

CHAPTER 7 TECHNIQUES OF INTEGRATION

7.1 Integration by Parts (page 287)

Integration by parts aims to exchange a difficult problem for a possibly longer but probably easier one. It is up to you to make the problem easier! The key lies in choosing “ u ” and “ dv ” in the formula $\int u dv = uv - \int v du$. Try to pick u so that du is simple (or at least no worse than u). For $u = x$ or x^2 the derivative 1 or $2x$ is simpler. For $u = \sin x$ or $\cos x$ or e^x it is no worse. On the other hand, choose “ dv ” to have a nice integral. Good choices are $dv = \sin x dx$ or $\cos x dx$ or $e^x dx$.

Of course the selection of u also decides dv (since $u dv$ is the given integration problem). Notice that $u = \ln x$ is a good choice because $du = \frac{1}{x} dx$ is simpler. On the other hand, $\ln x dx$ is usually a poor choice for dv , because its integral $x \ln x - x$ is more complicated. Here are more suggestions:

Good choices for u : $\ln x$, inverse trig functions, x^n , $\cos x$, $\sin x$, e^x or e^{cx} .

These are just suggestions. It's a free country. Integrate 1 – 6 by parts:

1. $\int x e^{-x} dx$.

- Pick $u = x$ because $\frac{du}{dx} = 1$ is simpler. Then $dv = e^{-x} dx$ gives $v = -e^{-x}$. Watch all the minus signs:

$$\int u dv = \underbrace{x}_{u} \underbrace{(-e^{-x})}_{v} - \int \underbrace{(-e^{-x})}_{v} \underbrace{dx}_{du} = -x e^{-x} - e^{-x} + C$$

2. $\int x \sec^{-1} x dx$.

- If we choose $u = x$, we are faced with $dv = \sec^{-1} x dx$. Its integral is difficult. Better to try $u = \sec^{-1} x$, so that $du = \frac{dx}{|x|\sqrt{x^2-1}}$. Is that simpler? It leaves $dv = x dx$, so that $v = \frac{1}{2}x^2$. Our integral is now $uv - \int v du$:

$$\begin{aligned} (\sec^{-1} x) \left(\frac{1}{2}x^2\right) - \int \frac{1}{2}x^2 \cdot \frac{dx}{|x|\sqrt{x^2-1}} &= \frac{1}{2}x^2 \sec^{-1} x \pm \frac{1}{2} \int \frac{x dx}{\sqrt{x^2-1}} \\ &= \frac{1}{2}x^2 \sec^{-1} x \pm \frac{1}{2}\sqrt{x^2-1} + C. \end{aligned}$$

The \pm sign comes from $|x|$; plus if $x > 0$ and minus if $x < 0$.

3. $\int e^x \sin x dx$. (Problem 7.1.9) This example requires *two* integrations by parts. First choose $u = e^x$ and $dv = \sin x dx$. This makes $du = e^x dx$ and $v = -\cos x$. The first integration by parts is $\int e^x \sin x dx = -e^x \cos x + \int e^x \cos x dx$. The new integral on the right is no simpler than the old one on the left. For the new one, $dv = \cos x dx$ brings back $v = \sin x$:

$$\int e^x \cos x dx = e^x \sin x - \int e^x \sin x dx.$$

Are we back where we started? Not quite. Put the second into the first:

$$\int e^x \sin x dx = -e^x \cos x + e^x \sin x - \int e^x \sin x dx.$$

The integrals are now the same. Move the one on the right side to the left side, and divide by 2:

$$\int e^x \sin x dx = \frac{1}{2}e^x(\sin x - \cos x) + C.$$

4. $\int x^2 \ln x \, dx$ (Problem 7.1.6). The function $\ln x$ (if it appears) is almost always the choice for u . Then $du = \frac{dx}{x}$. This leaves $dv = x^2 dx$ and $v = \frac{1}{3}x^3$. Therefore

$$\int x^2 \ln x \, dx = \frac{1}{3}x^3 \ln x - \int \frac{1}{3}x^2 dx = \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C.$$

5. $\int \frac{x^3 dx}{\sqrt{x^2+1}}$.

- Generally we choose u for a nice derivative, and dv is what's left. In this case it pays for dv to have a nice integral. We don't know $\int \frac{x^3}{\sqrt{x^2+1}} dx$ but we do know $\int \frac{x}{\sqrt{x^2+1}} dx = \sqrt{x^2+1}$. This leaves $u = x^2$ with $du = 2x \, dx$:

$$\begin{aligned} \int \frac{x^3 dx}{\sqrt{x^2+1}} &= x^2 \sqrt{x^2+1} - \int 2x \sqrt{x^2+1} \, dx \\ &= x^2 \sqrt{x^2+1} - \frac{2}{3}(x^2+1)^{3/2} + C. \end{aligned}$$

Note Integration by parts is not the only way to do this problem. You can directly substitute $u = x^2 + 1$ and $du = 2x \, dx$. Then x^2 is $u - 1$ and $x \, dx$ is $\frac{1}{2}du$. The integral is

$$\begin{aligned} \frac{1}{2} \int \frac{u-1}{\sqrt{u}} du &= \frac{1}{2} \int (u^{1/2} - u^{-1/2}) du = \frac{1}{3}u^{3/2} - u^{1/2} + C \\ &= \frac{1}{3}(x^2+1)^{3/2} - (x^2+1)^{1/2} + C \quad (\text{same answer in disguise}). \end{aligned}$$

6. Derive a reduction formula for $\int (\ln x)^n dx$.

- A reduction formula gives this integral in terms of an integral of $(\ln x)^{n-1}$. Let $u = (\ln x)^n$ so that $du = n(\ln x)^{n-1}(\frac{1}{x})dx$. Then $dv = dx$ gives $v = x$. This cancels the $\frac{1}{x}$ in du :

$$\int (\ln x)^n dx = x(\ln x)^n - \int n(\ln x)^{n-1} dx.$$

- 6'. Find a similar reduction from $\int x^n e^x dx$ to $\int x^{n-1} e^x dx$.

7. Use this reduction formula as often as necessary to find $\int (\ln x)^3 dx$.

- Start with $n = 3$ to get $\int (\ln x)^3 dx = x(\ln x)^3 - 3 \int (\ln x)^2 dx$. Now use the formula with $n = 2$. The last integral is $x(\ln x)^2 - 2 \int \ln x \, dx$. Finally $\int \ln x \, dx$ comes from $n = 1$: $\int \ln x \, dx = x(\ln x) - \int (\ln x)^0 dx = x(\ln x) - x$. Substitute everything back:

$$\begin{aligned} \int (\ln x)^3 dx &= x(\ln x)^3 - 3[x(\ln x)^2 - 2[x \ln x - x]] + C \\ &= x(\ln x)^3 - 3x(\ln x)^2 + 6x \ln x - 6x + C. \end{aligned}$$

Problems 8 and 9 are about the step function $U(x)$ and its derivative the delta function $\delta(x)$.

8. Find $\int_{-2}^6 (x^2 - 8)\delta(x) dx$

- Since $\delta(x) = 0$ everywhere except at $x = 0$, we are only interested in $v(x) = x^2 - 8$ at $x = 0$. At that point $v(0) = -8$. We separate the problem into two parts:

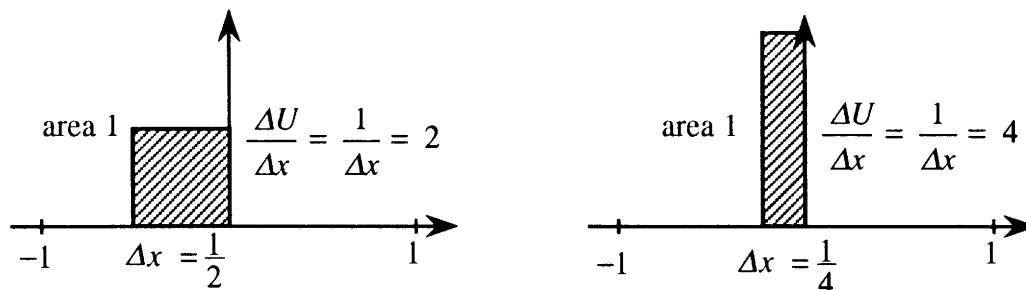
$$\int_{-2}^2 (x^2 - 8)\delta(x) dx + \int_2^6 (x^2 - 8)\delta(x) dx = -8 + 0.$$

The first integral is just like 7B, picking out $v(0)$. The second integral is zero since $\delta(x) = 0$ in the interval $[2,6]$. The answer is -8 .

9. (This is 7.1.54) Find the area under the graph of $\frac{\Delta U}{\Delta x} = [U(x + \Delta x) - U(x)]/\Delta x$.

- For the sake of this discussion let Δx be positive. The step function has $U(x) = 1$ if $x \geq 0$. In that case $U(x + \Delta x) = 1$ also. Subtraction $U(x + \Delta x) - U(x)$ leaves zero. The only time $U(x + \Delta x)$ is different from $U(x)$ is when $x + \Delta x \geq 0$ and $x < 0$. In that case

$$U(x + \Delta x) - U(x) = 1 - 0 = 1 \text{ and } \frac{U(x + \Delta x) - U(x)}{\Delta x} = \frac{1}{\Delta x}.$$



The sketches show the small interval $-\Delta x \leq x < 0$ where this happens. The base of the rectangle is Δx but the height is $\frac{1}{\Delta x}$. **The area stays constant at 1.**

The limit of $\frac{U(x+\Delta x)-U(x)}{\Delta x}$ is the slope of the step function. This is the delta function $U'(x) = \delta(x)$. Certainly $\delta(x) = 0$ except at $x = 0$. But the integral of the delta function across the spike at $x = 0$ is 1. (The area hasn't changed as $\Delta x \rightarrow 0$.) A strange function.

Read-throughs and selected even-numbered solutions :

Integration by parts is the reverse of the product rule. It changes $\int u dv$ into uv minus $\int v du$. In case $u = x$ and $dv = e^{2x} dx$, it changes $\int x e^{2x} dx$ to $\frac{1}{2} x e^{2x}$ minus $\int \frac{1}{2} e^{2x} dx$. The definite integral $\int_0^2 x e^{2x} dx$ becomes $\frac{3}{4} e^4$ minus $\frac{1}{4}$. In choosing u and dv , the derivative of u and the integral of dv/dx should be as simple as possible. Normally $\ln x$ goes into u and e^x goes into v . Prime candidates are $u = x$ or x^2 and $v = \sin x$ or $\cos x$ or e^x . When $u = x^2$ we need two integrations by parts. For $\int \sin^{-1} x dx$, the choice $dv = dx$ leads to $x \sin^{-1} x$ minus $\int x dx / \sqrt{1-x^2}$.

If U is the unit step function, $dU/dx = \delta$ is the unit delta function. The integral from $-A$ to A is $U(A) - U(-A) = 1$. The integral of $v(x)\delta(x)$ equals $v(0)$. The integral $\int_{-1}^1 \cos x \delta(x) dx$ equals 1. In engineering, the balance of forces $-dv/dx = f$ is multiplied by a displacement $u(x)$ and integrated to give a balance of work.

14 $\int \cos(\ln x) dx = uv - \int v du = \cos(\ln x)x + \int x \sin(\ln x) \frac{1}{x} dx$. Cancel x with $\frac{1}{x}$. Integrate by parts again to get $\cos(\ln x)x + \sin(\ln x)x - \int x \cos(\ln x) \frac{1}{x} dx$. Move the last integral to the left and divide by 2.

The answer is $\frac{x}{2}(\cos(\ln x) + \sin(\ln x)) + C$.

18 $uv - \int v du = \cos^{-1}(2x)x + \int x \frac{2 dx}{\sqrt{1-(2x)^2}} = x \cos^{-1}(2x) - \frac{1}{2}(1-4x^2)^{1/2} + C$.

22 $uv - \int v du = x^3(-\cos x) + \int (\cos x)3x^2 dx =$ (use Problem 5) $= -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C$.

28 $\int_0^1 e^{\sqrt{x}} dx = \int_{u=0}^1 e^u (2u du) = 2e^u(u-1)|_0^1 = 2$. 38 $\int x^n \sin x dx = -x^n \cos x + n \int x^{n-1} \cos x dx$.

44 (a) $e^0 = 1$; (b) $v(0)$ (c) 0 (limits do not enclose zero).

46 $\int_{-1}^1 \delta(2x) dx = \int_{u=-2}^2 \delta(u) \frac{du}{2} = \frac{1}{2}$. Apparently $\delta(2x)$ equals $\frac{1}{2}\delta(x)$; both are zero for $x \neq 0$.

48 $\int_0^1 \delta(x - \frac{1}{2}) dx = \int_{-1/2}^{1/2} \delta(u) du = 1$; $\int_0^1 e^x \delta(x - \frac{1}{2}) dx = \int_{-1/2}^{1/2} e^{u+\frac{1}{2}} \delta(u) du = e^{1/2}$; $\delta(x)\delta(x - \frac{1}{2}) = 0$.

60 $A = \int_1^e \ln x dx = [x \ln x - x]_1^e = 1$ is the area under $y = \ln x$. $B = \int_0^1 e^y dy = e - 1$ is the area to the left of $y = \ln x$. Together the area of the rectangle is $1 + (e - 1) = e$.

7.2 Trigonometric Integrals (page 293)

This section integrates powers and products of sines and cosines and tangents and secants. We are constantly using $\sin^2 x = 1 - \cos^2 x$. Starting with $\int \sin^3 x \, dx$, we convert it to $\int (1 - \cos^2 x) \sin x \, dx$. Are we unhappy about that one remaining $\sin x$? *Not at all*. It will be part of du , when we set $u = \cos x$. Odd powers are actually easier than even powers, because the extra term goes into du . For even powers use the double-angle formula in Problem 2 below.

1. $\int (\sin x)^{-3/2} (\cos x)^3 \, dx$ is a product of sines and cosines.

- The angles x are the same and *the power 3 is odd*. ($-\frac{3}{2}$ is neither even nor odd.) Change all but one of the cosines to sines by $\cos^2 x = 1 - \sin^2 x$. The problem is now

$$\int (\sin x)^{-3/2} (1 - \sin^2 x) \cos x \, dx = \int (u^{-3/2} - u^{1/2}) du.$$

Here $u = \sin x$ and $du = \cos x \, dx$. The answer is

$$-2u^{-1/2} - \frac{2}{3}u^{3/2} + C = -2(\sin x)^{-1/2} - \frac{2}{3}(\sin x)^{3/2} + C.$$

2. $\int \sin^4 3x \cos^2 3x \, dx$ has even powers 4 and 2, with the same angle $3x$.

- *Use the double-angle method*. Replace $\sin^2 3x$ with $\frac{1}{2}(1 - \cos 6x)$ and $\cos^2 3x$ with $\frac{1}{2}(1 + \cos 6x)$. The problem is now

$$\begin{aligned} \int \frac{(1 - \cos 6x)^2 (1 + \cos 6x)}{4 \cdot 2} dx &= \frac{1}{8} \int (1 - 2\cos 6x + \cos^2 6x)(1 + \cos 6x) dx \\ &= \frac{1}{8} \int (1 - \cos 6x - \cos^2 6x + \cos^3 6x) dx. \end{aligned}$$

The integrals of the first two terms are x and $\frac{1}{6} \sin 6x$. The third integral is another double angle:

$$\int \cos^2 6x \, dx = \int \frac{1}{2}(1 + \cos 12x) dx = \frac{1}{2}x + \frac{1}{24} \sin 12x.$$

For $\int \cos^3 6x \, dx$, with an odd power, change \cos^2 to $1 - \sin^2$:

$$\int \cos^3 6x \, dx = \int (1 - \sin^2 6x) \cos 6x \, dx = \int (1 - u^2) \frac{du}{6} = \frac{1}{6} \sin 6x - \frac{1}{18} \sin^3 6x.$$

Putting all these together, the final solution is

$$\frac{1}{8} \left[x - \frac{1}{6} \sin 6x - \left(\frac{1}{2}x + \frac{1}{24} \sin 12x \right) + \frac{1}{6} \sin 6x - \frac{1}{18} \sin^3 6x \right] = \frac{1}{16}x - \frac{1}{192} \sin 12x - \frac{1}{144} \sin^3 6x + C.$$

3. $\int \sin 10x \cos 4x \, dx$ has different angles $10x$ and $4x$. Use the identity $\sin 10x \cos 4x = \frac{1}{2} \sin(10 + 4)x + \frac{1}{2} \sin(10 - 4)x$. Now integrate:

$$\int \left(\frac{1}{2} \sin 14x + \frac{1}{2} \sin 6x \right) dx = -\frac{1}{28} \cos 14x - \frac{1}{12} \cos 6x + C.$$

4. $\int \cos x \cos 4x \cos 8x dx$ has three different angles!

- Use the identity $\cos 4x \cos 8x = \frac{1}{2} \cos(4+8)x + \frac{1}{2} \cos(4-8)x$. The integral is now $\frac{1}{2} \int (\cos x \cos 12x + \cos x \cos 4x) dx$. Apply the $\cos px \cos qx$ identity twice more to get

$$\frac{1}{2} \int \left(\frac{1}{2} \cos 13x + \frac{1}{2} \cos 11x + \frac{1}{2} \cos 5x + \frac{1}{2} \cos 3x \right) dx = \frac{1}{4} \left(\frac{\sin 13x}{13} + \frac{\sin 11x}{11} + \frac{\sin 5x}{5} + \frac{\sin 3x}{3} \right) + C.$$

5. $\int \tan^5 x \sec^4 x dx$. Here are three ways to deal with tangents and secants.

- *First:* Remember $d(\tan x) = \sec^2 x dx$ and convert the other $\sec^2 x$ to $1 + \tan^2 x$. The problem is

$$\int \tan^5 x (1 + \tan^2 x) \sec^2 x dx = \int (u^5 + u^7) du.$$

- *Second:* Remember $d(\sec x) = \sec x \tan x dx$ and convert $\tan^4 x$ to $(\sec^2 x - 1)^2$. The integral is

$$\int (\sec^2 x - 1)^2 \sec^3 x \sec x \tan x dx = \int (u^2 - 1)^2 u^3 du = \int (u^7 - 2u^5 + u^3) du.$$

- *Third:* Convert $\tan^5 x \sec^4 x$ to sines and cosines as $\frac{\sin^5 x}{\cos^9 x}$. Eventually take $u = \cos x$:

$$\begin{aligned} \int \frac{(1-\cos^2 x)^2}{\cos^9 x} \sin x dx &= \int (\cos^{-9} x - 2\cos^{-7} x + \cos^{-5} x) \sin x dx \\ &= \int (-u^{-9} + 2u^{-7} - u^{-5}) du. \end{aligned}$$

6. Use the substitution $u = \tan \frac{x}{2}$ in the text equation (11) to find $\int_0^{\pi/4} \frac{dx}{1-\sin x}$.

- The substitutions are $\sin x = \frac{2u}{1+u^2}$ and $dx = \frac{2du}{1+u^2}$. This gives

$$\int \frac{1}{1-\sin x} dx = \int \frac{1}{1-\frac{2u}{1+u^2}} \cdot \frac{2du}{1+u^2} = \int \frac{2du}{(1+u^2)-2u} = \int \frac{2du}{(1-u)^2} = \frac{2}{1-u}.$$

The definite integral is from $x = 0$ to $x = \frac{\pi}{4}$. Then $u = \tan \frac{x}{2}$ goes from 0 to $\tan \frac{\pi}{8}$. The answer is $\frac{2}{1-\tan \frac{\pi}{8}} - \frac{2}{1} \approx 1.41$.

7. Problem 7.2.26 asks for $\int_0^{\pi} \sin 3x \sin 5x dx$. First write $\sin 3x \sin 5x$ in terms of $\cos 8x$ and $\cos 2x$.

- The formula for $\sin px \sin qx$ gives

$$\int_0^{\pi} \left(-\frac{1}{2} \cos 8x + \frac{1}{2} \cos 2x \right) dx = \left[-\frac{1}{16} \sin 8x + \frac{1}{4} \sin 2x \right]_0^{\pi} = 0.$$

8. Problem 7.2.33 is the Fourier sine series $A \sin x + B \sin 2x + C \sin 3x + \dots$ that adds to x . Find A .

- Multiply both sides of $x = A \sin x + B \sin 2x + C \sin 3x + \dots$ by $\sin x$. Integrate from 0 to π :

$$\int_0^{\pi} x \sin x dx = \int_0^{\pi} A \sin^2 x dx + \int_0^{\pi} B \sin 2x \sin x dx + \int_0^{\pi} C \sin 3x \sin x dx + \dots$$

All of the definite integrals on the right are zero, except for $\int_0^{\pi} A \sin^2 x dx$. For example the integral of $\sin 2x \sin x$ is $[-\frac{1}{6} \sin 3x + \frac{1}{2} \sin x]_0^{\pi} = 0$. The only nonzero terms are $\int_0^{\pi} x \sin x dx = \int_0^{\pi} A \sin^2 x dx$. Integrate $x \sin x$ by parts to find one side of this equation for A :

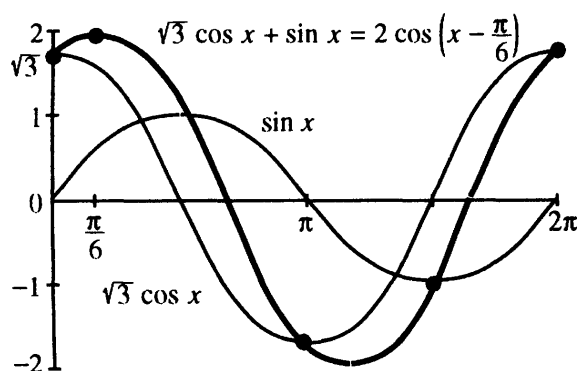
$$\int_0^{\pi} x \sin x dx = [-x \cos x]_0^{\pi} + \int_0^{\pi} \cos x dx = [-x \cos x + \sin x]_0^{\pi} = \pi.$$

On the other side $\int_0^{\pi} A \sin^2 x dx = \frac{A}{2} [x - \sin x \cos x]_0^{\pi} = \frac{A\pi}{2}$. Then $\frac{A\pi}{2} = \pi$ and $A = 2$.

- You should *memorize* those integrals $\int_0^\pi \sin^2 x dx = \int_0^\pi \cos^2 x dx = \frac{\pi}{2}$. They say that the average value of $\sin^2 x$ is $\frac{1}{2}$, and the average value of $\cos^2 x$ is $\frac{1}{2}$.
 - You would find B by multiplying the Fourier series by $\sin 2x$ instead of $\sin x$. This leads in the same way to $\int_0^\pi x \sin 2x dx = \int_0^\pi B \sin^2 2x dx = B \frac{\pi}{2}$ because all other integrals are zero.
9. When a sine and a cosine are added, the resulting wave is best expressed as a single cosine: $a \cos x + b \sin x = \sqrt{a^2 + b^2} \cos(x - \alpha)$. Show that this is correct and find the angle α (Problem 7.2.56).
- Expand $\cos(x - \alpha)$ into $\cos x \cos \alpha + \sin x \sin \alpha$. Choose α so that $\cos \alpha = \frac{a}{\sqrt{a^2 + b^2}}$ and $\sin \alpha = \frac{b}{\sqrt{a^2 + b^2}}$. Our formula becomes correct. The reason for $\sqrt{a^2 + b^2}$ is to ensure that $\cos^2 \alpha + \sin^2 \alpha = \frac{a^2 + b^2}{a^2 + b^2} = 1$. Dividing $\sin \alpha$ by $\cos \alpha$ gives $\tan \alpha = \frac{b}{a}$ or $\alpha = \tan^{-1} \frac{b}{a}$. Thus $3 \cos x + 4 \sin x = 5 \cos(x - \tan^{-1} \frac{4}{3})$.
10. Use the previous answer (Problem 9) to find $\int \frac{dx}{\sqrt{3} \cos x + \sin x}$.
- With $a = \sqrt{3}$ and $b = 1$ we have $\sqrt{a^2 + b^2} = \sqrt{3 + 1} = 2$ and $\alpha = \tan^{-1} \frac{1}{\sqrt{3}} = \frac{\pi}{6}$. Therefore

$$\int \frac{dx}{\sqrt{3} \cos x + \sin x} = \int \frac{dx}{2 \cos(x - \pi/6)} = \frac{1}{2} \int \sec(x - \frac{\pi}{6}) dx = \frac{1}{2} \ln |\sec(x - \frac{\pi}{6}) + \tan(x - \frac{\pi}{6})| + C.$$

The figure shows the waves $\sqrt{3} \cos x$ and $\sin x$ adding to $2 \cos(x - \frac{\pi}{6})$.



11. What is the distance from the equator to latitude 45° on a Mercator world map? From 45° to 70° ?
- The distance north is the integral of $\sec x$, multiplied by the radius R of the earth (on your map). See Figure 7.3 in the text. The equator is at 0° . The distance to $45^\circ = \frac{\pi}{4}$ radians is

$$R \int_0^{\pi/4} \sec x dx = R \ln(\sec x + \tan x) \Big|_0^{\pi/4} = R \ln(\sqrt{2} + 1) - R \ln 1 \approx 0.88R.$$

The distance from 45° to 70° is almost the same: $R \ln |\sec x + \tan x| \Big|_{45^\circ}^{70^\circ} \approx 0.85R$.

Read-throughs and selected even-numbered solutions :

To integrate $\sin^4 x \cos^3 x$, replace $\cos^2 x$ by $1 - \sin^2 x$. Then $(\sin^4 x - \sin^6 x) \cos x dx$ is $(u^4 - u^6) du$. In terms of $u = \sin x$ the integral is $\frac{1}{5}u^5 - \frac{1}{7}u^7$. This idea works for $\sin^m x \cos^n x$ if m or n is odd.

If both m and n are even, one method is integration by parts. For $\int \sin^4 x dx$, split off $dv = \sin x dx$.

Then $-\int v du$ is $\int 3 \sin^2 x \cos^2 x$. Replacing $\cos^2 x$ by $1 - \sin^2 x$ creates a new $\sin^4 x dx$ that combines with the original one. The result is a *reduction* to $\int \sin^2 x dx$, which is known to equal $\frac{1}{2}(x - \sin x \cos x)$.

The second method uses the double-angle formula $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$. Then $\sin^4 x$ involves $\cos^2 2x$. Another doubling comes from $\cos^2 2x = \frac{1}{2}(1 + \cos 4x)$. The integral contains the sine of $4x$.

To integrate $\sin 6x \cos 4x$, rewrite it as $\frac{1}{2} \sin 10x + \frac{1}{2} \sin 2x$. The integral is $-\frac{1}{20} \cos 10x - \frac{1}{4} \cos 2x$. The definite integral from 0 to 2π is zero. The product $\cos px \cos qx$ is written as $\frac{1}{2} \cos(p+q)x + \frac{1}{2} \cos(p-q)x$. Its integral is also zero, except if $p = q$ when the answer is π .

With $u = \tan x$, the integral of $\tan^9 x \sec^2 x$ is $\frac{1}{10} \tan^{10} x$. Similarly $\int \sec^9 x (\sec x \tan x dx) = \frac{1}{10} \sec^{10} x$. For the combination $\tan^m x \sec^n x$ we apply the identity $\tan^2 x = 1 + \sec^2 x$. After reduction we may need $\int \tan x dx = -\ln \cos x$ and $\int \sec x dx = \ln(\sec x + \tan x)$.

$$6 \int \sin^3 x \cos^3 x dx = \int \sin^3 x (1 - \sin^2 x) \cos x dx = \frac{1}{4} \sin^4 x - \frac{1}{6} \sin^6 x + C$$

$$10 \int \sin^2 ax \cos ax dx = \frac{\sin^3 ax}{3a} + C \text{ and } \int \sin ax \cos ax dx = \frac{\sin^2 ax}{2a} + C$$

$$16 \int \sin^2 x \cos^2 2x dx = \int \frac{1 - \cos 2x}{2} \cos^2 2x dx = \int \left(\frac{1 + \cos 4x}{4} - \frac{\cos 2x}{2} (1 - \sin^2 2x) \right) dx = \frac{x}{4} + \frac{\sin 4x}{16} - \frac{\sin 2x}{4} + \frac{\sin^3 2x}{12} + C. \text{ This is a hard one.}$$

$$18 \text{ Equation (7) gives } \int_0^{\pi/2} \cos^n x dx = \left[\frac{\cos^{n-1} x \sin x}{n} \right]_0^{\pi/2} + \frac{n-1}{n} \int_0^{\pi/2} \cos^{n-2} x dx. \text{ The integrated term is zero because } \cos \frac{\pi}{2} = 0 \text{ and } \sin 0 = 0. \text{ The exception is } n = 1, \text{ when the integral is } [\sin x]_0^{\pi/2} = 1.$$

$$26 \int_0^{\pi} \sin 3x \sin 5x dx = \int_0^{\pi} \frac{-\cos 8x + \cos 2x}{2} dx = \left[-\frac{\sin 8x}{16} + \frac{\sin 2x}{4} \right]_0^{\pi} = 0.$$

$$30 \int_0^{2\pi} \sin x \sin 2x \sin 3x dx = \int_0^{2\pi} \sin 2x \left(\frac{-\cos 4x + \cos 2x}{2} \right) dx = \int_0^{2\pi} \sin 2x \left(\frac{1 - 2\cos^2 2x + \cos 2x}{2} \right) dx = \left[-\frac{\cos 2x}{4} + \frac{\cos^3 2x}{6} - \frac{\cos^2 2x}{8} \right]_0^{2\pi} = 0. \text{ Note: The integral has other forms.}$$

$$32 \int_0^{\pi} x \cos x dx = [x \sin x]_0^{\pi} - \int_0^{\pi} \sin x dx = [x \sin x + \cos x]_0^{\pi} = -2.$$

$$34 \int_0^{\pi} 1 \sin 3x dx = \int_0^{\pi} (A \sin x + B \sin 2x + C \sin 3x + \dots) \sin 3x dx \text{ reduces to } \left[-\frac{\cos 3x}{3} \right]_0^{\pi} = 0 + 0 + C \int_0^{\pi} \sin^2 3x dx. \text{ Then } \frac{2}{3} = C \left(\frac{\pi}{2} \right) \text{ and } C = \frac{4}{3\pi}.$$

$$44 \text{ First by substituting for } \tan^2 x: \int \tan^2 x \sec x dx = \int \sec^3 x dx - \int \sec x dx. \text{ Use Problem 62 to integrate } \sec^3 x: \text{ final answer } \frac{1}{2}(\sec x \tan x - \ln|\sec x + \tan x|) + C. \text{ Second method from line 1 of Example 11: } \int \tan^2 x \sec x dx = \sec x \tan x - \int \sec^3 x dx. \text{ Same final answer.}$$

$$52 \text{ This should have an asterisk! } \int \frac{\sin^6 x dx}{\cos^3 x} = \int \frac{(1 - \cos^2 x)^3 dx}{\cos^3 x} = \int (\sec^3 x - 3 \sec x + 3 \cos x - \cos^3 x) dx = \text{use Example 11} = \text{Problem 62 for } \int \sec^3 x dx \text{ and change } \int \cos^3 x dx \text{ to } \int (1 - \sin^2 x) \cos x dx.$$

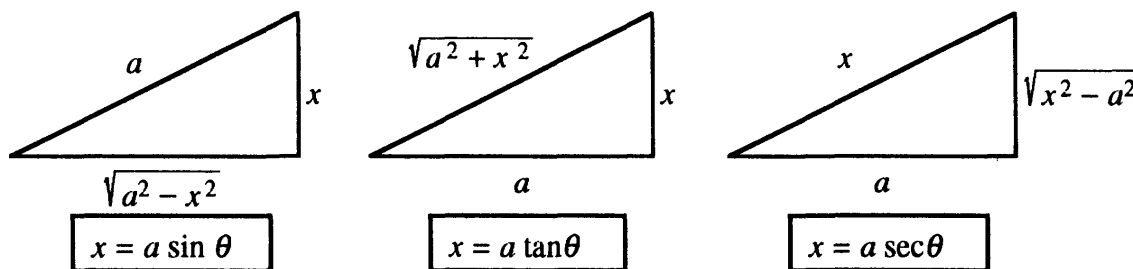
$$\text{Final answer } \frac{\sec x \tan x}{2} - \frac{5}{2} \ln|\sec x + \tan x| + 2 \sin x + \frac{\sin^3 x}{3} + C.$$

$$54 A = 2: 2 \cos(x + \frac{\pi}{3}) = 2 \cos x \cos \frac{\pi}{3} - 2 \sin x \sin \frac{\pi}{3} = \cos x - \sqrt{3} \sin x. \text{ Therefore } \int \frac{dx}{(\cos x - \sqrt{3} \sin x)^2} = \int \frac{dx}{4 \cos^2(x + \frac{\pi}{3})} = \frac{1}{4} \tan(x + \frac{\pi}{3}) + C.$$

$$58 \text{ When lengths are scaled by } \sec x, \text{ area is scaled by } \sec^2 x. \text{ The area from the equator to latitude } x \text{ is then proportional to } \int \sec^2 x dx = \tan x.$$

7.3 Trigonometric Substitutions (page 299)

The substitutions may be easier to remember from these right triangles:



Each triangle obeys Pythagoras. The squares of the legs add to the square of the hypotenuse. The first triangle has $\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{x}{a}$. Thus $x = a \sin \theta$ and $dx = a \cos \theta d\theta$.

Use these triangles in Problems 1–3 or use the table of substitutions in the text.

1. $\int_1^4 \frac{dx}{x^2 \sqrt{x^2 + 9}}$ has a plus sign in the square root (*second triangle*).

- Choose the second triangle with $a = 3$. Then $x = 3 \tan \theta$ and $dx = 3 \sec^2 \theta d\theta$ and $\sqrt{x^2 + 9} = 3 \sec \theta$. Substitute and then write $\sec \theta = \frac{1}{\cos \theta}$ and $\tan \theta = \frac{\sin \theta}{\cos \theta}$:

$$\int \frac{dx}{x^2 \sqrt{x^2 + 9}} = \int \frac{3 \sec^2 \theta d\theta}{(9 \tan^2 \theta)(3 \sec \theta)} = \int \frac{\cos \theta d\theta}{9 \sin^2 \theta} = \frac{-1}{9 \sin \theta}.$$

The integral was $\frac{1}{9} u^{-2} du$ with $u = \sin \theta$. The original limits of integration are $x = 1$ and $x = 4$. Instead of converting them to θ , we convert $\sin \theta$ back to x . The second triangle above shows

$$\frac{1}{\sin \theta} = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{\sqrt{x^2 + 9}}{x} \text{ and then } \left[\frac{-\sqrt{x^2 + 9}}{9x} \right]_1^4 = \frac{-5}{36} + \frac{\sqrt{10}}{9} \approx 0.212.$$

2. $\int \sqrt{100 - x^2} dx$ contains the square root of $a^2 - x^2$ with $a = 10$.

- Choose the *first* triangle: $x = 10 \sin \theta$ and $\sqrt{100 - x^2} = 10 \cos \theta$ and $dx = 10 \cos \theta d\theta$:

$$\int \sqrt{100 - x^2} dx = \int (10 \cos \theta) 10 \cos \theta d\theta = 100 \int \cos^2 \theta d\theta = 100 \cdot \frac{1}{2} (\theta + \sin \theta \cos \theta) + C.$$

Returning to x this is $50(\sin^{-1} \frac{x}{10} + \frac{x}{10} \cdot \frac{\sqrt{100 - x^2}}{10}) = 50 \sin^{-1} \frac{x}{10} + \frac{1}{2} x \sqrt{100 - x^2}$.

3. $\int \frac{dx}{x^2 \sqrt{9x^2 - 25}}$ does not exactly contain $x^2 - a^2$. But try the third triangle.

- Factor $\sqrt{9} = 3$ from the square root to leave $\sqrt{x^2 - \frac{25}{9}}$. Then $a^2 = \frac{25}{9}$ and $a = \frac{5}{3}$. The third triangle has $x = \frac{5}{3} \sec \theta$ and $dx = \frac{5}{3} \sec \theta \tan \theta d\theta$ and $\sqrt{9x^2 - 25} = 5 \tan \theta$. The problem is now

$$\int \frac{\frac{5}{3} \sec \theta \tan \theta d\theta}{(\frac{5}{3})^2 \sec^2 \theta (5 \tan \theta)} = \frac{3}{25} \int \cos \theta d\theta = \frac{3}{25} \sin \theta + C.$$

The third triangle converts $\frac{3}{25} \sin \theta$ back to $\frac{\sqrt{9x^2 - 25}}{25x}$.

4. For $\int \frac{x^3 dx}{\sqrt{1-x^2}}$ the substitution $x = \sin \theta$ will work. But try $u = 1 - x^2$.

- Then $du = -2x dx$ and $x^2 = 1 - u$. The problem becomes $-\frac{1}{2} \int \frac{(1-u)du}{u^{1/2}} = \frac{1}{2} \int (u^{-1/2} - u^{1/2}) du$. In this case the old way is simpler than the new.

Problems 5 and 6 require *completing the square* before a trig substitution.

5. $\int \frac{dx}{\sqrt{x^2 - 8x + 6}}$ requires us to complete $(x - 4)^2$. We need $4^2 = 16$ so add and subtract 10 :

$$x^2 - 8x + 6 = (x^2 - 8x + 16) - 10 = (x - 4)^2 - 10.$$

This has the form $u^2 - a^2$ with $u = x - 4$ and $a = \sqrt{10}$. Finally set $u = \sqrt{10} \sec \theta$:

$$\int \frac{dx}{\sqrt{x^2 - 8x + 6}} = \int \frac{du}{\sqrt{u^2 - 10}} = \int \frac{\sqrt{10} \sec \theta \tan \theta d\theta}{\sqrt{10} \tan \theta} = \int \sec \theta d\theta.$$

6. $\int \frac{\sqrt{2x-x^2}}{x} dx$ requires us to complete $2x - x^2$ (watch the minus sign):

$$2x - x^2 = -(x^2 - 2x) = -(x^2 - 2x + 1) + 1 = 1 - (x - 1)^2.$$

This is $1 - u^2$ with $u = x - 1$ and $x = 1 + u$. The trig substitution is $u = \sin \theta$:

$$\int \frac{\sqrt{2x-x^2}}{x} = \int \frac{\sqrt{1-u^2}}{1+u} du = \int \frac{\cos^2 \theta d\theta}{1+\sin \theta} = \int \frac{(1-\sin^2 \theta)}{1+\sin \theta} d\theta = \int (1-\sin \theta) d\theta.$$

Read-throughs and selected even-numbered solutions :

The function $\sqrt{1-x^2}$ suggests the substitution $x = \sin \theta$. The square root becomes $\cos \theta$ and dx changes to $\cos \theta d\theta$. The integral $\int (1-x^2)^{3/2} dx$ becomes $\int \cos^4 \theta d\theta$. The interval $\frac{1}{2} \leq x \leq 1$ changes to $\frac{\pi}{6} \leq \theta \leq \frac{\pi}{2}$.

For $\sqrt{a^2 - x^2}$ the substitution is $x = a \sin \theta$ with $dx = a \cos \theta d\theta$. For $x^2 - a^2$ we use $x = a \sec \theta$ with $dx = a \sec \theta \tan \theta d\theta$. (Insert: For $x^2 + a^2$ use $x = a \tan \theta$). Then $\int dx/(1+x^2)$ becomes $\int d\theta$, because $1 + \tan^2 \theta = \sec^2 \theta$. The answer is $\theta = \tan^{-1} x$. We already knew that $\frac{1}{1+x^2}$ is the derivative of $\tan^{-1} x$.

The quadratic $x^2 + 2bx + c$ contains a linear term $2bx$. To remove it we **complete the square**. This gives $(x+b)^2 + C$ with $C = c - b^2$. The example $x^2 + 4x + 9$ becomes $(x+2)^2 + 5$. Then $u = x+2$. In case x^2 enters with a minus sign, $-x^2 + 4x + 9$ becomes $-(x-2)^2 + 13$. When the quadratic contains $4x^2$, start by factoring out 4.

$$2 \quad x = a \sec \theta, x^2 - a^2 = a^2 \tan^2 \theta, \int \frac{dx}{\sqrt{x^2 - a^2}} = \int \frac{a \sec \theta \tan \theta d\theta}{a \tan \theta} = \ln |\sec \theta + \tan \theta| = \ln \left| \frac{x}{a} + \sqrt{\frac{x^2}{a^2} - 1} \right| + C$$

$$4 \quad x = \frac{1}{3} \tan \theta, 1 + 9x^2 = \sec^2 \theta, \int \frac{dx}{1+9x^2} = \int \frac{\frac{1}{3} \sec^2 \theta d\theta}{\sec^2 \theta} = \frac{\theta}{3} = \frac{1}{3} \tan^{-1} 3x + C.$$

$$12 \quad \text{Write } \sqrt{x^6 - x^8} = x^3 \sqrt{1 - x^2} \text{ and set } x = \sin \theta : \int \sqrt{x^6 - x^8} dx = \int \sin^3 \theta \cos \theta (\cos \theta d\theta) = \int \sin \theta (\cos^2 \theta - \cos^4 \theta) d\theta = -\frac{\cos^3 \theta}{3} + \frac{\cos^5 \theta}{5} = -\frac{1}{3} (1 - x^2)^{3/2} + \frac{1}{5} (1 - x^2)^{5/2} + C$$

$$14 \quad x = \sin \theta, \int \frac{dx}{(1-x^2)^{3/2}} = \int \frac{\cos \theta d\theta}{\cos^3 \theta} = \tan \theta + C = \frac{x}{\sqrt{1-x^2}} + C.$$

32 First use geometry: $\int_{1/2}^1 \sqrt{1-x^2} dx$ = half the area of the unit circle beyond $x = \frac{1}{2}$ which breaks into

$$\frac{1}{2}(120^\circ \text{ wedge minus } 120^\circ \text{ triangle}) = \frac{1}{2}\left(\frac{\pi}{3} - \frac{1}{2} \cdot \frac{1}{2} \cdot 2\sqrt{1 - \left(\frac{1}{2}\right)^2}\right) = \frac{\pi}{6} - \frac{\sqrt{3}}{8}.$$

$$\text{Check by integration: } \int_{1/2}^1 \sqrt{1-x^2} dx = \left[\frac{1}{2}(x\sqrt{1-x^2} + \sin^{-1} x)\right]_{1/2}^1 = \frac{1}{2}\left(\frac{\pi}{2} - \frac{1}{2}\frac{\sqrt{3}}{2} - \frac{\pi}{6}\right) = \frac{\pi}{6} - \frac{\sqrt{3}}{8}.$$

$$\mathbf{34} \int \frac{dx}{\cos x} = \int \sec x dx = \ln |\sec x + \tan x| + C; \int \frac{dx}{1+\cos x} \left(\frac{1-\cos x}{1-\cos x}\right) = \int \frac{dx}{\sin^2 x} - \int \frac{\cos x dx}{\sin^2 x} = \int \csc^2 x dx - \int \frac{du}{u^2} = -\cot x + \frac{1}{\sin x} = \frac{1-\cos x}{\sin x} + C; \int \frac{dx}{\sqrt{1+\cos x}} = \int \frac{dx}{\sqrt{2\cos \frac{x}{2}}} = \sqrt{2} \ln |\sec \frac{x}{2} + \tan \frac{x}{2}| + C$$

$$\mathbf{40} x = \cosh \theta: \int \frac{\sqrt{x^2-1}}{x^2} dx = \int \frac{\sinh \theta}{\cosh^2 \theta} \sinh \theta d\theta = \int \tanh^2 \theta d\theta = \int (1 - \operatorname{sech}^2 \theta) d\theta = \theta - \tanh \theta = \cosh^{-1} x - \frac{\sqrt{x^2-1}}{x} + C$$

$$\mathbf{44} -x^2 + 2x + 8 = -(x-1)^2 + 9$$

$$\mathbf{50} \int \frac{dx}{\sqrt{9-(x-1)^2}} = \int \frac{du}{\sqrt{9-u^2}}. \text{ Set } u = 3 \sin \theta: \int \frac{\cos \theta d\theta}{\cos \theta} = \theta = \sin^{-1} \frac{u}{3} = \sin^{-1} \frac{x-1}{3} + C;$$

$$\int \frac{dx}{10-x^2} = \frac{1}{2\sqrt{10}} \ln \frac{x-\sqrt{10}}{x+\sqrt{10}} + C; \int \frac{dx}{(x+2)^2-16} = \int \frac{du}{u^2-16} = \frac{1}{8} \ln \frac{2u-8}{2u+8} = \frac{1}{8} \ln \frac{x-2}{x+6} + C$$

$$\mathbf{52} \text{ (a) } u = x - 2 \text{ (b) } u = x + 1 \text{ (c) } u = x - 5 \text{ (d) } u = x - \frac{1}{4}$$

7.4 Partial Fractions (page 304)

This method applies to ratios $\frac{P(x)}{Q(x)}$, where P and Q are polynomials. The goal is to split the ratio into pieces that are easier to integrate. We begin by comparing this method with substitutions, on some basic problems where both methods give the answer.

$$1. \int \frac{2x}{x^2-1} dx. \text{ The substitution } u = x^2 - 1 \text{ produces } \int \frac{du}{u} = \ln |u| = \ln |x^2 - 1|.$$

- Partial fractions breaks up this problem into smaller pieces:

$$\frac{2x}{x^2-1} = \frac{2x}{(x+1)(x-1)} \text{ splits into } \frac{A}{x+1} + \frac{B}{x-1} = \frac{1}{x+1} + \frac{1}{x-1}.$$

Now integrate the pieces to get $\ln |x+1| + \ln |x-1|$. This equals $\ln |x^2-1|$, the answer from substitution. We review how to find the numbers A and B starting from $\frac{2x}{x^2-1}$.

- First, factor $x^2 - 1$ to get the denominators $x + 1$ and $x - 1$. Second, cover up $(x - 1)$ and set $x = 1$:

$$\frac{2x}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1} \text{ becomes } \frac{2x}{(x+1)} = \frac{2}{2} = B. \text{ Thus } B = 1.$$

Third, cover up $(x + 1)$ and set $x = -1$ to find A :

$$\text{At } x = -1 \text{ we get } A = \frac{2x}{(x-1)} = \frac{-2}{-2} = 1.$$

That is it. Both methods are good. Use substitution or partial fractions.

$$2. \int \frac{1}{x^2-1} dx. \text{ The substitution } x = \sec \theta \text{ gives } \int \frac{4 \sec \theta \tan \theta}{\tan^2 \theta} d\theta = \int \frac{4}{\sin \theta} d\theta.$$

- The integral of $\frac{1}{\sin \theta}$ is not good. This time partial fractions look better:

$$\frac{4}{x^2-1} = \frac{4}{(x+1)(x-1)} \text{ splits into } \frac{A}{x+1} + \frac{B}{x-1} = \frac{-2}{x+1} + \frac{2}{x-1}.$$

The integral is $-2 \ln |x+1| + 2 \ln |x-1| = 2 \ln \left| \frac{x-1}{x+1} \right|$. Remember the cover-up:

$$x = 1 \text{ gives } B = \frac{4}{(x+1)} = 2. \quad x = -1 \text{ gives } A = \frac{4}{(x-1)} = -2.$$

3. $\int \frac{2x+4}{x^2-1} dx$ is the sum of the previous two integrals. Add A 's and B 's:

$$\frac{2x+4}{x^2-1} = \frac{A}{x+1} + \frac{B}{x-1} = \frac{-1}{x+1} + \frac{3}{x-1}.$$

In practice I would find $A = -1$ and $B = 3$ by the usual cover-up:

$$x = 1 \text{ gives } B = \frac{2x+4}{(x+1)} = \frac{6}{2}. \quad x = -1 \text{ gives } A = \frac{2x+4}{(x-1)} = \frac{2}{-2}.$$

The integral is immediately $-\ln|x+1| + 3\ln|x-1|$. In this problem partial fractions is much better than substitutions. This case is $\frac{\text{linear}}{\text{quadratic}} = \frac{\text{degree } 1}{\text{degree } 2}$. *That is where partial fractions work best.*

The text solves the logistic equation by partial fractions. Here are more difficult ratios $\frac{P(x)}{Q(x)}$.

- It is the algebra, not the calculus, that can make $\frac{P(x)}{Q(x)}$ difficult. A reminder about division of polynomials may be helpful. If the degree of $P(x)$ is greater than or equal to the degree of $Q(x)$, *you first divide Q into P* . The example $\frac{x^3}{x^2+2x+1}$ requires long division:

$$\begin{array}{r} x^2 + 2x + 1 \quad \sqrt{x^3} \quad \text{divide } x^2 \text{ into } x^3 \text{ to get } x \\ \underline{x^3 + 2x^2 + x} \quad \text{multiply } x^2 + 2x + 1 \text{ by } x \\ -2x^2 - x \quad \text{subtract from } x^3 \end{array}$$

The first part of the division gives x . If we stop there, division leaves $\frac{x^3}{x^2+2x+1} = x + \frac{-2x^2-x}{x^2+2x+1}$. This new fraction is $\frac{\text{degree } 2}{\text{degree } 2}$. So the division has to continue one more step:

$$\begin{array}{r} x^2 + 2x + 1 \quad \sqrt{x^3} \quad \text{divide } x^2 \text{ into } -2x^2 \text{ to get } -2 \\ \underline{x^3 + 2x^2 + x} \\ -2x^2 - x \\ \underline{-2x^2 - 4x - 2} \quad \text{multiply } x^2 + 2x + 1 \text{ by } -2 \\ 3x + 2 \quad \text{subtract to find remainder} \end{array}$$

Now stop. The remainder $3x + 2$ has lower degree than $x^2 + 2x + 1$:

$$\frac{x^3}{x^2+2x+1} = x - 2 + \frac{3x+2}{x^2+2x+1} \text{ is ready for partial fractions.}$$

Factor $x^2 + 2x + 1$ into $(x+1)^2$. Since $x+1$ is repeated, we look for

$$\frac{3x+2}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} \quad (\text{notice this form!})$$

Multiply through by $(x+1)^2$ to get $3x+2 = A(x+1) + B$. Set $x = -1$ to get $B = -1$. Set $x = 0$ to get $A + B = 2$. This makes $A = 3$. The algebra is done and we integrate:

$$\begin{aligned} \int \frac{x^3}{x^2+2x+1} dx &= \int \left(x - 2 + \frac{3x+2}{x^2+2x+1} \right) dx = \int \left(x - 2 + \frac{3}{x+1} - \frac{1}{(x+1)^2} \right) dx \\ &= \frac{1}{2}x^2 - 2x + 3\ln|x+1| + (x+1)^{-1} + C. \end{aligned}$$

4. $\int \frac{x^2+x}{x^2-4} dx$ also needs long division. The top and bottom have equal degree 2:

$$\begin{array}{r} x^2 + 0x - 4 \quad \frac{1}{\sqrt{x^2 + x}} \quad \text{divide } x^2 \text{ into } x^2 \text{ to get } 1 \\ \underline{x^2 + 0x - 4} \quad \text{multiply } x^2 - 4 \text{ by } 1 \\ x + 4 \quad \text{subtract to find remainder } x + 4 \end{array}$$

This says that $\frac{x^2+x}{x^2-4} = 1 + \frac{x+4}{x^2-4} = 1 + \frac{x+4}{(x-2)(x+2)}$. To decompose the remaining fraction, let

$$\frac{x+4}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2}.$$

Multiply by $x-2$ so the problem is $\frac{x+4}{x+2} = A + \frac{B(x-2)}{x+2}$. Set $x=2$ to get $A = \frac{6}{4} = \frac{3}{2}$. Cover up $x+2$ to get (in the mind's eye) $\frac{x+4}{x-2} = \frac{A(x+2)}{(x-2)} + B$. Set $x=-2$ to get $B = -\frac{1}{2}$. All together we have

$$\int \frac{x^2+x}{x^2-4} dx = \int \left(1 + \frac{\frac{3}{2}}{x-2} - \frac{\frac{1}{2}}{x+2}\right) dx = x + \frac{3}{2} \ln|x-2| - \frac{1}{2} \ln|x+2| + C.$$

5. $\int \frac{7x^2+14x+15}{(x^2+3)(x+7)} dx$ requires no division. Why not? We have degree 2 over degree 3. Also x^2+3 cannot be factored further, so there are just two partial fractions:

$$\frac{7x^2+14x+15}{(x^2+3)(x+7)} = \frac{A}{x+7} + \frac{Bx+C}{x^2+3}. \quad \text{Use } Bx+C \text{ over a quadratic, not just } B!$$

Cover up $x+7$ and set $x=-7$ to get $\frac{260}{52} = A$, or $A=5$. So far we have

$$\frac{7x^2+14x+15}{(x^2+3)(x+7)} = \frac{5}{x+7} + \frac{Bx+C}{x^2+3}.$$

We can set $x=0$ (because zero is easy) to get $\frac{15}{21} = \frac{5}{7} + \frac{C}{3}$, or $C=0$. Then set $x=-1$ to get $\frac{8}{24} = \frac{5}{6} + \frac{-B}{4}$. Thus $B=2$. Our integration problem is $\int \left(\frac{5}{x+7} + \frac{2x}{x^2+3}\right) dx = 5 \ln|x+7| + \ln(x^2+3) + C$.

6. (Problem 7.5.25) By substitution change $\int \frac{1+e^x}{1-e^x}$ to $\int \frac{P(u)}{Q(u)} du$. Then integrate.

- The ratio $\frac{1+e^x}{1-e^x} dx$ does not contain polynomials. Substitute $u = e^x$, $du = e^x dx$, and $dx = \frac{du}{u}$ to get $\frac{(1+u)du}{(1-u)u}$. A perfect set-up for partial fractions!

$$\frac{1+u}{u(1-u)} = \frac{A}{u} + \frac{B}{1-u} = \frac{1}{u} + \frac{2}{1-u}.$$

The integral is $\ln|u| - 2 \ln|1-u| = x - 2 \ln|1-e^x| + C$.

Read-throughs and selected even-numbered solutions :

The idea of **partial fractions** is to express $P(x)/Q(x)$ as a **sum** of simpler terms, each one easy to integrate. To begin, the degree of P should be **less than** the degree of Q . Then Q is split into **linear** factors like $x-5$ (possibly repeated) and quadratic factors like x^2+x+1 (possibly repeated). The quadratic factors have two **complex** roots, and do not allow real linear factors.

A factor like $x-5$ contributes a fraction $A/(x-5)$. Its integral is **$A \ln|x-5|$** . To compute A , cover up $x-5$ in the denominator of P/Q . Then set $x=5$, and the rest of P/Q becomes A . An equivalent method puts all

fractions over a common denominator (which is Q). Then match the **numerators**. At the same point ($x = 5$) this matching gives A .

A repeated linear factor $(x - 5)^2$ contributes not only $A/(x - 5)$ but also $B/(x - 5)^2$. A quadratic factor like $x^2 + x + 1$ contributes a fraction $(Cx + D)/(x^2 + x + 1)$ involving C and D . A repeated quadratic factor or a triple linear factor would bring in $(Ex + F)/(x^2 + x + 1)^2$ or $G/(x - 5)^3$. The conclusion is that any P/Q can be split into **partial fractions**, which can always be integrated.

- 6 $\frac{1}{x(x-1)(x+1)} = -\frac{1}{x} + \frac{1/2}{x-1} + \frac{1/2}{x+1}$
 14 $x + 1\sqrt{x^2 + 0x + 1} \quad \frac{x^2+1}{x+1} = x - 1 + \frac{2}{x+1}$ 16 $\frac{1}{x^2(x-1)} = -\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x-1}$
 18 $\frac{x^2}{(x-3)(x+3)} = \frac{A(x+3)+B(x-3)}{(x-3)(x+3)}$ is impossible (no x^2 in the numerator on the right side).
 Divide first to rewrite $\frac{x^2}{(x-3)(x+3)} = 1 + \frac{9}{(x-3)(x+3)} =$ (now use partial fractions) $1 + \frac{3/2}{x-3} - \frac{3/2}{x+3}$.
 22 Set $u = \sqrt{x}$ so $u^2 = x$ and $2u \, du = dx$. Then $\int \frac{1-\sqrt{x}}{1+\sqrt{x}} dx = \int \frac{1-u}{1+u} 2u \, du =$ (divide $u + 1$ into $-2u^2 + 2u$) $= \int (-2u + 4 - \frac{4}{u+1}) du = -u^2 + 4u - 4 \ln|u+1| + C = -x + 4\sqrt{x} - 4 \ln(\sqrt{x} + 1) + C$.

7.5 Improper Integrals (page 309)

An improper integral is really a limit: $\int_0^\infty y(x) dx$ means $\lim_{b \rightarrow \infty} \int_0^b y(x) dx$. Usually we just integrate and substitute $b = \infty$. If the integral of $y(x)$ contains e^{-x} then $e^{-\infty} = 0$. If the integral contains $\frac{1}{x}$ or $\frac{1}{x+7}$ then $\frac{1}{\infty} = 0$. If the integral contains $\tan^{-1} x$ then $\tan^{-1} \infty = \frac{\pi}{2}$. The numbers are often convenient when the upper limit is $b = \infty$.

Similarly $\int_{-\infty}^\infty y(x) dx$ is really the sum of two limits. You have to use a and b to keep those limits separate: $\lim_{a \rightarrow -\infty} \int_a^0 y(x) dx + \lim_{b \rightarrow \infty} \int_0^b y(x) dx$. Normally just integrate $y(x)$ and substitute $a = -\infty$ and $b = \infty$.

EXAMPLE 1 $\int_{-\infty}^\infty \frac{dx}{x^2+5} = \left[\frac{1}{5} \tan^{-1} \frac{x}{5} \right]_{-\infty}^\infty = \frac{1}{5} \left(\frac{\pi}{2} \right) - \frac{1}{5} \left(-\frac{\pi}{2} \right) = \frac{\pi}{5}$.

Notice the lower limit, where $\tan^{-1} \frac{a}{5}$ approaches $-\frac{\pi}{2}$ as a approaches $-\infty$. Strictly speaking the solution should have separated the limits $a = -\infty$ and $b = \infty$:

$$\int_{-\infty}^\infty \frac{dx}{x^2+5} = \lim_{a \rightarrow -\infty} \left[\frac{1}{5} \tan^{-1} \frac{x}{5} \right]_a^0 + \lim_{b \rightarrow \infty} \left[\frac{1}{5} \tan^{-1} \frac{x}{5} \right]_0^b.$$

If $y(x)$ blows up *inside* the interval, the integral is really the sum of a left-hand limit and a right-hand limit.

EXAMPLE 2 $\int_{-2}^3 \frac{dx}{x^3}$ blows up at $x = 0$ inside the interval.

- If this was not in a section labeled “improper integrals,” would your answer have been $\left[-\frac{1}{2} x^{-2} \right]_{-2}^3 = \frac{1}{2} \left(\frac{1}{9} - \frac{1}{4} \right) = \frac{5}{72}$? This is a very easy mistake to make. But since $\frac{1}{x^3}$ is infinite at $x = 0$, the integral is improper. Separate it into the part up to $x = 0$ and the part beyond $x = 0$.

The integral of $\frac{1}{x^3}$ is $-\frac{1}{2x^2}$ which blows up at $x = 0$. Those integrals from -2 to 0 and from 0 to 3 are both infinite. This improper integral *diverges*.

- Notice: The question is whether the *integral* blows up, not whether $y(x)$ blows up. $\int_0^1 \frac{1}{\sqrt{x}} dx$ is OK.

Lots of times you only need to know whether or not the integral converges. This is where the comparison test comes in. Assuming $y(x)$ is positive, try to show that its unknown integral is smaller than a known finite integral (or greater than a known divergent integral).

EXAMPLE 3 $\int_1^\infty \frac{2+\cos x}{x^3} dx$ has $2+\cos x$ between 1 and 3. Therefore $\frac{2+\cos x}{x^3} \leq \frac{3}{x^3}$. Since $\int_1^\infty \frac{3}{x^3} dx$ converges to a finite answer, the original integral must converge. You could have started with $\frac{2+\cos x}{x^3} \geq \frac{1}{x^3}$. This is true, but it is not helpful! It only shows that the integral is greater than a convergent integral. The greater one could converge or diverge – this comparison doesn't tell.

EXAMPLE 4 $\int_5^\infty \frac{\sqrt{x}}{x+8} dx$ has $\frac{\sqrt{x}}{x+8} \approx \frac{\sqrt{x}}{x} = \frac{1}{x^{1/2}}$ for large x .

We suspect divergence (the area under $x^{-1/2}$ is infinite). To show a comparison, note $\frac{\sqrt{x}}{x+8} > \frac{\sqrt{x}}{3x}$. This is because 8 is smaller than $2x$ beyond our lower limit $x = 5$. Increasing the denominator to $3x$ makes the fraction smaller. The official reasoning is

$$\int_5^\infty \frac{\sqrt{x}}{x+8} dx > \int_5^\infty \frac{\sqrt{x}}{3x} dx = \int_5^\infty \frac{dx}{3x^{1/2}} = \lim_{b \rightarrow \infty} \frac{2}{3} x^{1/2} \Big|_5^b = \infty.$$

5. (Problem 7.5.37) What is improper about the area between $y = \sec x$ and $y = \tan x$?

The area under the secant graph minus the area under the tangent graph is

$$\int_0^{\pi/2} \sec x dx - \int_0^{\pi/2} \tan x dx = \ln(\sec x + \tan x) \Big|_0^{\pi/2} + \ln(\cos x) \Big|_0^{\pi/2} = \infty - \infty.$$

The separate areas are infinite! However we can subtract before integrating:

$$\begin{aligned} \int_0^{\pi/2} (\sec x - \tan x) dx &= [\ln(\sec x + \tan x) + \ln(\cos x)]_0^{\pi/2} \\ &= [\ln(\cos x)(\sec x + \tan x)]_0^{\pi/2} = [\ln(1 + \sin x)]_0^{\pi/2} = \ln 2 - \ln 1 = \ln 2. \end{aligned}$$

This is perfectly correct. The difference of areas comes in Section 8.1.

Read-throughs and selected even-numbered solutions :

An improper integral $\int_a^b y(x) dx$ has lower limit $a = -\infty$ or upper limit $b = \infty$ or y becomes infinite in the interval $a \leq x \leq b$. The example $\int_1^\infty dx/x^3$ is improper because $b = \infty$. We should study the limit of $\int_1^b dx/x^3$ as $b \rightarrow \infty$. In practice we work directly with $-\frac{1}{2}x^{-2} \Big|_1^\infty = \frac{1}{2}$. For $p > 1$ the improper integral $\int_p^\infty x^{-p} dx$ is finite. For $p < 1$ the improper integral $\int_0^1 x^{-p} dx$ is finite. For $y = e^{-x}$ the integral from 0 to ∞ is 1.

Suppose $0 \leq u(x) \leq v(x)$ for all x . The convergence of $\int v(x) dx$ implies the convergence of $\int u(x) dx$. The divergence of $\int u(x) dx$ implies the divergence of $\int v(x) dx$. From $-\infty$ to ∞ , the integral of $1/(e^x + e^{-x})$ converges by comparison with $1/e^{|x|}$. Strictly speaking we split $(-\infty, \infty)$ into $(-\infty, 0)$ and $(0, \infty)$. Changing

to $1/(e^x - e^{-x})$ gives divergence, because $e^x = e^{-x}$ at $x = 0$. Also $\int_{-\pi}^{\pi} dx/\sin x$ diverges by comparison with $\int dx/x$. The regions left and right of zero don't cancel because $\infty - \infty$ is **not zero**.

2 $\int_0^1 \frac{dx}{x^\pi} = \left[\frac{x^{1-\pi}}{1-\pi}\right]_0^1$ diverges at $x = 0$: infinite area

8 $\int_{-\infty}^{\infty} \sin x dx$ is not defined because $\int_a^b \sin x dx = \cos a - \cos b$ does not approach a limit as $b \rightarrow \infty$ and $a \rightarrow -\infty$

16 $\int_0^{\infty} \frac{e^x dx}{(e^x - 1)^p} = (\text{set } u = e^x - 1) \int_0^{\infty} \frac{du}{u^p}$ which is infinite: diverges at $u = 0$ if $p \geq 1$, diverges at $u = \infty$ if $p \leq 1$.

18 $\int_0^1 \frac{dx}{x^6 + 1} < \int_0^1 \frac{dx}{1} = 1$: convergence

24 $\int_0^1 \sqrt{-\ln x} dx < \int_0^{1/e} (-\ln x) dx + \int_{1/e}^1 1 dx = [-x \ln x + x]_0^{1/e} + [x]_{1/e}^1 = \frac{1}{e} + 1$: convergence
(note $x \ln x \rightarrow 0$ as $x \rightarrow 0$)

36 $\int_a^b \frac{x dx}{1+x^2} = \left[\frac{1}{2} \ln(1+x^2)\right]_a^b = \frac{1}{2} \ln(1+b^2) - \frac{1}{2} \ln(1+a^2)$. As $b \rightarrow \infty$ or as $a \rightarrow -\infty$ (separately!) there is no limiting value. If $a = -b$ then the answer is zero - but we are not allowed to connect a and b .

40 The red area in the right figure has an extra unit square (area 1) compared to the red area on the left.

7 Chapter Review Problems

Review Problems

R1 Why is $\int u(x)v(x)dx$ not equal to $(\int u(x)dx)(\int v(x)dx)$? What formula is correct?

R2 What method of integration would you use for these integrals?

$$\int x \cos(2x^2 + 1) dx \quad \int x \cos(2x + 1) \quad \int \cos^2(2x + 1) dx \quad \int \cos(2x + 1) \sin(2x + 1) dx$$

$$\int \cos^3(2x + 1) \sin^5(2x + 1) dx \quad \int \cos^4(2x + 1) \sin^2(2x + 1) dx \quad \int \cos 2x \sin 3x dx \quad \int \frac{\cos(2x+1)}{\sin(2x+1)} dx$$

R3 Which eight methods will succeed for these eight integrals?

$$\int \frac{x}{\sqrt{3+x^2}} dx \quad \int \frac{dx}{\sqrt{3+x^2}} \quad \int \frac{dx}{\sqrt{3-x^2}} \quad \int \frac{dx}{x^2-3}$$

$$\int \frac{dx}{x^2+2x-3} \quad \int \frac{dx}{\sqrt{x^2+2x}} \quad \int \frac{x dx}{x^2-3} \quad \int \frac{x^3 dx}{x^2-3}$$

R4 What is an improper integral? Show by example four ways a definite integral can be improper.

R5 Explain with two pictures the comparison tests for convergence and divergence of improper integrals.

Drill Problems

D1 $\int x^2 \ln x \, dx$

D2 $\int e^x \sin 2x \, dx$

D3 $\int x^3 \sqrt{1-x^2} \, dx$

D4 $\int \frac{x^3 dx}{x^2+4x+3}$

D5 $\int \frac{\ln x}{\sqrt{x}} \, dx$

D6 $\int \tan^3 2x \sec^2 2x \, dx$

D7 $\int e^{e^x} e^x \, dx$

D8 $\int \frac{dx}{\sqrt{3-2x-x^2}}$

D9 $\int \sin(\ln x) \, dx$

D10 $\int \frac{6x+4}{x^3-4x} \, dx$

D11 $\int \sin^{-1} \sqrt{x} \, dx$

D12 $\int \cos^4 2x \sin^2 2x \, dx$

D13 $\int \frac{dx}{(4-x^2)^{3/2}}$

D14 $\int \frac{9x+36}{x^3+6x^2+9x} \, dx$

Evaluate the improper integrals **D15** to **D20** or show that they diverge.

D15 $\int_0^{\pi/2} \frac{\cos x \, dx}{1-\sin x}$

D16 $\int_1^{\infty} \frac{\ln x}{\sqrt{x}} \, dx$

D17 $\int_0^{\infty} x e^{-x} \, dx$

D18 $\int_{-1}^{\infty} x^{-1/3} \, dx$

D19 $\int_0^{33} \frac{dx}{(1-x)^{3/5}}$

D20 $\int_e^{\infty} \frac{dx}{x(\ln x)^{3/2}}$

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