CHAPTER 4  DERIVATIVES BY THE CHAIN RULE

4.1 The Chain Rule  (page 158)

The function \( \sin(3x + 2) \) is "composed" out of two functions. The inner function is \( u(x) = 3x + 2 \). The outer function is \( \sin u \). I don't write \( \sin x \) because that would throw me off. The derivative of \( \sin(3x + 2) \) is not \( \cos x \) or even \( \cos(3x + 2) \). The chain rule produces the extra factor \( \frac{du}{dx} \), which in this case is the number 3. The derivative of \( \sin(3x + 2) \) is \( \cos(3x + 2) \) times 3.

Notice again: Because the sine was evaluated at \( u \) (not at \( x \)), its derivative is also evaluated at \( u \). We have \( \cos(3x + 2) \) not \( \cos x \). The extra factor 3 comes because \( u \) changes as \( x \) changes:

\[
\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}
\]

These letters can and will change. Many many functions are chains of simpler functions.

1. Rewrite each function below as a composite function \( y = f(u(x)) \). Then find \( \frac{dy}{dx} = f'(u) \frac{du}{dx} \) or \( \frac{dy}{dx} = \frac{du}{dx} \).

   (a) \( y = \tan(3x + 2) \)

   * \( y = \tan(3x + 2) \) is the chain \( y = \tan u \) with \( u = 3x + 2 \). The chain rule gives \( \frac{dy}{dx} = \sec^2(u) \cdot \frac{du}{dx} \) where \( \sec^2(u) = \frac{1}{\cos^2(u)} \).

   Substituting back for \( u \) gives \( \frac{dy}{dx} = \sec^2(3x + 2) \cdot \cos(3x + 2) \) times 3.

   (b) \( y = \cos(3x + 2) \)

   * \( y = \cos(3x + 2) \) separates into \( \cos u \) with \( u = 2x - 5 \). The chain rule gives \( \frac{dy}{dx} = (-\sin u) \cdot \frac{du}{dx} \) where \( \sin(2x - 5) = -2\sin(x) \cdot \cos(x) \).

   * \( y = \frac{1}{(2x - 5)^2} \) is \( y = \frac{1}{u} \) with \( u = 2x - 5 \). The chain rule gives \( \frac{dy}{dx} = \frac{1}{u^2} \cdot \frac{du}{dx} \) where \( \frac{1}{(2x - 5)^2} = 2(2x - 5)^{-2} \) times 3.

2. Write \( y = \sin \sqrt{3x^2 - 5} \) and \( y = \frac{1}{x} \) as triple chains \( y = f(g(3x - 5)) \). Then find \( \frac{dy}{dx} = f'(g(u)) \cdot g'(u) \cdot \frac{du}{dx} \).

   You could write the chain as \( y = f(w), u = g(v) \). Then you see the slope as a product of three factors, \( \frac{dy}{dx} = \left( \frac{dy}{dw} \right) \left( \frac{dw}{du} \right) \left( \frac{du}{dx} \right) \).

   * For \( y(x) = \sin \sqrt{3x^2 - 5} \) the triple chain is \( y = \sin w \), where \( w = \sqrt{u} \) and \( u = 3x^2 - 5 \). The chain rule is \( \frac{dy}{dx} = \left( \frac{dy}{dw} \right) \left( \frac{dw}{du} \right) \left( \frac{du}{dx} \right) = \left( \cos w \right) \left( \frac{1}{2\sqrt{u}} \right) \left( 6x \right) \).

   Substitute to get back to \( x \):

   \[
   \frac{dy}{dx} = \cos \sqrt{3x^2 - 5} \cdot \frac{1}{2\sqrt{3x^2 - 5}} \cdot 6x = \frac{6x \cos \sqrt{3x^2 - 5}}{2\sqrt{3x^2 - 5}}.
   \]

   * For \( y(x) = \frac{1}{\sqrt{x}} \) let \( u = \frac{1}{x} \). Let \( w = 1 - u \). Then \( y = \frac{1}{w} \). The derivative is

   \[
   \frac{dy}{dx} = \left( \frac{dy}{dw} \right) \left( \frac{dw}{du} \right) \left( \frac{du}{dx} \right) = \left( -\frac{1}{w^2} \right) \left( -\frac{1}{u^2} \right) \left( \frac{1}{x^2} \right) = \frac{-1}{(1 - \frac{1}{x})^2x^2} = \frac{-1}{x^2(x - 1)^2}.
   \]

With practice, you should get to the point where it is not necessary to write down \( u \) and \( w \) in full detail. Try this with exercises 1 - 22, doing as many as you need to get good at it. Problems 45 - 54 are excellent practice, too.

Questions 3 - 6 are based on the following table, which gives the values of functions \( f \) and \( f' \) and \( g \) and \( g' \) at a few points. You do not know what these functions are!
4.1 The Chain Rule (page 158)

<table>
<thead>
<tr>
<th>x</th>
<th>f(x)</th>
<th>f'(x)</th>
<th>g(x)</th>
<th>g'(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>undefined</td>
</tr>
<tr>
<td>1/2</td>
<td>1/2</td>
<td>-1/2</td>
<td>1/2</td>
<td>1/2</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

3. Find: \( f(g(4)) \) and \( f(g(1)) \) and \( f(g(0)) \).

- \( g(4) = 2 \) and \( f(2) = \frac{1}{3} \) so \( f(g(4)) = \frac{1}{3} \). Also \( g(1) = 1 \) so \( f(g(1)) = f(1) = \frac{1}{2} \). Then \( f(g(0)) = f(0) = 0 \).

4. Find: \( g(f(1)) \) and \( g(f(2)) \) and \( g(f(0)) \).

- Since \( f(1) = \frac{1}{2} \), the chain \( g(f(1)) \) is \( g(\frac{1}{2}) = \sqrt{2} \). Also \( g(f(2)) = g(\frac{1}{2}) = \sqrt{2} \). Then \( g(f(0)) = g(0) \).

5. If \( y = f(g(x)) \) find \( \frac{dy}{dx} \) at \( x = 9 \).

- The chain rule says that \( \frac{dy}{dx} = f'(g(x)) \cdot g'(x) \). At \( x = 9 \) we have \( g(9) = 3 \) and \( g'(9) = \frac{1}{6} \). Therefore at \( x = 9 \), \( \frac{dy}{dx} = f'(g(9)) \cdot g'(9) = -\frac{1}{16} \cdot \frac{1}{6} = -\frac{1}{96} \).

6. If \( y = g(f(x)) \) find \( \frac{dy}{dx} \) at \( x = 2 \). This chain repeats the same function \( f = g \). It is "iteration."

7. If \( y = f(f(x)) \) find \( \frac{dy}{dx} \) at \( x = 2 \). The chain repeats the same function \( f = g \). It is 8.

8. \( \frac{dz}{dx} \) is evaluated at \( y = x^2 - 1 \) (not at \( y = z \)). For \( x \) in the table gives \( u = \frac{1}{3} \).

9. The proof of the chain rule begins with \( \Delta z/\Delta x = (\Delta z/\Delta y) (\Delta y/\Delta x) \) and ends with \( dz/dx = (dz/dy)(dy/dx) \).

10. Changing letters, \( y = \cos u(x) \) has \( dy/dx = -\sin u(x) \frac{du}{dx} \). The power rule for \( y = |u(x)|^{1/n} \) is the chain rule \( dy/dx = nu^{n-1} \frac{du}{dx} \). The slope of \( 5g(x) \) is \( 5g'(x) \) and the slope of \( g(5x) \) is \( 5g'(5x) \). When \( f = \cosine \) and \( g = \sin x \) and \( x = 0 \), the numbers \( f(g(x)) \) and \( g(f(x)) \) and \( f(x)g(x) \) are 1 and \( \sin 1 \) and 0.

Read-throughs and selected even-numbered solutions:

- \( z = f(g(x)) \) comes from \( z = f(y) \) and \( y = g(x) \). At \( x = 2 \) the chain \( (x^2 - 1)^3 \) equals \( 3^3 = 27 \). Its inside function is \( y = x^2 - 1 \), its outside function is \( z = y^3 \). Then \( dz/dx \) is evaluated at \( x = 0 \) (not at \( y = z \)). Also \( f'(2) = 1 \) and \( \sin 1 \) and 0.

11. The chain rule produces \( -1 \) so derivatives of even functions are odd functions.

12. False (The chain rule produces \( -1 \) so derivatives of even functions are odd functions)

13. False (The chain rule produces \( -1 \) so derivatives of even functions are odd functions)

14. False (The chain rule produces \( -1 \) so derivatives of even functions are odd functions)

15. False (The chain rule produces \( -1 \) so derivatives of even functions are odd functions)
4.2 Implicit Differentiation and Related Rates (page 163)

Questions 1 – 5 are examples using implicit differentiation (ID).

1. Find \( \frac{dy}{dx} \) from the equation \( x^2 + xy = 2 \). Take the \( x \) derivative of all terms.
   - The derivative of \( x^2 \) is \( 2x \). The derivative of \( xy \) (a product) is \( x \frac{dy}{dx} + y \). The derivative of \( 2 \) is 0. Thus \( 2x + x \frac{dy}{dx} + y = 0 \), and \( \frac{dy}{dx} = -\frac{y + 2x}{x} \).

   In this example the original equation can be solved for \( y = \frac{1}{x}(2 - x^2) \). Ordinary explicit differentiation yields \( \frac{dy}{dx} = -\frac{2}{x^2} - 1 \). This must agree with our answer from ID.

2. Find \( \frac{dy}{dx} \) from \( (x + y)^3 = x^4 + y^4 \). This time we cannot solve for \( y \).
   - The chain rule tells us that the \( x \)-derivative of \( (x + y)^3 \) is \( 3(x + y)^2 (1 + \frac{dy}{dx}) \). Therefore ID gives \( 3(x + y)^2 (1 + \frac{dy}{dx}) = 4x^3 + 4y^3 \frac{dy}{dx} \). Now algebra separates out \( \frac{dy}{dx} = \frac{3(x+y)^2 - 4y^3}{4x^3 - 3(x+y)^2} \).

3. Use ID to find \( \frac{dy}{dx} \) for \( y = x\sqrt{1 - x} \).
   - Implicit differentiation (ID for short) is not necessary, but you might appreciate how it makes the problem easier. Square both sides to eliminate the square root: \( y^2 = x^2(1 - x) = x^2 - x^3 \), so that
     \[
     2y \frac{dy}{dx} = 2x - 3x^2 \quad \text{and} \quad \frac{dy}{dx} = \frac{2x - 3x^2}{2y} = \frac{2x - 3x^2}{2\sqrt{1 - x}} = \frac{2 - 3x}{2\sqrt{1 - x}}.
     \]

4. Find \( \frac{d^2y}{dx^2} \) when \( xy + y^2 = 1 \). Apply ID twice to this equation.
   - First derivative: \( x \frac{dy}{dx} + y + 2y \frac{dy}{dx} = 0 \). Rewrite this as \( \frac{dy}{dx} = -\frac{y}{x + 2y} \). Now take the derivative again.
     The second form needs the quotient rule, so I prefer to use ID on the first derivative equation:
     \[
     x \frac{d^2y}{dx^2} + \frac{dy}{dx} + 2y \frac{d^2y}{dx^2} + 2(\frac{dy}{dx})^2 = 0 \quad \text{or} \quad \frac{d^2y}{dx^2} = -\frac{2 \frac{dy}{dx} + (\frac{dy}{dx})^2}{x + 2y}.
     \]
     Now substitute \( -\frac{y}{x + 2y} \) for \( \frac{dy}{dx} \) and simplify the answer to \( \frac{d^2y}{dx^2} = -\frac{2}{(x + 2y)^3} \).

5. Find the equation of the tangent line to the ellipse \( x^2 + xy + y^2 = 1 \) through the point (1,0).
   - The line has equation \( y = m(x - 1) \) where \( m \) is the slope at (1,0). To find that slope, apply ID to the equation of the ellipse: \( 2x + x \frac{dy}{dx} + y + 2y \frac{dy}{dx} = 0 \). Do not bother to solve this for \( \frac{dy}{dx} \). Just plug in \( x = 1 \) and \( y = 0 \) to obtain \( 2 + \frac{dy}{dx} = 0 \). Then \( m = \frac{dy}{dx} = -2 \) and the tangent equation is \( y = -2(x - 1) \).
Questions 6–8 are problems about related rates. The slope of one function is known, we want the slope of a related function. Of course slope = rate = derivative. You must find the relation between functions.

6. Two cars leave point A at the same time \( t = 0 \). One travels north at 65 miles/hour, the other travels east at 55 miles/hour. How fast is the distance \( D \) between the cars changing at \( t = 2 \)?

- The distance satisfies \( D^2 = x^2 + y^2 \). This is the relation between our functions! Find the rate of change (take the derivative): \( 2D \frac{dD}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt} \). We need to know \( \frac{dD}{dt} \) at \( t = 2 \). We already know \( \frac{dx}{dt} = 55 \) and \( \frac{dy}{dt} = 65 \). At \( t = 2 \) the cars have traveled for two hours: \( x = 2(55) = 110 \), \( y = 2(65) = 130 \) and \( D = \sqrt{110^2 + 130^2} \approx 170.3 \).

Substituting these values gives \( 2(170.3) \frac{dD}{dt} = 2(110)(55) + 2(130)(65) \), so \( \frac{dD}{dt} \approx 85 \) miles/hour.

7. Sand pours out from a conical funnel at the rate of 5 cubic inches per second. The funnel is 6" wide at the top and 6" high. At what rate is the sand height falling when the remaining sand is 1" high?

- Ask yourself what rate(s) you know and what rate you want to know. In this case you know \( \frac{dV}{dt} = -5 \) (V is the volume of the sand). You want to know \( \frac{dh}{dt} \) when \( h = 1 \) (h is the height of the sand). Can you get an equation relating \( V \) and \( h \)? This is usually the crux of the problem.

The volume of a cone is \( V = \frac{1}{3} \pi r^2 h \). If we could eliminate \( r \), then \( V \) would be related to \( h \). Look at the figure. By similar triangles \( \frac{r}{h} = \frac{3}{6} \), so \( r = \frac{1}{2} h \). This means that \( V = \frac{1}{3} \pi \left(\frac{h}{2}\right)^2 h = \frac{1}{12} \pi h^3 \).

Now take the \( t \) derivative: \( \frac{dV}{dt} = \frac{1}{12} \pi (3h^2) \frac{dh}{dt} \). After the derivative has been taken, substitute what is known at \( h = 1 \): \( -5 = \frac{1}{12} \pi (3) \frac{dh}{dt} \), so \( \frac{dh}{dt} = \frac{-20}{\pi} \) in/sec \( \approx -6.4 \) in/sec.

8. (This is Problem 4.2.21) The bottom of a 10-foot ladder moves away from the wall at 2 ft/sec. How fast is the top going down the wall when the top is (a) 6 feet high? (b) 5 feet high? (c) zero feet high?

- We are given \( \frac{dx}{dt} = 2 \). We want to know \( \frac{dy}{dt} \). The equation relating \( x \) and \( y \) is \( x^2 + y^2 = 100 \). This gives \( 2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0 \). Substitute \( \frac{dx}{dt} = 2 \) to find \( \frac{dy}{dt} = -\frac{2x}{y} \).

(a) If \( y = 6 \), then \( x = 8 \) (use \( x^2 + y^2 = 100 \)) and \( \frac{dx}{dt} = -\frac{8}{3} \) ft/sec.

(b) If \( y = 5 \), then \( x = 5\sqrt{3} \) (use \( x^2 + y^2 = 100 \)) and \( \frac{dx}{dt} = -2\sqrt{3} \) ft/sec.

(c) If \( y = 0 \), then we are dividing by zero: \( \frac{dy}{dx} = -\frac{8}{0} \). Is the speed infinite? How is this possible?

Read-throughs and selected even-numbered solutions:

For \( x^2 + y^2 = 2 \) the derivative \( dy/dx \) comes from implicit differentiation. We don’t have to solve for \( y \). Term by term the derivative is \( 2x + 2y \frac{dy}{dx} = 0 \). Solving for \( dy/dx \) gives \(-x^2/y^2 \). At \( x = y = 1 \) this slope is \(-1 \). The equation of the tangent line is \( y - 1 = -1(x - 1) \).

A second example is \( y^2 = x \). The \( x \) derivative of this equation is \( 2y \frac{dy}{dx} = 1 \). Therefore \( dy/dx = 1/2y \).

Replacing \( y \) by \( \sqrt{x} \) this is \( dy/dx = 1/2\sqrt{x} \).

In related rates, we are given \( dg/dt \) and we want \( df/dt \). We need a relation between \( f \) and \( g \). If \( f = g^2 \), then \( (df/dt) = 2g(dg/dt) \). If \( f^2 + g^2 = 1 \), then \( df/dt = -g \frac{dg}{dt} \). If the sides of a cube grow by \( ds/dt = 2 \), then its volume grows by \( dV/dt = 3s^2(2) = 6s^2 \). To find a number (8 is wrong), you also need to know \( s \).
4.3 Inverse Functions and Their Derivatives (page 170)

6 \( f'(x) + F'(y) \frac{dy}{dx} = y + x \frac{dy}{dx} \) so \( \frac{dx}{dy} = \frac{y-F'(x)}{y'} \)

12 \( 2(x-2) + 2y \frac{dy}{dx} = 0 \) gives \( \frac{dy}{dx} = 1 \) at \((1,1)\); \( 2x + 2(y-2) \frac{dy}{dx} = 0 \) also gives \( \frac{dy}{dx} = 1 \).

20 \( x \) is a constant (fixed at 7) and therefore a change \( \Delta x \) is not allowed.

24 Distance to you is \( \sqrt{x^2 + y^2} \), rate of change is \( \frac{x}{\sqrt{x^2 + y^2}} \) and \( \frac{dx}{dt} = 560 \). (a) Distance is 16 and \( x = 8\sqrt{3} \) and rate is \( \frac{8\sqrt{3}}{16} \) of 560 = 280 \( \sqrt{3} \); (b) \( x = 8 \) and rate is \( \frac{8}{\sqrt{x^2 + y^2}} \) of 560 = 280 \( \sqrt{2} \); (c) \( x = 0 \) and rate is 0.

28 Volume = \( \frac{4}{3} \pi r^3 \) has \( \frac{dv}{dt} = 4\pi r^2 \frac{dr}{dt} \). If this equals twice the surface area \( 4\pi r^2 \) (with minus for evaporation) than \( \frac{dr}{dt} = -2 \).

4.3 Inverse Functions and Their Derivatives (page 170)

The vertical line test and the horizontal line test are good for visualising the meaning of "function" and "invertible." If a vertical line hits the graph twice, we have two \( y \)'s for the same \( x \). Not a function. If a horizontal line hits the graph twice, we have two \( x \)'s for the same \( y \). Not invertible. This means that the inverse is not a function.

These tests tell you that the sideways parabola \( x = y^2 \) does not give \( y \) as a function of \( x \). (Vertical lines intersect the graph twice. There are two square roots \( y = \sqrt{x} \) and \( y = -\sqrt{x} \).) Similarly the function \( y = x^2 \) has no inverse. This is an ordinary parabola − horizontal lines cross it twice. If \( y = 4 \) then \( x = f^{-1}(4) \) has two answers \( x = 2 \) and \( x = -2 \). In questions 1 − 2 find the inverse function \( x = f^{-1}(y) \).

1. \( y = x^2 + 2 \). This function fails the horizontal line test. It has no inverse. Its graph is a parabola opening upward, which is crossed twice by some horizontal lines (and not crossed at all by other lines).

Here's another way to see why there is no inverse: \( x^2 = y - 2 \) leads to \( x = \pm \sqrt{y - 2} \). Then \( x^+ = \sqrt{y - 2} \) represents the right half of the parabola, and \( x = -\sqrt{y - 2} \) is the left half. We can get an inverse by reducing the domain of \( y = x^2 + 2 \) to \( x \geq 0 \). With this restriction, \( x = f^{-1}(y) = \sqrt{y - 2} \). The positive square root is the inverse. The domain of \( f(x) \) matches the range of \( f^{-1}(y) \).

2. \( y = f(x) = \frac{x}{x-1} \). (This is Problem 4.3.4) Find \( x \) as a function of \( y \).

- Write \( y = \frac{x}{x-1} \) as \( y(x-1) = x \) or \( yz - y = x \). We always have to solve for \( x \). We have \( yz - x = y \) or \( z(x-1) = y \) or \( x = \frac{y+1}{y-1} \). Therefore \( f^{-1}(y) = \frac{x}{y-1} \).

Note that \( f \) and \( f^{-1} \) are the same! If you graph \( y = f(x) \) and the line \( y = x \) you will see that \( f(x) \) is symmetric about the 45° line. In this unusual case, \( x = f(y) \) when \( y = f(x) \).
You might wonder at the statement that \( f(x) = \frac{x}{x-1} \) is the same as \( g(y) = \frac{y}{y-1} \). The definition of a function does not depend on the particular choice of letters. The functions \( h(r) = \frac{r}{r-1} \) and \( F(t) = \frac{t}{t-1} \) and \( G(x) = \frac{x}{x-1} \) are also the same. To graph them, you would put \( r, t, \) or \( x \) on the horizontal axis—they are the input (domain) variables. Then \( h(r) \), \( F(t) \), \( G(x) \) would be on the vertical axis as output variables.

The function \( y = f(x) = 3x \) and its inverse \( x = f^{-1}(y) = \frac{1}{3}y \) (absolutely not \( \frac{1}{3}y \)) are graphed on page 167. For \( f(x) = 3x \), the domain variable \( x \) is on the horizontal axis. For \( f^{-1}(y) = \frac{1}{3}y \), the domain variable for \( f^{-1} \) is \( y \).

This can be confusing since we are so accustomed to seeing \( x \) along the horizontal axis. The advantage of \( f(x) = 3x \) is that it allows you to keep \( x \) on the horizontal and to stick with \( x \) for domain (input). The advantage of \( f^{-1}(y) = \frac{1}{3}y \) is that it emphasizes: \( f \) takes \( x \) to \( y \) and \( f^{-1} \) takes \( y \) back to \( x \).

3. (This is 4.3.34) Graph \( y = |x| - 2x \) and its inverse on separate graphs.

- \( y = |x| - 2x \) should be analyzed in two parts: positive \( x \) and negative \( x \). When \( x \geq 0 \) we have \( |x| = x \).
  - The function is \( y = x - 2x = -x \). When \( x \) is negative we have \( |x| = -x \). Then \( y = -x \) on the right of the \( y \) axis and \( y = -3x \) on the left. Inverses \( x = -y \) and \( x = -\frac{y}{3} \). The second graph shows the inverse function.

4. Find \( \frac{dx}{dy} \) when \( y = x^2 + z \). Compare implicit differentiation with \( \frac{1}{dy/dx} \).

- The \( x \) derivative of \( y = x^2 + z \) is \( \frac{dy}{dx} = 2x + 1 \). Therefore \( \frac{dx}{dy} = \frac{1}{2x+1} \).
- The \( y \) derivative of \( y = x^2 + z \) is \( 1 = 2x \frac{dx}{dy} + \frac{dz}{dy} = (2x + 1) \frac{dz}{dy} \). This also gives \( \frac{dx}{dy} = \frac{1}{2x+1} \).
- It might be desirable to know \( \frac{dx}{dy} \) as a function of \( y \), not \( x \). In that case solve the quadratic equation \( x^2 + x - y = 0 \) to get \( x = \frac{-1 \pm \sqrt{1+4y}}{2} \). Substitute this into \( \frac{dx}{dy} = \frac{1}{2x+1} = \frac{1}{\pm1 + \sqrt{1+4y}} \).
- Now we know \( x = \frac{-1 \pm \sqrt{1+4y}}{2} \) (this is the inverse function). So we can directly compute \( \frac{dz}{dy} = \pm \frac{1}{2} \frac{1}{\pm1 \sqrt{1+4y}} \frac{1}{\pm1 + \sqrt{1+4y}} \). Same answer four ways!

5. Find \( \frac{dz}{dy} \) at \( x = \pi \) for \( y = \cos x + x^2 \).

\[ \frac{dy}{dx} = -\sin x + 2x. \] Substitute \( x = \pi \) to find \( \frac{dy}{dx} = -\sin \pi + 2\pi = 2\pi \). Therefore \( \frac{dz}{dy} = \frac{1}{2\pi} \).

Read-throughs and selected even-numbered solutions:

The functions \( g(x) = x - 4 \) and \( f(y) = y + 4 \) are inverse functions, because \( f(g(x)) = x \). Also \( g(f(y)) = y \). The notation is \( f = g^{-1} \) and \( g = f^{-1} \). The composition of \( f \) and \( f^{-1} \) is the identity function. By definition
4.4 Inverses of Trigonometric Functions

4.4 Inverses

The function $g$ must be steadily increasing or steadily decreasing.

The chain rule applied to $f(g(x)) = x$ gives $(df/dy)(dg/dx) = 1$. More directly $dx/dy = 1/(dy/dx)$. For $y = 2x + 1$ and $x = \frac{1}{2}(y - 1)$, the slopes are $dy/dx = 2$ and $dx/dy = \frac{1}{2}$. For $y = x^2$ and $x = \sqrt{y}$, the slopes are $dy/dx = 2x$ and $dx/dy = 1/2\sqrt{y}$. Substituting $x^2$ for $y$ gives $dx/dy = 1/2x$. Then $(dx/dy)(dy/dx) = 1$.

The graph of $y = g(x)$ is also the graph of $x = g^{-1}(y)$, but with $x$ across and $y$ up. For an ordinary graph of $g^{-1}$, take the reflection in the line $y = x$. If $(3,8)$ is on the graph of $g$, then its mirror image $(8,3)$ is on the graph of $g^{-1}$. Those particular points satisfy $8 = 2^3$ and $3 = \log_2 8$.

The inverse of the chain $z = h(g(z))$ is the chain $x = g^{-1}(h^{-1}(z))$. If $g(x) = 3x$ and $h(y) = y^3$ then $z = (3x)^3 = 27x^3$. Its inverse is $x = \frac{1}{3}y^{1/3}$, which is the composition of $g^{-1}(y) = \frac{1}{3}y$ and $h^{-1}(z) = z^{1/3}$.

The table on page 175 summarizes what you need to know - the six inverse trig functions, their domains, and their derivatives. The table gives $\frac{dx}{dy}$ since the inverse functions have input $y$ and output $x$. The input $y$ is a number and the output $x$ is an angle. Watch the restrictions on $y$ and $x$ (to permit an inverse).

1. Compute (a) $\sin^{-1}(\sin \frac{\pi}{4})$ (b) $\cos^{-1}(\sin \frac{\pi}{3})$ (c) $\sin^{-1}(\sin \pi)$ (d) $\tan^{-1}(\cos 0)$ (e) $\cos^{-1}(\cos(-\frac{\pi}{2}))$

- (a) $\sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$ and $\sin^{-1} \frac{\sqrt{2}}{2}$ brings us back to $\frac{\pi}{4}$.
- (b) $\sin \frac{\pi}{3} = \frac{1}{2}$ and then $\cos^{-1}(\frac{1}{2}) = \pm \frac{\pi}{3}$. Note that $\frac{\pi}{3} + \frac{2\pi}{3} = \frac{5\pi}{3}$. The angles $\frac{\pi}{3}$ and $\frac{2\pi}{3}$ are complementary (they add to 90° or $\frac{\pi}{2}$). Always $\sin^{-1} y + \cos^{-1} y = \frac{\pi}{2}$.
- (c) $\sin^{-1}(\sin \pi)$ is not $\pi$! Certainly $\sin \pi = 0$. But $\sin^{-1}(0) = 0$. The $\sin^{-1}$ function or arccos function only yields angles between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.
- (d) $\tan^{-1}(\cos 0) = \tan^{-1} 1 = \frac{\pi}{4}$
- (e) $\cos^{-1}(\cos(-\frac{\pi}{2}))$ looks like $-\frac{\pi}{2}$. But $\cos(-\frac{\pi}{2}) = 0$ and then $\cos^{-1}(0) = \frac{\pi}{2}$.
2. Find \( \frac{dx}{dy} \) if \( x = \sin^{-1} 3y \). What are the restrictions on \( y \)?

We know that \( x = \sin^{-1} u \) yields \( \frac{dx}{du} = \frac{1}{\sqrt{1-u^2}} \). Set \( u = 3y \) and use the chain rule: \( \frac{dx}{dy} = \frac{3}{\sqrt{1-u^2}} = \frac{3}{\sqrt{1-9y^2}} \). The restriction \( |u| \leq 1 \) on sines means that \( |3y| \leq 1 \) and \( |y| \leq \frac{1}{3} \).

3. Find \( \frac{dx}{dz} \) when \( z = \cos^{-1}\left(\frac{1}{x}\right) \). What are the restrictions on \( x \)?

\( \cos^{-1} \) accepts inputs between \(-1\) and \(1\), inclusive. For this reason \( \frac{1}{x} \leq 1 \) and \( |x| \geq 1 \). To find the derivative, use the chain rule with \( z = \cos^{-1} u \) and \( u = \frac{1}{x} \):

\[
\frac{dz}{dz} = \frac{dz}{du} \cdot \frac{du}{dz} = \frac{-1}{\sqrt{1-u^2}} \cdot \frac{1}{x^2} = \frac{1}{x\sqrt{x^2-1}}.
\]

Here is another way to do this problem. Since \( y = \sec^{-1} \sqrt{x^2+1} \), we have \( \sec y = \sqrt{x^2+1} \) and \( \sec^2 y = x^2 + 1 \). This is a trig identity provided \( z = \pm \tan y \). Then \( y = \pm \tan^{-1} x \) and \( \frac{dy}{dx} = \pm \frac{1}{x^2+1} \).

4. Find \( \frac{dy}{dz} \) when \( y = \sec^{-1} \sqrt{z^2+1} \). (This is Problem 4.4.23)

a. The derivative of \( y = \sec^{-1} u \) is \( \frac{1}{|u|\sqrt{u^2-1}} \). In this problem \( u = \sqrt{z^2+1} \). Then

\[
\frac{dy}{dz} = \frac{dy}{du} \cdot \frac{du}{dz} = \frac{1}{|u|\sqrt{u^2-1}} \cdot \frac{x}{\sqrt{z^2+1}} = \frac{1}{x\sqrt{x^2+1}}.
\]

Here is another way to do this problem. Since \( y = \sec^{-1} \sqrt{x^2+1} \), we have \( \sec y = \sqrt{x^2+1} \) and \( \sec^2 y = x^2 + 1 \). This is a trig identity provided \( z = \pm \tan y \). Then \( y = \pm \tan^{-1} x \) and \( \frac{dy}{dz} = \pm \frac{1}{x^2+1} \).

5. Find \( \frac{dy}{dx} \) if \( y = \tan^{-1} \frac{2}{x} - \cot^{-1} \frac{x}{2} \). Explain zero.

a. The derivative of \( \tan^{-1} \frac{2}{x} \) is \( \frac{1}{1+\left(\frac{2}{x}\right)^2} \cdot \frac{-2}{x^2} = \frac{-2}{x^2+4} \). The derivative of \( \cot^{-1} \frac{x}{2} \) is \( -\frac{1}{1+\left(\frac{x}{2}\right)^2} \cdot \frac{1}{2} = -\frac{2}{x^2+4} \).

By subtraction \( \frac{dy}{dx} = 0 \). Why do \( \tan^{-1} \frac{2}{x} \) and \( \cot^{-1} \frac{x}{2} \) have the same derivative? Are they equal?

Think about domain and range before you answer that one.

The relation \( x = \sin^{-1} y \) means that \( y \) is the sine of \( x \). Thus \( x \) is the angle whose sine is \( y \). The number \( y \) lies between \(-1\) and \(1\). The angle \( x \) lies between \(-\pi/2\) and \(\pi/2\). (If we want the inverse to exist, there cannot be two angles with the same sine.) The cosine of the angle \( \sin^{-1} y \) is \( \sqrt{1-y^2} \). The derivative of \( x = \sin^{-1} y \) is \( \frac{dx}{dy} = 1/\sqrt{1-y^2} \).

The relation \( x = \cos^{-1} y \) means that \( y \) equals \( \cos x \). Again the number \( y \) lies between \(-1\) and \(1\). This time the angle \( x \) lies between \(0\) and \(\pi\) (so that each \( y \) comes from only one angle \( x \)). The sum \( \sin^{-1} y + \cos^{-1} y = \pi/2 \). (The angles are called complementary, and they add to a right angle.) Therefore the derivative of \( x = \cos^{-1} y \) is \( \frac{dx}{dy} = -1/\sqrt{1-y^2} \), the same as for \( \sin^{-1} y \) except for a minus sign.

The relation \( x = \tan^{-1} y \) means that \( y = \tan x \). The number \( y \) lies between \(-\infty\) and \(\infty\). The angle \( x \) lies between \(-\pi/2\) and \(\pi/2\). The derivative is \( \frac{dx}{dy} = 1/(1+y^2) \). Since \( \tan^{-1} y + \cot^{-1} y = \pi/2 \), the derivative of \( \cot^{-1} y \) is the same except for a minus sign.

The relation \( x = \sec^{-1} y \) means that \( y = \sec x \). The number \( y \) never lies between \(-1\) and \(1\). The angle \( x \) lies between \(0\) and \(\pi\), but never at \( x = \pi/2 \). The derivative of \( x = \sec^{-1} y \) is \( \frac{dx}{dy} = 1/|y|\sqrt{y^2-1} \).
The sides of the triangle are \( y, \sqrt{1-y^2}, \) and 1. The tangent is \( \frac{y}{\sqrt{1-y^2}}. \)

14 \( \frac{d(sin^{-1}y)}{dy}|_{y=0} = 1, \frac{d(cos^{-1}y)}{dy}|_{y=0} = -\infty; \frac{d(tan^{-1}y)}{dy}|_{y=0} = 1, \frac{d(sin^{-1}y)}{dy}|_{y=1} = \frac{1}{\cos 1}, \frac{d(cos^{-1}y)}{dy}|_{y=1} = \frac{1}{\sin 1} \)

16 \( cos^{-1}(sin x) \) is the complementary angle \( \frac{\pi}{2} - x. \) The tangent of that angle is \( \frac{\cos x}{\sin x} = \cot x. \)

S4 The requirement is \( u' = \frac{1}{1+y^2}. \) To satisfy this requirement take \( u = \tan^{-1}t. \)

36 \( u = \tan^{-1} y \) has \( \frac{du}{dy} = -\frac{1}{1+y^2} \) and \( \frac{d^2u}{dy^2} = \frac{-2y}{(1+y^2)^2}. \)

42 By the product rule \( \frac{d}{dx}(\cos x)(\sin^{-1}x) + \frac{1}{\sqrt{1-x^2}}. \) Note that \( x \neq \pm \) and \( \frac{d}{dx} \neq 1. \)

48 \( u(x) = \frac{1}{2}\tan^{-1}2x \) (need \( \frac{1}{2} \) to cancel 2 from the chain rule).

50 \( u(x) = \frac{x-1}{x+1} \) has \( \frac{du}{dx} = \frac{1}{(x+1)^2} = \frac{1}{x+1}. \) Then \( \frac{d}{dx}\tan^{-1}u(x) = \frac{1}{1+u^2} \frac{du}{dx} = \frac{1}{1+(\frac{1}{x+1})^2} \frac{1}{x+1} = \frac{1}{x^2+1}. \) This is also the derivative of \( \tan^{-1}x! \) So \( \tan^{-1}u(x) - \tan^{-1}x \) is a constant.

4 Chapter Review Problems

Review Problems

R1 Give the domain and range of the six inverse trigonometric functions.

R2 Is the derivative of \( u(u(x)) \) ever equal to the derivative of \( u(x)u(x)? \)

R3 Find \( y' \) and the second derivative \( y'' \) by implicit differentiation when \( y^2 = x^2 + xy. \)

R4 Show that \( y = x + 1 \) is the tangent line to the graph of \( y = x + \cos xy \) through the point \((0,1).\)

R5 If the graph of \( y = f(x) \) passes through the point \((a, b)\) with slope \( m, \) then the graph of \( y = f^{-1}(x) \) passes through the point ______ with slope ______.

R6 Where does the graph of \( y = \cos x \) intersect the graph of \( y = \cos^{-1}x? \) Give an equation for \( x \) and show that \( x \approx .7391 \) in Section 3.6 is a solution.

R7 Show that the curves \( xy = 4 \) and \( x^2 - y^2 = 15 \) intersect at right angles.

R8 "The curve \( y^2 + x^2 + 1 = 0 \) has \( 2y \frac{dy}{dx} + 2x = 0 \) so its slope is \(-x/y.\)" What is the problem with that statement?

R9 Gas is escaping from a spherical balloon at 2 cubic feet/minute. How fast is the surface area shrinking when the area is 576\( \pi \) square feet?
R10  A 50 foot rope goes up over a pulley 18 feet high and diagonally down to a truck. The truck drives away at 9 ft/sec. How fast is the other end of the rope rising from the ground?

R11  Two concentric circles are expanding, the outer radius at 2 cm/sec and the inner radius at 5 cm/sec. When the radii are 10 cm and 3 cm, how fast is the area between them increasing (or decreasing)?

R12  A swimming pool is 25 feet wide and 100 feet long. The bottom slopes steadily down from a depth of 3 feet to 10 feet. The pool is being filled at 100 cubic feet/minute. How fast is the water level rising when it is 6 feet deep at the deep end?

R13  A five-foot woman walks at night toward a 12-foot street lamp. Her speed is 4 ft/sec. Show that her shadow is shortening by \( \frac{20}{7} \) ft/sec when she is 3 feet from the lamp.

R14  A 40 inch string goes around an 8 by 12 rectangle – but we are changing its shape (same string). If the 8 inch sides are being lengthened by 1 inch/second, how fast are the 12 inch sides being shortened? Show that the area is increasing at 4 square inches per second. (For some reason it will take two seconds before the area increases from 96 to 100.)

R15  The volume of a sphere (when we know the radius) is \( V(r) = \frac{4}{3} \pi r^3 \). The radius of a sphere (when we know the volume) is \( r(V) = \left( \frac{3V}{4\pi} \right)^{1/3} \). This is the inverse! The surface area of a sphere is \( A(r) = 4\pi r^2 \). The radius (when we know the area) is \( r(A) = \). The chain \( r(A(r)) = \).

R16  The surface area of a sphere (when we know the area) is \( A(V) = \frac{4\pi (3V/4\pi)^{2/3}}{3} \). The volume (when we know the area) is \( V(A) = \).

Drill Problems   (Find \( dy/dx \) in Problems D1 to D6).

D1  \[ y = t^3 - t^2 + 2 \text{ with } t = \sqrt{x} \]
D2  \[ y = \sin^3(2x - \pi) \]
D3  \[ y = \tan^{-1}(4x^2 + 7x) \]
D4  \[ y = \csc \sqrt{x} \]
D5  \[ y = \sin(\sin^{-1} x) \text{ for } |x| \leq 1 \]
D6  \[ y = \sin u \cos u \text{ with } u = \cos^{-1} x \]

In D7 to D10 find \( y' \) by implicit differentiation.

D7  \[ x^2 - 2xy + y^2 = 4 \]
D8  \[ y = \sin(xy) + x \]
D9  \[ 9x^2 + 16y^2 = 144 \]
D10 \[ 9y - 6x + y^4 = 0 \]

D11  The area of a circle is \( A(r) = \pi r^2 \). Find the radius \( r \) when you know the area \( A \). (This is the inverse function \( r(A) \)). The derivative of \( A = \pi r^2 \) is \( dA/dr = 2\pi r \). Find \( dr/dA \).