has $(uv)' = uv' + \mathbf{u'v} = \mathbf{\cos^2 x - \sin^2 x}$. The derivative of $1/v$ is $\mathbf{-v'/v^2}$, so the slope of sec x is $\mathbf{\sin x/\cos^2 x}$. The derivative of u/v is $\mathbf{(vu' - uv')/v^2}$ so the slope of $\tan x$ is $\mathbf{(\cos^2 x + \sin^2 x)/\cos^2 x = \sec^2 x}$. The derivative of $\tan^3 x$ is $\mathbf{3 \tan^2 x \sec^2 x}$. The slope of x^n is $\mathbf{nx^{n-1}}$ and the slope of $(u(x))^n$ is $\mathbf{nu^{n-1}du/dx}$. With $n = -1$ the derivative of $(\cos x)^{-1}$ is $\mathbf{-1(\cos x)^{-2}(-\sin x)}$, which agrees with the rule for $\sec x$.

Even simpler is the rule of **linearity,** which applies to $au(x) + bv(x)$. The derivative is $\mathbf{au'(x) + bv'(x)}$. The slope of $3 \sin x + 4 \cos x$ is $\mathbf{3 \cos x - 4 \sin x}$. The derivative of $(3 \sin x + 4 \cos x)^2$ is $\mathbf{2(3 \sin x + 4 \cos x)}$ $\mathbf{(3 \cos x - 4 \sin x)}$. The derivative of $\mathbf{\sin^4 x}$ is $4 \sin^3 x \cos x$.

2 $\frac{dy}{dx} = (x^2+1)(2x) + (x^2-1)(2x) = \mathbf{4x^3}$ **4** $\mathbf{\frac{-2x}{(1+x^2)^2} + \frac{-(-\cos x)}{(1-\sin x)^2}}$.

6 $(x-1)^2 2(x-2) + (x-2)^2 2(x-1) = 2(x-1)(x-2)(x-1+x-2) = \mathbf{2(x-1)(x-2)(2x-3)}$.

8 $x^{1/2}(1 + \cos x) + (x + \sin x)\frac{1}{2}x^{-1/2}$ or $\frac{3}{2}x^{1/2} + x^{1/2}\cos x + \frac{1}{2}x^{-1/2}\sin x$

10 $\frac{(x^2-1)2x-(x^2+1)2x}{(x^2-1)^2} + \frac{\cos x(\cos x)-\sin x(-\sin x)}{\cos^2 x} = \mathbf{\frac{-4x}{(x^2-1)^2} + \frac{1}{\cos^2 x}}$.

12 $x^{3/2}(3\sin^2 x \cos x) + \frac{3}{2}x^{1/2}\sin^3 x + \frac{3}{2}(\sin x)^{1/2}\cos x$

14 $\sqrt{x}(\sqrt{x}+1)\frac{1}{2}x^{-1/2} + \sqrt{x}(\sqrt{x}+2)\frac{1}{2}x^{-1/2} + (\sqrt{x}+1)(\sqrt{x}+2)\frac{1}{2}x^{-1/2} = (3x + 6\sqrt{x}+2)\frac{1}{2}x^{-1/2}$ (or other form).

16 $10(x-6)^9 + 10\sin^9 x \cos x$.

18 $\csc^2 x - \cot^2 x = \frac{1}{\sin^2 x} - \frac{\cos^2 x}{\sin^2 x} = \frac{\sin^2 x}{\sin^2 x} = 1$ so the derivative is **zero.**

20 $\frac{(\sin x+\cos x)(\cos x+\sin x)-(\sin x-\cos x)(\cos x-\sin x)}{(\sin x+\cos x)^2} = \frac{2\sin^2 x+2\cos^2 x}{(\sin x+\cos x)^2} = \mathbf{\frac{2}{(\sin x+\cos x)^2}}$

22 $\frac{x\cos x}{\sin x}$ has derivative $\frac{\sin x(-x\sin x+\cos x)-x\cos x(\cos x)}{\sin^2 x} = \frac{-x+\sin x\cos x}{\sin^2 x}$ (or other form).
24 $[u(x)]^2(2v(x)\frac{dv}{dx}) + [v(x)]^2(2u(x)\frac{du}{dx})$
26 $x\cos x + \sin x - \sin x = \mathbf{x\cos x}$ (we now have a function with derivative $x\cos x$).
34 (a) $y = \frac{1}{4}x^4$ (b) $y = -\frac{1}{2}x^{-2}$ (c) $y = -\frac{2}{5}(1-x)^{5/2}$ (This one is more difficult.) (d) $y = -\frac{1}{3}\cos^3 x$

2.6 Limits (page 84)

Limits are not seen in algebra. They are special to calculus. You do use algebra to simplify an expression beforehand. But that final gasp of "taking the limit" needs a definition, which involves epsilon (ϵ) and delta (δ).

The idea that $3x+2$ approaches 5 as x approaches 1 is pretty clear. We pin this down (and make it look difficult) by following through on the epsilon-delta definition

$3x+2$ is near 5 (as near as we want) when x is near 1
$(3x+2)-5$ is small (as small as we want) when $x-1$ is small
$|(3x+2)-5| < \epsilon$ (for any fixed $\epsilon > 0$) when $0 < |x-1| < \delta$ (δ depends on ϵ).

This example wants to achieve $|3x-3| < \epsilon$. This will be true if $|x-1| < \frac{1}{3}\epsilon$. So choose δ to be $\frac{1}{3}\epsilon$. By making that particular choice of δ we can say: If $|x-1| < \delta$ then $|(3x+2)-5| < \epsilon$. The number ϵ can be as small as we like. Therefore $L = 5$ is the correct limit.

A small point. Are we saying that "$f(x)$ comes *closer* to L as x comes closer to 1"? No! That is true in this example, but not for all limits. It gives the idea but it is not exactly right. Invent the function $y = x\sin\frac{1}{x}$. Since the sine stays below 1, we have $|y| < \epsilon$ if $|x| < \epsilon$. (This is extra confusing because we can choose $\delta = \epsilon$. The limit of $y = x\sin\frac{1}{x}$ is $L = 0$ as $x \to 0$). The point is that this function actually *hits* zero many times, and then moves *away* from zero, as the number x gets small. It hits zero when $\sin\frac{1}{x} = 0$. It doesn't move far, because y never gets larger than x.

This example does not get *steadily* closer to $L = 0$. It oscillates around its limit. But it converges.

Take $\epsilon = \frac{1}{1000}$ in Questions 1 and 2. Choose δ so that $|f(x) - L| < \frac{1}{1000}$ if $|x| < \delta$.

1. Show that $f(x) = 2x+3$ approaches $L = 3$ as $x \to 0$.

   - Here $|f(x) - L| = |2x|$. We want $|2x| < \epsilon = \frac{1}{1000}$. So we need $|x| < \frac{1}{2000}$. Choose $\delta = \frac{1}{2000}$ or any smaller number like $\delta = \frac{1}{5000}$. Our margin of error on $f(x)$ is $\frac{1}{1000}$ if our margin of error on x is $\frac{1}{2000}$.

2. Show that $\lim_{x\to 0} x^2 = 0$. In other words $x^2 \to 0$ as $x \to 0$. Don't say obvious.

   - $|f(x) - L| = |x^2 - 0| = |x^2|$. We want $|x^2| < \epsilon = \frac{1}{1000}$. This is guaranteed if $|x| < \sqrt{\frac{1}{1000}}$. This square root is a satisfactory δ.

Find the limits in 3–6 if they exist. If direct substitution leads to $\frac{0}{0}$, you need to do more work!

3. $\lim_{x\to 3} \frac{x^2-9}{x-3}$. Substituting $x = 3$ gives $\frac{0}{0}$ which is meaningless.

   - Since $\frac{x^2-9}{x-3} = \frac{(x-3)(x+3)}{(x-3)} = x+3$, substituting $x = 3$ now tells us that the limit is 6.

4. $\lim_{x\to 2} \frac{x^2-4}{\sin(x-2)}$. At $x = 2$ this is $\frac{0}{0}$. But note $x^2 - 4 = (x-2)(x+2)$.

- Write the function as $\frac{(x-2)}{\sin(x-2)}(x+2)$. Since $(x-2) \to 0$ as $x \to 2$, the fraction behaves like $\lim_{\theta\to 0} \frac{\theta}{\sin\theta} = 1$. The other factor $x+2$ goes to 4. The overall limit is $(1)(4) = 4$.

5. $\lim_{x\to 1} \frac{\sqrt{x+8}-3}{x-1}$. Both top and bottom go to zero at $x = 1$.
   - Here is a trick from algebra: Multiply top and bottom by $(\sqrt{x+8}+3)$. This gives $\frac{(x+8)-9}{(x-1)(\sqrt{x+8}+3)}$. The numerator is $x-1$. So cancel that above and below. The fraction approaches $\frac{1}{6}$ when $x \to 1$.

6. $\lim_{x\to 2} \frac{4}{x^2-4} - \frac{1}{x-2}$. Substitution gives "undefined minus undefined." Combine the two fractions into one:

$$\lim_{x\to 2} \frac{4-(x+2)}{(x-2)(x+2)} = \lim_{x\to 2} \frac{-x+2}{(x-2)(x+2)} = \lim_{x\to 2} \frac{-1}{(x+2)} = \frac{-1}{4}.$$

Buried under exercise 34 on page 85 is a handy "important rule" for limits as $x \to \infty$. Use it for 7–8.

7. Find $\lim_{x\to 0} \frac{4x^9-2x^7+18x}{3x^{12}-7x^2+6}$.
   - When x is large, the expression is very like $\frac{4x^9}{3x^{12}}$. This is $\frac{4}{3x^3}$. As $x \to \infty$, this limit is 0. Whenever the top has lower degree than the bottom, the limit as $x \to \infty$ is 0.

8. Find $\lim_{x\to\infty} \frac{(4x^3-3)^6}{25x^{17}}$. You don't have to multiply out $(4x^3-3)^6$.
   - You just have to know that if you did, *the leading term would be* 4^6x^{18}. The limit of $\frac{4^6x^{18}}{25x^{17}} = \frac{4^6x}{25}$ is ∞. This is the limit of the original problem.

9. (This is Problem 2.5.35c) Prove that $\lim_{x\to\infty} x\sin\frac{1}{x} = 1$. (The limit at $x = 0$ was zero!)
   - It is useless to say that $x\sin\frac{1}{x} \to \infty \cdot 0$. Infinity times zero is meaningless. The trick is to write $x\sin\frac{1}{x} = \frac{\sin\frac{1}{x}}{\frac{1}{x}}$. In other words, move x into the denominator as $\frac{1}{x}$. Since $\frac{1}{x} \to 0$ as $x \to \infty$, the limit is $\frac{\sin(\frac{1}{x})}{(\frac{1}{x})} \to 1$.

Read-throughs and selected even-numbered solutions :

The limit of $a_n = (\sin n)/n$ is **zero.** The limit of $a_n = n^4/2^n$ is **zero.** The limit of $a_n = (-1)^n$ is **not defined.** The meaning of $a_n \to 0$ is: Only **finitely many** of the numbers $|a_n|$ can be **greater than** ϵ **(an arbitrary positive number).** The meaning of $a_n \to L$ is: For every ϵ there is an $\mathbf{N}$ such that $|\mathbf{a_n} - \mathbf{L}| < \epsilon$ if $n > \mathbf{N}$. The sequence $1, 1+\frac{1}{2}, 1+\frac{1}{2}+\frac{1}{3}, \cdots$ is not **convergent** because eventually those sums go past **any number L.**

The limit of $f(x) = \sin x$ as $x \to a$ is $\mathbf{\sin a}$. The limit of $f(x) = x/|x|$ as $x \to -2$ is $\mathbf{-1}$, but the limit as $x \to 0$ does not **exist.** This function only has **one-sided** limits. The meaning of $\lim_{x\to a} f(x) = L$ is: For every ϵ there is a δ such that $|f(x) - L| < \epsilon$ whenever $\mathbf{0} < |\mathbf{x} - \mathbf{a}| < \delta$.

Two rules for limits, when $a_n \to L$ and $b_n \to M$, are $a_n + b_n \to \mathbf{L+M}$ and $a_nb_n \to \mathbf{LM}$. The corresponding rules for functions, when $f(x) \to L$ and $g(x) \to M$ as $x \to a$, are $\mathbf{f(x)+g(x)} \to \mathbf{L+M}$ and $\mathbf{f(x)g(x)} \to \mathbf{LM}$. In all limits, $|a_n - L|$ or $|f(x) - L|$ must eventually go below and **stay below** any positive **number** ϵ.

$A \Rightarrow B$ means that A is a **sufficient** condition for B. Then B is true **if** A is true. $A \Leftrightarrow B$ means that A is a **necessary and sufficient** condition for B. Then B is true **if and only if** A is true.

2 (a) is false when $L = 0$: $\mathbf{a_n} = \frac{\mathbf{1}}{\mathbf{n}} \to 0$ and $\mathbf{b_n} = \frac{\mathbf{1}}{\mathbf{n^2}} \to 0$ but $\frac{a_n}{b_n} = n \to \infty$ (b) It is true that: If $a_n \to L$ then $a_n^2 \to L^2$. It is false that: If $a_n^2 \to L^2$ then $a_n \to L$: $\mathbf{a_n}$ **could approach** $-L$ or $a_n = L, -L, L, -L, \cdots$ has no limit. (c) $a_n = -\frac{1}{n}$ is negative but the limit $L = 0$ is **not** negative (d) $1, \frac{1}{2}, 1, \frac{1}{3}, 1, \frac{1}{4}, \cdots$ has infinitely many a_n in every strip around zero but a_n does not approach zero.
8 No limit **10** Limits equals $\mathbf{f'(1)}$ **if the derivative exists.** **12** $\frac{2x \tan x}{\sin x} = \frac{2x}{\cos x} \to \frac{0}{1} = \mathbf{0}$
14 $|x| = -x$ when x is negative; the limit of $\frac{-x}{x}$ is $\mathbf{-1}$. **16** $\frac{f(c)-f(a)}{c-a} \to \mathbf{f'(a)}$ if the derivative exists.
18 $\frac{x^2-25}{x-5} = x + 5$ approaches **10** as $x \to 5$ **20** $\frac{\sqrt{4-x}}{\sqrt{6+x}}$ approaches $\frac{\sqrt{2}}{\sqrt{8}} = \frac{\sqrt{2}}{2\sqrt{2}} = \frac{\mathbf{1}}{\mathbf{2}}$ as $x \to 2$
22 $\sec x - \tan x = \frac{1-\sin x}{\cos x} = \frac{1-\sin x}{\cos x}\left(\frac{1+\sin x}{1+\sin x}\right) = \frac{1-\sin^2 x}{\cos x(1+\sin x)} = \frac{\cos x}{1+\sin x}$ which approaches $\frac{0}{2} = \mathbf{0}$ at $x = \frac{\pi}{2}$.
24 $\frac{\sin(x-1)}{x-1}\left(\frac{1}{x+1}\right)$ approaches $1 \cdot \frac{1}{2} = \frac{\mathbf{1}}{\mathbf{2}}$ as $x \to 1$
28 Given any $\epsilon > 0$ there is an X such that $|f(x)| < \epsilon$ if $x < X$.
32 The limit is $e = 2.718\cdots$

2.7 Continuous Functions (page 89)

Notes on the text: "blows up" means "approaches infinity." Even mathematicians use slang. To understand the Extreme Value Property, place two dots on your paper, and connect them with any function you like. Do not lift your pencil from the paper. The left dot is $(a, f(a))$, the right dot is $(b, f(b))$. Since you did not lift your pencil, your function is continuous on $[a, b]$. The function reaches a maximum (high point) and minimum (low point) somewhere on this closed interval. These extreme points are called $(x_{\max}, M)$ and $(x_{\min}, m)$. It is quite possible that the min or max is reached more than once.

The Extreme Value Property states that m and M are reached *at least once.* Now take a ruler and draw a horizontal line anywhere you like between m and M. The Intermediate Value Property says that, because $f(x)$ is continuous, your line and graph cross at least once.

In 1–3, decide if $f(x)$ is continuous for all x. If not, which requirement is not met? Can $f(x)$ be "fixed" to be continuous?

1. $f(x) = \frac{x^2-9}{x+3}$. This is a standard type of example. At $x = -3$ it gives $\frac{0}{0}$.

   - Note that $f(x) = \frac{(x-3)(x+3)}{(x+3)} = x - 3$. We can remove the difficulty (undefined value) by $f(-3) = -6$.

2. The "sign function" is 1 for positive x and -1 for negative x (and $f(0) = 0$ at the jump).

   - The sign function is continuous except at $x = 0$, where it jumps from -1 to 1. There is no way to redefine $f(0)$ to make this continuous. This $f(x)$ is not "continuable."

3. Suppose $f(x) = 3 + |x|$ except $f(0) = 0$. Is this function "continuable"?

   - Yes. At $x = 0$ the limit $L = 3$ does not equal $f(0) = 0$. Change to $f(0) = 3$.

Exercises 1–18 are excellent for understanding continuity and differentiability. A few solutions are worked out here. Find a number c (*if possible*) to make the function continuous and differentiable.

4. Problem 2.7.5 has $f(x) = \begin{cases} c+x & \text{for } x < 0 \\ c^2+x^2 & \text{for } x \geq 0. \end{cases}$ The graph is a straight line then a parabola.

   - For continuity, the line $y = c + x$ must be made to meet the parabola $y = c^2 + x^2$ at $x = 0$. This means $c = c^2$, so $c = 0$ or 1. The slope is 1 from the left and 0 from the right. This function cannot be made differentiable at $x = 0$.

5. Problem 2.7.7 has $f(x) = \begin{cases} 2x, & x < c \\ x+1, & x \geq c. \end{cases}$ A line then another line.

   - $y = 2x$ meets $y = x + 1$ at $x = 1$. If $c = 1$ then $f(x)$ is continuous. It is not differentiable because the lines have different slopes 2 and 1. If $c \neq 1$ then $f(x)$ is not even continuous. The lines don't meet.

6. Problem 2.7.12 has $f(x) = \begin{cases} c & x \leq 0 \\ \sec x & x > 0. \end{cases}$ A constant (horizontal line) and a curve.

   - At $x = 0$ the limit of $\sec x = \frac{1}{\cos x}$ is $\frac{1}{1} = 1$. So if $c = 1$, the function is continuous at $x = 0$. The slope of $f(x) = \sec x$ is $\sec x \ \tan x$, which is 0 at $x = 0$. Therefore the function is differentiable at 0 if $c = 1$. However, $\sec x$ is undefined at $x = \frac{\pi}{2}, \frac{3\pi}{2}, \cdots$.

7. Problem 2.7.13 has $f(x) = \begin{cases} \frac{x^2+c}{x-1}, & x \neq 1 \\ 2 & x = 1. \end{cases}$ The expression $\frac{x^2+c}{x-1}$ is undefined at $x = 1$.

   - If we choose $c = -1$, the fraction $\frac{x^2+x}{x-1}$ reduces to $x + 1$. This is good. The value at $x = 1$ agrees with $f(1) = 2$. Then $f(x)$ is both continuous and differentiable if $c = -1$. Remember that $f(x)$ must be continuous if it is differentiable. *Not vice versa*!

Read-throughs and selected even-numbered solutions :

Continuity requires the **limit** of $f(x)$ to exist as $x \to a$ and to agree with $\mathbf{f(a)}$. The reason that $x/|x|$ is not continuous at $x = 0$ is : **it jumps from −1 to 1.** This function does have **one-sided** limits. The reason that $1/\cos x$ is discontinuous at $\mathbf{x = \pi/2}$ is **that it approaches infinity.** The reason that $\cos(1/x)$ is discontinuous at $x = 0$ is **infinite oscillation.** The function $f(x) = \mathbf{\frac{1}{x-3}}$ has a simple pole at $x = 3$, where f^2 has a **double** pole.

The power x^n is continuous at all x provided n is **positive.** It has no derivative at $x = 0$ when n is **between 0 and 1.** $f(x) = \sin(-x)/x$ approaches **−1** as $x \to 0$, so this is a **continuous** function provided we define $f(0) = \mathbf{-1}$. A "continuous function" must be continuous at all **points in its domain.** A "continuable function" can be extended to every point x so that **it is continuous.**

If f has a derivative at $x = a$ then f is necessarily **continuous** at $x = a$. The derivative controls the speed at which $f(x)$ approaches $\mathbf{f(a)}$. On a closed interval $[a, b]$, a continuous f has the **extreme** value property and the **intermediate** value property. It reaches its **maximum** M and its **minimum** m, and it takes on every value **in between.**

8 $c > 0$ gives $f(x) = x^c$: For $0 < c < 1$ this is not differentiable at $x = 0$ but is continuous for $(x \geq 0)$. For $c \geq 1$ this is continuous and differentiable where it is defined ($x \geq 0$ for noninteger c).

10 Need $x + c = 1$ at $x = c$ which gives $2c = 1$ or $\mathbf{c} = \mathbf{\frac{1}{2}}$. Then $x + \frac{1}{2}$ matches 1 at $x = \frac{1}{2}$ (continuous but not differentiable).

16 At $x = c$ continuity requires $c^2 = 2c$. Then $\mathbf{c = 0}$ **or 2**. At $x = c$ the derivative jumps from $2x$ to 2.

36 $\cos x$ is greater than $2x$ at $x = 0$; $\cos x$ is less than $2x$ at $x = 1$. The continuous function $\cos x - 2x$ changes from positive to negative. By the **intermediate value theorem** there is a point where $\cos x - 2x = 0$.

38 $x \sin \frac{1}{x}$ approaches zero as $x \to 0$ (so it is continuous) because $|\sin \frac{1}{x}| < 1$. There is no derivative because $\frac{f(h)-f(0)}{h} = \frac{h}{h} \sin \frac{1}{h} = \sin \frac{1}{h}$ has no limit (infinite oscillation).

40 A continuous function is continuous at each point x in its domain (where $f(x)$ is defined). **A continuable** function can be defined at all other points x in such a way that it is continuous there too. $f(x) = \frac{1}{x}$ is continuous away from $x = 0$ but not continuable.

42 $f(x) = x$ if x is a fraction, $f(x) = 0$ otherwise

44 Suppose L is the limit of $f(x)$ as $x \to a$. To prove continuity we have to show that $f(a) = L$.
For any ϵ we can obtain $|f(x) - L| < \epsilon$, and this applies **at** $\mathbf{x = a}$ (since that point is not excluded any more).
Since ϵ is arbitrarily small we reach $f(a) = L$: the function has the right value at $x = a$.

2 Chapter Review Problems

Review Problems

R1 The average slope of the graph of $y(x)$ between two points x and $x + \Delta x$ is ____ .
The slope at the point x is ____ .

R2 For a distance function $f(t)$, the average velocity between times t and $t + \Delta t$ is ____ .
The instantaneous velocity at time t is ____.

R3 Identify these limits as derivatives at specific points and compute them:

(a) $\lim_{x \to 1} \frac{x^6 - 1}{x - 1}$ (b) $\lim_{x \to -1} \frac{x^8 - 1}{x + 1}$ (c) $\lim_{x \to \pi} \frac{\sin x}{x - \pi}$ (d) $\lim_{x \to t} \frac{\cos x - \cos t}{x - t}$

R4 Write down the six terms of $(x + h)^5$. Subtract x^5. Divide by h. Set $h = 0$ to find ____ .

R5 When u increases by Δu and v increases by Δv, how much does uv increase by?
Divide that increase by Δx and let $\Delta x \to 0$ to find $\frac{d}{dx}(uv) =$ ____ .

R6 What is the power rule for the derivative of $1/f(x)$ and specifically of $1/(x^2 + 1)$?

R7 The tangent line to the graph of $y = \tan x - x$ at $x = \frac{\pi}{4}$ is $y =$ ____ .

R8 Find the slope and the equation of the normal line perpendicular to the graph of $y = \sqrt{1 + x^2}$ at $x = 3$.

R9 $f(x) = 0$ for $x \leq 1$ and $f(x) = (x - 1)^2$ for $x > 1$. Find the derivatives $f'(1)$ and $f''(1)$ if they exist.

R10 The limit as $x \to 2$ of $f(x) = x^2 - 4x + 10$ is 6. Find a number δ so that $|f(x) - 6| < .01$ if $|x - 2| < \delta$.

Drill Problems Find the derivative $\frac{dy}{dx}$ in **D1 – D7**.

D1 $y = 3x^5 + 8x^2 - \frac{10}{x} + 7$

D2 $y = (\cos 2x)(\tan \frac{x}{8})$

D3 $y = \sin^3(x-3)$

D4 $y = (x^3+2)/(3-x^2)$

D5 $y = \sqrt{1+\sqrt{x}}$

D6 $y = x \tan x \sin x$

D7 $y = \frac{x}{\sin x} + \frac{\sin x}{x}$

D8 Compute $\lim_{x\to 0} \frac{\sin 2x}{x}$ and $\lim_{x\to 0} \frac{\sin x^2}{1-\cos x}$ and $\lim_{x\to 2} \frac{\sin x}{x}$.

D9 Evaluate the limits as $x \to \infty$ of $\frac{5-x^3}{x^2}$ and $\frac{5-x^2}{x^2}$ and $\frac{5-x}{x^2}$.

D10 For what values of x is $f(x)$ continuous? $f(x) = \frac{5x^3+12}{x^2-8}$ and $f(x) = \sqrt{\frac{x-3}{x^2-9}}$ and $f(x) = \frac{x}{|x|}$.

D11 Draw any curve $y = f(x)$ that goes up and down and up again between $x = 0$ and $x = 4$. Then aligned below it sketch the derivative of $f(x)$. Then aligned below that sketch the second derivative.

Resource: Calculus
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