CHAPTER 11 VECTORS AND MATRICES

11.1 Vectors and Dot Products

One side of the brain thinks of a vector as a pair of numbers \((x, y)\) or a triple of numbers \((x, y, z)\) or a string \(n\) numbers \((x_1, x_2, \ldots, x_n)\). This is the algebra side. The other hemisphere of the brain deals with geometry. It thinks of a vector as a point (or maybe an arrow.) For \((x, y)\) the point is in a plane and for \((x, y, z)\) the point is in 3-dimensional space. The picture gets fuzzy in \(n\) dimensions but our intuition is backed up by algebra.

The big operation on vectors is to take linear combinations \(au + bv\). The second biggest operation is to take dot products \(u \cdot v\). Those are the key steps in matrix multiplication and they are best seen by example!

\[
\begin{bmatrix} 3 & 1 \end{bmatrix} + 1 \begin{bmatrix} 4 & 5 \end{bmatrix} = \begin{bmatrix} 7 & 11 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 5 \end{bmatrix} \cdot \begin{bmatrix} 4 & 5 \end{bmatrix} = 14.
\]

Linear combinations produce vectors. Dot products yield numbers. The dot product of a vector with itself is the length squared. The dot product of a vector with another vector reveals the angle between them:

\[
v = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{has} \quad v \cdot v = |v|^2 = 5. \quad \text{The length is} \quad |v| = \sqrt{5}.
\]

\[
v = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad w = \begin{bmatrix} 4 \\ 5 \end{bmatrix} \quad \text{have} \quad \cos \theta = \frac{v \cdot w}{|v||w|} = \frac{14}{\sqrt{5}\sqrt{41}} = .9778. \quad \text{The angle is} \quad \theta = \cos^{-1}.9778 = .21
\]

Since cosines are never above 1, dot products \(v \cdot w\) are never above \(|v|\) times \(|w|\). This is the Cauchy-Schwarz inequality.

1. Find the lengths of \(v = \begin{bmatrix} 3 \\ 4 \end{bmatrix}\) and \(w = \begin{bmatrix} 4 \\ -3 \end{bmatrix}\). What is the angle between them?

   - Since \(v \cdot v = 9 + 16 = 25\), the length is \(|v| = \sqrt{25} = 5\). Similarly \(w \cdot w = 25\) and the length is \(|w| = 5\). The dot product is \(v \cdot w = 12 - 12 = 0\). The cosine of \(\theta\) is zero! Therefore \(\theta = 90^\circ\) and these two vectors are perpendicular.

The arrow from the origin to the point \(v = (3, 4)\) has slope \(\frac{4}{3}\). The arrow to the other point \(w = (4, -3)\) has slope \(-\frac{3}{4}\). The slopes multiply to give \(-1\). This is our old test for perpendicular lines. Dot products give a new and better test \(v \cdot w = 0\). This applies in three dimensions or in \(n\) dimensions.

Read-throughs and selected even-numbered solutions:

A vector has length and direction. If \(v\) has components 6 and \(-8\), its length is \(|v| = 10\) and its direction vector is \(u = .6i - .8j\). The product of \(|v|\) with \(u\) is \(v\). This vector goes from \((0,0)\) to the point \(x = 6, y = -8\). A combination of the coordinate vectors \(i = (1, 0)\) and \(j = (0, 1)\) produces \(v = x i + y j\).

To add vectors we add their components. The sum of \((6, -8)\) and \((1,0)\) is \((7, -8)\). To see \(v + i\) geometrically, put the tail of \(i\) at the head of \(v\). The vectors form a parallelogram with diagonal \(v + i\). (The other diagonal is \(v - i\).) The vectors \(2v\) and \(-v\) are \((12, -16)\) and \((-6, 8)\). Their lengths are 20 and 10.

In a space without axes and coordinates, the tail of \(V\) can be placed anywhere. Two vectors with the same components or the same length and direction are the same. If a triangle starts with \(V\) and continues with \(W\), the third side is \(V + W\). The vector connecting the midpoint of \(V\) to the midpoint of \(W\) is \(\frac{1}{2}(V + W)\). That vector is half of the third side. In this coordinate-free form the dot product is \(V \cdot W = |V||W| \cos \theta\).

Using components, \(V \cdot W = V_1 W_1 + V_2 W_2 + V_3 W_3\) and \((1,2,1) \cdot (2,-3,7) = 3\). The vectors are perpendicular if \(V \cdot W = 0\). The vectors are parallel if \(V\) is a multiple of \(W\). \(V \cdot V\) is the same as \(|V|^2\). The dot
product of $U + V$ with $W$ equals $U \cdot W + V \cdot W$. The angle between $V$ and $W$ has $\cos \theta = V \cdot W / |V||W|$. When $V \cdot W$ is negative then $\theta$ is greater than $90^\circ$. The angle between $i + j$ and $i + k$ is $\pi/3$ with cosine $\frac{1}{2}$.

The Cauchy-Schwarz inequality is $|V \cdot W| \leq |V||W|$, and for $V = i + j$ and $W = i + k$ it becomes $1 \leq 2$.

12 We want $V \cdot (W - cV) = 0$ or $V \cdot W = cV \cdot V$. Then $c = \frac{5}{2} = 2$ and $W - cV = (-1, 0, 1)$.

14 (a) Try two possibilities: keep clock vectors 1 through 5 or 1 through 6. The five add to $2 \cos 30^\circ + 2 \cos 60^\circ = 2 \sqrt{3} = 3.73$ (in the direction of 3:00). The six add to $2 \cos 15^\circ + 2 \cos 45^\circ + 2 \cos 75^\circ = 3.86$ which is longer (in the direction of 3:30). (b) The 12 o'clock vector (call it $j$ because it is vertical) is subtracted from all twelve clock vectors. So the sum changes from $V = 0$ to $V^* = -12j$.

18 (a) The points $tB$ form a line from the origin in the direction of $B$. (b) $aA + tB$ forms a line from $A$ in the direction of $B$. (c) $sA + tB$ forms a plane containing $A$ and $B$.

(d) $v - A = v \cdot B$ means $\cos \theta_1 = |B|$, where $\theta_1$ and $\theta_2$ are the angles from $v$ to $A$ and $B$.

Then $v$ is on the plane through the origin that gives this fixed number. (If $|A| = |B|$ the plane bisects the angle between those vectors.)

28 $I \cdot J = \frac{1+1}{2} = \frac{1-1}{2} = 0$. Add $i + j = \sqrt{2}I$ to $i - j = \sqrt{2}J$ to find $i = \frac{\sqrt{2}}{2}(I + J)$. Substitute back to find $j = \frac{\sqrt{2}}{2}(I - J)$. Then $A = 2i + 3j = \sqrt{2}(I + J) + \frac{3\sqrt{2}}{2}(I - J) = aI + bJ$ with $a = \sqrt{2} + \frac{3\sqrt{2}}{2}$ and $b = \sqrt{2} - \frac{3\sqrt{2}}{2}$.

34 The diagonals are $A + B$ and $B - A$. Suppose $|A + B|^2 = A \cdot A + A \cdot B + B \cdot A + B \cdot B$ equals $|B - A|^2 = B \cdot B - A \cdot B - B \cdot A + B \cdot B$. After cancelling this is $4A \cdot B = 0$ (note that $A \cdot B$ is the same as $B \cdot A$).

The region is a rectangle.

40 Choose $W = (1, 1, 1)$. Then $V \cdot W = V_1 + V_2 + V_3$. The Schwarz inequality $|V \cdot W|^2 \leq |V|^2 |W|^2$ is $(V_1 + V_2 + V_3)^2 \leq 3(V_1^2 + V_2^2 + V_3^2)$.

42 $|A + B| \leq |A| + |B|$ or $|C| \leq |A| + |B|$ says that any side length is less than the sum of the other two side lengths. Proof: $|A + B|^2 \leq (\text{using Schwarz for } A \cdot B)|A|^2 + 2|A||B| + |B|^2 = (|A| + |B|)^2$.

11.2 Planes and Projections

The main point is to understand the equation of a plane (in three dimensions). In general it is $ax + by + cz = d$. Specific examples are $2x + 4y + z = 14$ and $2x + 4y + z = 0$. Those planes are parallel. They have the same normal vector $N = (2, 4, 1)$. The equations of the planes can be written as $N \cdot P = d$. A typical point in $NP = 14$ is $(x, y, z) = (2, 2, 2)$ because then $2x + 4y + z = 14$. A typical point in $NP = 0$ is $(2, -1, 0)$ because then $2x + 4y + z = 0$.

From $NP = 0$ we see that $N$ is perpendicular ( = normal ) to every vector $P$ in the plane. The components of $P$ are $(x, y, z)$ and the vector is $P = x\hat{i} + y\hat{j} + z\hat{k}$. This plane goes through the origin, because $x = 0, y = 0, z = 0$ satisfies the equation $2x + 4y + z = 0$. The other plane $NP = 14$ does not go through the origin. Its equation can be written as $NP = NP_0$ where $P_0$ is any particular point on the plane (since every one of those points has $NP_0 = 14$). Then the plane equation is $N \cdot (P - P_0) = 0$, which shows that the normal vector $N = (2, 4, 1)$ is perpendicular to every vector $P - P_0$ lying in the plane.

To repeat: $P$ and $P_0$ are vectors to the plane, starting at $(0,0,0)$. $P - P_0$ is a vector in the plane.
The distance to the plane is the shortest vector $P$ with $\mathbf{N} \cdot \mathbf{P} = 14$. The shortest $P$ is in the same direction as $\mathbf{N}$. Call it $\mathbf{P} = t \mathbf{N}$. Then $\mathbf{N} \cdot \mathbf{P} = 14$ becomes $t \mathbf{N} \cdot \mathbf{N} = 14$. In this example $\mathbf{N} \cdot \mathbf{N} = 2^2 + 4^2 + 1^2 = 21$. Therefore $21t = 14$ and $t = \frac{2}{3}$. The shortest $P$ is $\frac{2}{3} \mathbf{N}$.

The general formula for distance from $(0,0,0)$ to $\mathbf{N} \cdot \mathbf{P} = d$ is $|d|$ divided by $|\mathbf{N}|$. The distance to $\mathbf{N} \cdot \mathbf{P} = 0$ is 0.

1. Find the plane through $P_0 = (1,2,2)$ perpendicular to $\mathbf{N} = (3,1,4)$.
   - Because of $\mathbf{N}$ the equation must be $3x + y + 4z = d$. Because of $P_0$ the number $d$ must be $3 \cdot 1 + 2 + 4 \cdot 2 = 13$. The plane is $3x + y + 4z = 13$ or $\mathbf{N} \cdot \mathbf{P} = \mathbf{N} \cdot P_0$.

2. Find the plane parallel to $x + y + z = 4$ but going through $(1,4,5)$.
   - To be parallel the equation must be $x + y + z = d$. To go through $(1,4,5)$ we need $d = 10$. The plane $x + y + z = 10$ has the desired normal vector $\mathbf{N} = (1,1,1)$. The distance from the origin is $|d| = \frac{10}{\sqrt{3}}$.

Projection onto a line We look for the projection of a vector $\mathbf{B}$ in the direction of $\mathbf{A}$. The projection is an unknown number $x$ times $\mathbf{A}$. Geometry says that the vector from $\mathbf{B}$ down to $x\mathbf{A}$ should be perpendicular to $\mathbf{A}$:

$$\mathbf{A} \cdot (\mathbf{B} - x\mathbf{A}) = 0 \quad \text{or} \quad \mathbf{A} \cdot \mathbf{B} = x\mathbf{A} \cdot \mathbf{A}.$$ Therefore $x = \frac{\mathbf{A} \cdot \mathbf{B}}{\mathbf{A} \cdot \mathbf{A}}$ and $\mathbf{P} = x\mathbf{A} = \frac{\mathbf{A} \cdot \mathbf{B}}{\mathbf{A} \cdot \mathbf{A}} \mathbf{A}$.

The projection is $\mathbf{P}$ (a new use for this letter). Its length is $|\mathbf{B}| \cos \theta$. This is the length of $\mathbf{B}$ "along $\mathbf{A}$.

3. In terms of $\mathbf{A}$ and $\mathbf{B}$, find the length of the projection $\mathbf{P}$.
   - The length is $|\mathbf{B}| \cos \theta = |\mathbf{B}| \frac{|\mathbf{A} \cdot \mathbf{B}|}{|\mathbf{A}||\mathbf{B}|} = \frac{|\mathbf{A} \cdot \mathbf{B}|}{|\mathbf{A}|}$.
   - From the formula $\mathbf{P} = \frac{\mathbf{A} \cdot \mathbf{B}}{\mathbf{A} \cdot \mathbf{A}} \mathbf{A}$ the length is again $|\mathbf{A} \cdot \mathbf{B}|$.  

4. Project the vector $\mathbf{B} = (1,3,1)$ onto the vector $\mathbf{A} = (2,1,2)$. Find the component $\mathbf{P}$ along $\mathbf{A}$ and also find the component $\mathbf{B} - \mathbf{P}$ perpendicular to $\mathbf{A}$.
   - The dot product is $\mathbf{A} \cdot \mathbf{B} = 7$. Also $\mathbf{A} \cdot \mathbf{A} = 9$. Therefore $\mathbf{P} = \frac{7}{9} \mathbf{A} = \left( \frac{14}{9}, \frac{7}{9}, \frac{14}{9} \right)$.

The perpendicular component is $\mathbf{B} - \mathbf{P} = (1,3,1) - \mathbf{P} = (-\frac{8}{9}, \frac{20}{9}, \frac{8}{9})$. Check that it really is perpendicular: $\mathbf{A} \cdot (\mathbf{B} - \mathbf{P}) = -\frac{10}{9} + \frac{20}{9} - \frac{10}{9} = 0$.

Read-throughs and selected even-numbered solutions:

A plane in space is determined by a point $P_0 = (x_0, y_0, z_0)$ and a normal vector $\mathbf{N}$ with components $(a, b, c)$. The point $P = (x, y, z)$ is on the plane if the dot product of $\mathbf{N}$ with $\mathbf{P} - \mathbf{P}_0$ is zero. (That answer was not $P$!) The equation of this plane is $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$. The equation is also written as $ax + by + cz = d$, where $d$ equals $ax_0 + by_0 + cz_0$ or $\mathbf{N} \cdot \mathbf{P}_0$. A parallel plane has the same $\mathbf{N}$ and a different $d$. A plane through the origin has $d = 0$.

The equation of the plane through $P_0 = (2,1,0)$ perpendicular to $\mathbf{N} = (3,4,5)$ is $3x + 4y + 5z = 10$. A second point in the plane is $P = (0,0,2)$. The vector from $P_0$ to $P$ is $(-2, -1, 2)$, and it is perpendicular to $\mathbf{N}$. (Check by dot product). The plane through $P_0 = (2,1,0)$ perpendicular to the $z$ axis has $\mathbf{N} = (0,0,1)$ and equation $z = 0$. 

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The component of \( \mathbf{B} \) in the direction of \( \mathbf{A} \) is \( |\mathbf{B}| \cos \theta \), where \( \theta \) is the angle between the vectors. This is \( \mathbf{A} \cdot \mathbf{B} \) divided by \( |\mathbf{A}| \). The projection vector \( \mathbf{P} \) is \( |\mathbf{B}| \cos \theta \) times a unit vector in the direction of \( \mathbf{A} \). Then \( \mathbf{P} = (|\mathbf{B}| \cos \theta)(\mathbf{A}/|\mathbf{A}|) \) simplifies to \((\mathbf{A} \cdot \mathbf{B})\mathbf{A}/|\mathbf{A}|^2 \). When \( \mathbf{B} \) is doubled, \( \mathbf{P} \) is doubled. When \( \mathbf{A} \) is doubled, \( \mathbf{P} \) is not changed. If \( \mathbf{B} \) reverses direction, so does \( \mathbf{P} \). If \( \mathbf{A} \) reverses direction, then \( \mathbf{P} \) stays the same.

When \( \mathbf{B} \) is a velocity vector, \( \mathbf{P} \) represents the velocity in the \( \mathbf{A} \) direction. When \( \mathbf{B} \) is a force vector, \( \mathbf{P} \) is the force component along \( \mathbf{A} \). The component of \( \mathbf{B} \) perpendicular to \( \mathbf{A} \) equals \( \mathbf{B} - \mathbf{P} \). The shortest distance from \((0,0,0)\) to the plane \( ax + by + cz = d \) is along the normal vector. The distance from the origin is \( |d|/\sqrt{a^2 + b^2 + c^2} \) and the point on the plane closest to the origin is \( \mathbf{P} = (da/db/dc)/(a^2 + b^2 + c^2) \). The distance from \( \mathbf{Q} = (x_1, y_1, z_1) \) to the plane is \( |d - ax_1 - by_1 - cz_1|/\sqrt{a^2 + b^2 + c^2} \).

6 The plane \( y - z = 0 \) contains the given points \((0,0,0)\) and \((1,0,0)\) and \((0,1,1)\). The normal vector is \( \mathbf{N} = \mathbf{j} - \mathbf{k} \).

12 (a) No: the line where the planes (or walls) meet is not perpendicular to itself. (b) A third plane perpendicular to the first plane could make any angle with the second plane.

22 If \( \mathbf{B} \) makes a 60° angle with \( \mathbf{A} \) then the length of \( \mathbf{P} \) is \( |\mathbf{B}| \cos 60° = 2 \cdot \frac{1}{2} = 1 \). Since \( \mathbf{P} \) is in the direction of \( \mathbf{A} \) it must be \( \frac{\mathbf{A}}{|\mathbf{A}|} \).

32 The points at distance 1 from the plane \( x + 2y + 2z = 3 \) fill two parallel planes \( x + 2y + 2z = 6 \) and \( x + 2y + 2z = 0 \). Check: The point \((0,0,0)\) on the last plane is a distance \( |d|/|\mathbf{N}| = \frac{3}{3} = 1 \) from the plane.

38 The point \( \mathbf{P} = \mathbf{Q} + t\mathbf{N} = (3 + t, 3 + 2t) \) lies on the line \( x + 2y = 4 \) if \((3 + t) + 2(3 + 2t) = 4 \) or \( 9 + 5t = 4 \) or \( t = -1 \). Then \( \mathbf{P} = (2, 1) \).

40 The drug runner takes \( \frac{1}{2} \) second to go the 4 meters. You have 5 meters to travel in the same \( \frac{1}{2} \) second.

Your speed must be 10 meters per second. The projection of your velocity (a vector) onto the drug runner’s velocity equals the drug runner’s velocity.

11.3 Cross Products and Determinants (page 423)

This section is mostly in three dimensions. We take the cross product of vectors \( \mathbf{A} \) and \( \mathbf{B} \). That produces a third vector \( \mathbf{A} \times \mathbf{B} \) perpendicular to the first two. The length of this cross product is adjusted to equal \( |\mathbf{A}| |\mathbf{B}| \sin \theta \). This leads to a (fairly) neat formula

\[
\mathbf{A} \times \mathbf{B} = \begin{vmatrix}
i & j & k \\
ar_1 & a_2 & a_3 \\
b_1 & b_2 & b_3 \\
\end{vmatrix} = (a_2b_3 - a_3b_2)i + (a_3b_1 - a_1b_3)j + (a_1b_2 - a_2b_1)k.
\]

1. Find the cross product of \( \mathbf{A} = \mathbf{i} \) and \( \mathbf{B} = \mathbf{k} \).

- The perpendicular direction is \( \mathbf{j} \). But the cross product \( \mathbf{i} \times \mathbf{k} \) is \( -\mathbf{j} \). This comes from the formula above, when you substitute 1,0,0 in the second row and 0,0,1 in the third row. The minus sign also comes from the right hand rule. The length of \( \mathbf{A} \times \mathbf{B} \) is \( |\mathbf{A}| |\mathbf{B}| \sin \theta \). The vectors \( \mathbf{A} = \mathbf{i} \) and \( \mathbf{B} = \mathbf{k} \) are perpendicular so \( \theta = 90° \) and \( \sin \theta = 1 \).
2. Find the cross product of $k$ and $i$.
   - In this opposite order the cross product changes sign: $k \times i = -j$.

3. Find $(i + k) \times (i + k)$. Also split this into $i \times i + i \times k + k \times i + k \times k$.
   - $i \times i$ is the zero vector. So is $k \times k$. So is any $A \times A$ because the angle is $\theta = 0$ and its sine is zero. The splitting also gives $0 - j + j + 0 = 0$.

Most determinants do not involve $i, j, k$. They are determinants of ordinary matrices. For a 2 by 2 matrix we get the area of a parallelogram. For a 3 by 3 matrix we get the volume of a parallelepiped (a box):

$$
\begin{vmatrix}
  a & b \\
  c & d \\
\end{vmatrix} = ad - bc = \text{area with corners } (0,0), (a,b), (c,d), (a+c,b+d)
$$

$$
\begin{vmatrix}
  a_1 & a_2 & a_3 \\
  b_1 & b_2 & b_3 \\
  c_1 & c_2 & c_3 \\
\end{vmatrix} = A \cdot (B \times C) = a_1b_2c_3 + \text{five other terms} = \text{volume of box.}
$$

A 3 by 3 determinant has six terms. An $n$ by $n$ determinant has $n!$ terms. Half the signs are minus.

4. Compute these determinants. Interpret as areas or volumes:

$$
\begin{array}{c|c|c}
  (a) & 2 & 5 \\
  4 & 10 & \\
\end{array} \quad \begin{array}{c|c|c}
  (b) & 0 & 1 \\
  0 & 0 & 1 \\
  1 & 0 & 0 \\
\end{array} \quad \begin{array}{c|c|c}
  (c) & 1 & 2 & 3 \\
  1 & 4 & 5 \\
  2 & 6 & 8 \\
\end{array}
$$

   - (a) The determinant is $(2)(10) - (-5)(4) = 0$. Because $(2,5)$ is parallel to $(4,10)$, the parallelogram is crushed. It has no area. The matrix has no inverse.
   - (b) The determinant is $(1)(1)(1) = 1$ with a plus sign. This is also $j \cdot (k \times i)$ which is $j \cdot j = 1$. It is the volume of a unit cube. Exchange two rows and the determinant is $-1$ (left-handed cube).
   - (c) The six terms in the determinant are $+32 + 18 + 20 - 24 - 30 - 16$. This gives zero! The volume is zero because the 3-dimension box is completely squashed into a plane. The third row of this matrix is the sum of the first two rows. $A \times B$ is perpendicular to that plane, $C$ is in that plane, so the determinant $(A \times B) \cdot C$ is zero.

**Read-throughs and selected even-numbered solutions**

The cross product $A \times B$ is a vector whose length is $|A||B|\sin \theta$. Its direction is perpendicular to $A$ and $B$. That length is the area of a parallelogram, whose base is $|A|$ and whose height is $|B|\sin \theta$. When $A = a_1i + a_2j$ and $B = b_1i + b_2j$, the area is $|a_1b_2 - a_2b_1|$. This equals a 2 by 2 determinant. In general $|A \cdot B|^2 + |A \times B|^2 = |A|^2|B|^2$.

The rules for cross products are $A \times A = 0$ and $A \times B = -(B \times A)$ and $A \times (B + C) = A \times B + A \times C$. In particular $A \times B$ needs the right-hand rule to decide its direction. If the fingers curl from $A$ towards $B$ (not more than $180^\circ$), then $A \times B$ points along the right thumb. By this rule $i \times j = k$ and $i \times k = -j$ and $j \times k = i$.

The vectors $a_1i + a_2j + a_3k$ and $b_1i + b_2j + b_3k$ have cross product $(a_2b_3 - a_3b_2)i + (a_3b_1 - a_1b_3)j + (a_1b_2 - a_2b_1)k$. The vectors $A = i + j + k$ and $B = i + j$ have $A \times B = -i + j$. *(This is also the 3 by 3 3*
11.4 Matrices and Linear Equations

A system of linear equations is written $Ax = b$ or in this book $Au = d$. The coefficient matrix $A$ multiplies the unknown vector $u$. This matrix-vector multiplication is set up so that it reproduces the given linear equations:

$$\begin{align*}
2x + 5y &= 9 \\
x + 3y &= 5
\end{align*}$$

becomes

$$\begin{bmatrix}
2 & 5 \\
1 & 3
\end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \end{bmatrix} = d$$

Then the solution is $u = A^{-1}d$. It can be found in three or more ways:

(a) Find $A^{-1}$ explicitly. (This matrix satisfies $A A^{-1} = I$) Then multiply $A^{-1}$ times $d$.

(b) Use Cramer's Rule. Then $x$ and $y$ are ratios of determinants.
(c) Eliminate $x$ from the second equation by subtracting $\frac{1}{2}$ of the first equation. From $3 - \frac{1}{2}(5)$ and $5 - \frac{1}{2}(9)$ you get $\frac{1}{2}y = \frac{1}{2}$. Then $y = 1$. Back substitution gives $x = 2$.

Elimination (c) is preferred for larger problems. For a $2 \times 2$ problem it is reasonable to know $A^{-1}$:

$$
\begin{bmatrix}
  a & b \\
  c & d
\end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix}
  d & -b \\
  -c & a
\end{bmatrix}
$$

and

$$
\begin{bmatrix}
  2 & 5 \\
  1 & 3
\end{bmatrix}^{-1} = \frac{1}{1} \begin{bmatrix}
  3 & -5 \\
  -1 & 2
\end{bmatrix}.
$$

The multiplication $A^{-1}d = \begin{bmatrix}
  3 \\
  -1
\end{bmatrix}$ gives $u = \begin{bmatrix}
  x \\
  y
\end{bmatrix} = \begin{bmatrix}
  2 \\
  1
\end{bmatrix}$ which agrees with elimination.

Multiply by dot products $3 \cdot 9 - 5 \cdot 5 = 2$ and $-1 \cdot 9 + 2 \cdot 5 = 1$ or by linear combination of columns:

$$
9 \begin{bmatrix}
  3 \\
  -1
\end{bmatrix} + 5 \begin{bmatrix}
  -5 \\
  2
\end{bmatrix} = \begin{bmatrix}
  2 \\
  1
\end{bmatrix}.
$$

1. Write the equations

$$
\begin{align*}
  x + 3y &= 0 \\
  2x + 4y &= 10
\end{align*}
$$

as $Au = d$. Solve by $A^{-1}$ and also by elimination.

- $A = \begin{bmatrix}
  1 & 3 \\
  2 & 4
\end{bmatrix}$ has $A^{-1} = \frac{1}{2} \begin{bmatrix}
  -2 & 3 \\
  -1 & 1
\end{bmatrix}$. Multiply by $d = \begin{bmatrix}
  0 \\
  10
\end{bmatrix}$ to get $u = \begin{bmatrix}
  15 \\
  -5
\end{bmatrix}$.

- Eliminate $x$ by subtracting two times the first equation from the second equation. This leaves $-2y = 10$. Then $y = -5$. Back substitution gives $x = 15$.

**Projection onto a plane** We look for the combination $p = x \ a + y \ b$ that is closest to a given vector $d$. It is the projection of $d$ onto the plane of $a$ and $b$. The numbers $x$ and $y$ come from the "normal equations"

$$
\begin{align*}
(a \cdot a) \ x + (a \cdot b) \ y &= a \cdot d \\
(b \cdot a) \ x + (b \cdot b) \ y &= b \cdot d
\end{align*}
$$

2. Project the vector $d = (1,2,4)$ onto the plane of $a = (1,1,1)$ and $b = (1,2,3)$. Interpret this projection as least squares fitting by a straight line.

- After computing dot products, the normal equations are $3x + 6y = 7$ and $6x + 14y = 17$. Subtract 2 times the first to get $2y = 3$. Then $y = \frac{3}{2}$ and $x = -\frac{5}{3}$. The projection is $p = -\frac{2}{3}(1,1,1) + \frac{3}{2}(1,2,3)$.

- To see the line-fitting problem, write down $x \ a + y \ b \approx d$ (this has no solution!):

$$
\begin{align*}
x \begin{bmatrix} 1 \\ 1 \end{bmatrix} + y \begin{bmatrix} 2 \\ 4 \end{bmatrix} &\approx \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{or} \quad x + 2y \approx 2 \\
x + 3y &\approx 4.
\end{align*}
$$

The straight line $f(t) = x + yt$ has intercept $x$ and slope $y$. We are trying to make it go through the three points $(1,1), (2,2)$, and $(3,4)$. This is doomed to failure. If those points were on a line, we could solve our three equations. We can't. The best solution (the least squares solution) has $x = -\frac{2}{3}$ and $y = \frac{2}{3}$. That line $f(t) = -\frac{2}{3} + \frac{2}{3}t$ comes closest to the three points. It minimises the sum of squares of the three errors.

**Read-throughs and selected even-numbered solutions**

The equations $3x + y = 8$ and $x + y = 6$ combine into the vector equation $x[\begin{bmatrix} 3 \\ 1 \end{bmatrix} + y[\begin{bmatrix} 1 \\ 1 \end{bmatrix}] = \begin{bmatrix} 8 \\ 6 \end{bmatrix} = d$. The left side is $Au$ with coefficient matrix $A = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$ and unknown vector $u = \begin{bmatrix} x \\ y \end{bmatrix}$. The determinant of $A$ is 2, so this problem is not singular. The row picture shows two intersecting lines. The column picture shows...
11.4 Matrices and Linear Equations

\[ xa + yb = d, \text{ where } a = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \text{ and } b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \] The inverse matrix is \( A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -1 & 3 \end{bmatrix} \). The solution is \( u = A^{-1}d = \begin{bmatrix} 1 \\ 5 \end{bmatrix} \).

A matrix-vector multiplication produces a vector of dot products from the rows, and also a combination of the columns:

\[
\begin{bmatrix} A \\ B \end{bmatrix} \cdot u = [ A \cdot u ] \quad \text{and} \quad [ a \ b ] \cdot \begin{bmatrix} x \\ y \end{bmatrix} = [ xa + yb ] \quad \text{and} \quad [ 3 \ 1 ] \cdot [ 1 \ 5 ] = [ 8 \ 6 ].
\]

If the entries are \( a, b, c, d \), the determinant is \( D = ad - bc \). \( A^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \) divided by \( D \). Cramer's Rule shows components of \( u = A^{-1}d \) as ratios of determinants: \( x = (b_2d_1 - b_1d_2)/D \) and \( y = (a_1d_2 - a_2d_1)/D \).

A matrix-matrix multiplication \( MV \) yields a matrix of dot products, from the rows of \( M \) and columns of \( V \):

\[
\begin{bmatrix} A \\ B \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = [ A \cdot v_1 ] \quad \text{and} \quad [ a \ b ] \cdot \begin{bmatrix} x \\ y \end{bmatrix} = [ xa + yb ] \quad \text{and} \quad [ 3 \ 1 ] \cdot [ 1 \ 5 ] = [ 8 \ 6 ].
\]

The last line contains the identity matrix, denoted by \( I \). It has the property that \( IA = AI = A \) for every matrix \( A \), and \( Iu = u \) for every vector \( u \). The inverse matrix satisfies \( A^{-1}A = I \). Then \( Au = d \) is solved by multiplying both sides by \( A^{-1} \), to give \( u = A^{-1}d \). There is no inverse matrix when \( \det A = 0 \).

The combination \( xa + yb \) is the projection of \( d \) when the error \( d - xa - yb \) is perpendicular to \( a \) and \( b \). If \( a = (1,1,1), b = (1,2,3), \) and \( d = (0,8,4) \), the equations for \( x \) and \( y \) are \( 3x + 6y = 12 \) and \( 6x + 14y = 28 \). Solving them also gives the closest line to the data points \((1,0), (2,8),\) and \((3,4)\). The solution is \( x = 0, y = 2 \), which means the best line is horizontal. The projection is \( 0a + 2b = (2,4,6) \). The three error components are \((-2, 4, -2)\). Check perpendicularity: \((1,1,1) \cdot (-2, 4, -2) = 0 \) and \((1,2,3) \cdot (-2, 4, -2) = 0 \). Applying calculus to this problem, \( x \) and \( y \) minimize the sum of squares \( E = (x - y)^2 + (8 - x - 2y)^2 + (4 - x - 3y)^2 \).

8 The solution is \( x = \frac{d - b}{ad - bc}, y = \frac{a - c}{ad - bc} \) (ok to use Cramer's Rule). The solution breaks down if \( ad = bc \).

\[
\frac{d - b}{ad - bc} \begin{bmatrix} a \\ c \end{bmatrix} + \frac{a - c}{ad - bc} \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.
\]

12 With \( A = I \) the equations are \( 1x + 0y = d_1 \quad 0x + 1y = d_2 \). Then \( x = \begin{bmatrix} d_1 \\ d_2 \\ 0 \\ 1 \end{bmatrix} = d_1 \) and \( y = \begin{bmatrix} 1 & d_1 \\ 0 & d_2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = d_2 \).

22 Problem 21 is \( .96x + .02y = d_1 \quad .04x + .98y = d_2 \). The sums down the columns of \( A \) are \(.96 + .04 = 1\) and \(.02 + .98 = 1 \).

Reason: Everybody has to be accounted for. Nobody is lost or gained. Then \( x + y \) (total population before move) equals \( d_1 + d_2 \) (total population after move).

34 \( AB = \begin{bmatrix} 2 & 6 \\ 0 & 2 \end{bmatrix} \) has \( (AB)^{-1} = \frac{1}{4} \begin{bmatrix} 2 & -6 \\ 0 & 2 \end{bmatrix} \). Check that this is \( B^{-1}A^{-1} = \frac{1}{4} \begin{bmatrix} 1 & -2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & -4 \\ 0 & 1 \end{bmatrix} \).
Compute \( a \cdot a = 3 \) and \( a \cdot b = b \cdot a = 2 \) and \( b \cdot b = 6 \) and \( a \cdot d = 5 \) and \( b \cdot d = 6 \). The normal equation (14) is \( 3x + 2y = 5 \) with solution \( x = \frac{18}{14} = \frac{9}{7} \) and \( y = \frac{6}{14} = \frac{3}{7} \). The nearest combination \( 3x + 2y = 5 \) with solution \( x = \frac{9}{7} \) and \( y = \frac{3}{7} \). The vector of three errors is \( d - p = (\frac{2}{7}, -\frac{6}{7}, \frac{4}{7}) \). It is perpendicular to \( a \) and \( b \). The best straight line is \( f = x + yt = \frac{9}{7} + \frac{3}{7}t \).

11.5 Linear Algebra

This section is about 3 by 3 matrices \( A \), leading to three equations \( Au = d \) in three unknowns. The ideas of \( A^{-1} \) and determinants and elimination still apply – but the formulas are beginning to get complicated. The determinant formula (six terms) we know from Section 11.3. The inverse divides by this determinant \( D \). When \( D = 0 \) the matrix \( A \) has no inverse. Such a matrix is called singular. Otherwise see pages 438-439 for \( A^{-1} \).

Again we can solve \( Au = d \) in three ways: use \( A^{-1} \) which has \( D \) in the denominator, or use Cramer's Rule which has \( D \) in every denominator, or use elimination which will fail if \( D = 0 \).

1. Solve the three equations
   \[
   \begin{align*}
   x + y + z &= 5 \\
   x + 2y + 2z &= 9 \\
   y - z &= 0
   \end{align*}
   \]
   
   • My first choice is elimination. Subtract equation 1 from equation 2 to get \( y + z = 4 \). Subtract this from equation 3 to get \( -2z = -4 \). Then \( z = 2 \). Then \( y = 2 \). Then \( x = 1 \). The solution is \( u = (1, 2, 2) \).

   The determinant of \( A \) is \( D = -2 + 1 + 0 - 2 + 1 + 0 = -2 \). Find \( A^{-1} \) from the formulas:
   \[
   A^{-1} = \frac{1}{-2} \begin{bmatrix} -4 & 2 & 0 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix}.
   \]
   Check \( AA^{-1} = I \). Then multiply \( A^{-1}d = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \).

   The 1993 textbook "Introduction to Linear Algebra" by Gilbert Strang goes beyond these formulas. In calculus we are seeing lines and planes as special cases of curves and surfaces. In linear algebra we see lines and planes as special cases of \( n \)-dimensional vector spaces. Please take that course – it is truly useful.

Read-throughs and selected even-numbered solutions:

Three equations in three unknowns can be written as \( Au = d \). The vector \( u \) has components \( x, y, z \) and \( A \) is a \( 3 \times 3 \) matrix. The row picture has a plane for each equation. The first two planes intersect in a line, and all three planes intersect in a point, which is \( u \). The column picture starts with vectors \( a, b, c \) from the columns of \( A \) and combines them to produce \( xa + yb + xc \). The vector equation is \( xa + yb + xc = d \).

The determinant of \( A \) is the triple product \( a \cdot b \times c \). This is the volume of a box, whose edges from the origin are \( a, b, c \). If \( \det A = 0 \) then the system is singular. Otherwise there is an inverse matrix such that \( A^{-1}A = I \) (the identity matrix). In this case the solution to \( Au = d \) is \( u = A^{-1}d \).

The rows of \( A^{-1} \) are the cross products \( b \times c, c \times a, a \times b \), divided by \( D \). The entries of \( A^{-1} \) are 2 by 2 determinants, divided by \( D \). The upper left entry equals \( (b_2c_3 - b_3c_2)/D \). The 2 by 2 determinants needed...
for a row of \( A^{-1} \) do not use the corresponding column of \( A \).

The solution is \( u = A^{-1}d \). Its first component \( x \) is a ratio of determinants, \(|d b c|/|a b c|\). Cramer's Rule breaks down when \( \det A = 0 \). Then the columns \( a, b, c \) lie in the same plane. There is no solution to \( xa + yb + zc = d \), if \( d \) is not on that plane. In a singular row picture, the intersection of planes 1 and 2 is parallel to the third plane.

In practice \( u \) is computed by elimination. The algorithm starts by subtracting a multiple of row 1 to eliminate \( x \) from the second equation. If the first two equations are \( x - y = 1 \) and \( 3x + z = 7 \), this elimination step leaves \( 3y + z = 4 \). Similarly \( x \) is eliminated from the third equation, and then \( y \) is eliminated. The equations are solved by back substitution. When the system has no solution, we reach an impossible equation like \( 1 = 0 \). The example \( x - y = 1, 3x + z = 7 \) has no solution if the third equation is \( 3y + z = 5 \).

8 \( x + 2y + 2z = 0 \rightarrow x + 2y + 2z = 0 \rightarrow x + 2y + 2z = 0 \rightarrow x = -8 \)
\( 2x + 3y + 5z = 0 \rightarrow -y + z = 0 \rightarrow -y + z = 0 \rightarrow y = 2 \)
\( 2y + 2z = 8 \rightarrow 4x = 8 \rightarrow z = 2 \)

18 Choose \( d = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \) as right side. The same steps as in Problem 16 end with \( y - 3z = 0 \) and \( -y + 3z = 1 \).

Addition leaves \( 0 = 1 \). No solution. Note: The left sides of the three equations add to zero.

There is a solution only if the right sides (components of \( d \)) also add to zero.

20 \( BC = \begin{bmatrix} 0 & -1 & 3 \\ 1 & 0 & -6 \\ 2 & 2 & -18 \end{bmatrix} \) and \( CB = \begin{bmatrix} -20 & -13 & 1 \\ 4 & 2 & -1 \\ 16 & 11 & 0 \end{bmatrix} \). The columns of \( CB \) add to zero

(they are combinations of columns of \( C \), and those add to zero). \( BC \) and \( CB \) are singular because \( C \) is.

22 \( 2A = \begin{bmatrix} 2 & 8 & 0 \\ 0 & 4 & 12 \\ 0 & 0 & 6 \end{bmatrix} \) has determinant 48 which is 8 times \( \det A \). If an \( n \) by \( n \) matrix is multiplied by 2, the determinant is multiplied by \( 2^n \). Here \( 2^3 = 8 \).

28 \[
\begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1
\end{bmatrix}
\]
These are “even”

\[
\begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0
\end{bmatrix}
\]
These are “odd”

11 Chapter Review Problems

Review Problems

R1 If the vectors \( v \) and \( w \) form two sides of a triangle, the third side has squared length \((v - w) \cdot (v - w) = v \cdot v + w \cdot w - 2v \cdot w \). How is this connected to the Law of Cosines?

R2 Find unit vectors in the direction of \( i + j + k \) and \( 2i + 2j + k \).

R3 Explain why the derivative of a dot product \( v(t) \cdot w(t) \) is \( v(t) \cdot w'(t) + v'(t) \cdot w(t) \).
Chapter Review Problems

R4 Find a plane through \((1,1,1)\) that is perpendicular to the plane \(2x + 2y - z = 3\).

R5 How far is the origin from the plane through \((1,0,0), (0,2,0), (0,0,4)\)? What is the nearest point?

R6 Project \(B = (1,4,4)\) onto the vector \(A = (2,2,-1)\). Then project the projection back onto \(B\).

R7 Find the dot product and cross product of \(A = (3,1,2)\) and \(B = (1,-1,-1)\).

R8 Starting from a 2 by 2 matrix with entries \(a, b, c, d\), fill out the other 5 entries of a 3 by 3 matrix that has the same determinant.

R9 Multiply \(AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}\). Compute the determinant of \(AB\) and check that it equals the determinant of \(A\) times the determinant of \(B\).

R10 Fit the three points \((0,1), (1,3), (2,3)\) by the closest line \(f = x + yt\) (least squares).

Drill Problems

D1 Find the lengths and dot products of \(A = (1,2,1)\) and \(B = (1,1,2)\). Check the inequality \(|A \cdot B| \leq |A||B|\).

D2 Find the normal vector to the plane \(3x - y - z = 0\) and find a vector in the plane.

D3 Multiply \(A = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}\) times \(A^{-1}\) after computing \(A^{-1}\).

D4 For the same matrix compute \(A^2\) and \(2A\) and their determinants.

D5 Find the area of the parallelogram with \(u = i + 3j\) and \(v = 3i + j\) as two sides.

D6 Find the volume of the box with those sides \(u\) and \(v\) and third side \(w = j + 2k\).

D7 Write down two equations in \(x\) and \(y\) that have no solution. Change the right sides of those equations to produce infinitely many solutions. Change the left side to produce one unique solution and find it.
Resource: Calculus
Gilbert Strang

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