CHAPTER 9  POLAR COORDINATES AND COMPLEX NUMBERS

9.1  Polar Coordinates  

Polar coordinates \( r \) and \( \theta \) correspond to \( x = r \cos \theta \) and \( y = r \sin \theta \). The points with \( r > 0 \) and \( \theta = \pi \) are located on the negative \( x \) axis. The points with \( r = 1 \) and \( 0 \leq \theta \leq \pi \) are located on a semicircle. Reversing the sign of \( \theta \) moves the point \((x, y)\) to \((x, -y)\).

Given \( x \) and \( y \), the polar distance is \( r = \sqrt{x^2 + y^2} \). The tangent of \( \theta \) is \( y/x \). The point \((6, \theta)\) has \( r = 10 \) and \( \theta = \tan^{-1}(3) \). Another point with the same \( \theta \) is \((3, 4)\). Another point with the same \( r \) is \((10, 0)\). Another point with the same \( r \) and \( \tan \theta \) is \((-6, -8)\).

The polar equation \( r = \cos \theta \) produces a shifted circle. The top point is at \( \theta = \pi/4 \), which gives \( r = \sqrt{2}/2 \). When \( \theta \) goes from 0 to \( 2\pi \), we go two times around the graph. Rewriting as \( r^2 = r \cos \theta \) leads to the \( xy \) equation \( x^2 + y^2 = x \). Substituting \( r = \cos \theta \) into \( z = r \cos \theta \) yields \( z = \cos^2 \theta \) and similarly \( y = \cos \theta \sin \theta \). In this form \( x \) and \( y \) are functions of the parameter \( \theta \).

<table>
<thead>
<tr>
<th>( r )</th>
<th>( \theta )</th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>( \frac{\pi}{4} )</td>
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<tr>
<td>2</td>
<td>( \frac{\pi}{6} )</td>
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<tr>
<td>5</td>
<td>( \sqrt{2}, \frac{\pi}{4} )</td>
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<tr>
<td>7</td>
<td>( 0, \frac{\pi}{2} )</td>
</tr>
<tr>
<td>9</td>
<td>( \sqrt{10}, \sqrt{10} )</td>
</tr>
<tr>
<td>11</td>
<td>( \sqrt{3}, -1 )</td>
</tr>
</tbody>
</table>

2. \( z = -4, y = 0 \) has polar coordinates \( r = 4, \theta = \pi \)
3. \( z = -1, y = \sqrt{3} \) has polar coordinates \( r = 2, \theta = \frac{2\pi}{3} \).
4. \( z = 3, y = 4 \) has polar coordinates \( r = 5, \theta = \tan^{-1}(\frac{4}{3}) = .925 \).
5. \( r = 1, \theta = \frac{3\pi}{2} \) has rectangular coordinates \( x = 0, y = -1 \).
6. \( r = 3\pi, \theta = 3\pi \) has rectangular coordinates \( x = -3\pi, y = 0 \).
7. \( r = 2, \theta = \frac{5\pi}{6} \) has rectangular coordinates \( x = -\sqrt{3}, y = 1 \).

8. \( r = 1, \theta = \frac{\pi}{4} \) is a different point from \((3, \frac{\pi}{4})\) or \((-1, \frac{3\pi}{4})\) or \((-1, -\frac{\pi}{4})\) or \((-3, \frac{\pi}{4})\) or \((-3, -\frac{\pi}{4})\) or \((1, -\frac{\pi}{4})\) or \((1, \frac{\pi}{4})\) or \((-1, \frac{\pi}{4}) \) or \((1, -\frac{\pi}{4})\) or \((-1, \frac{\pi}{4})\) or \((-1, -\frac{\pi}{4})\) or \((-3, \frac{\pi}{4})\) or \((-3, -\frac{\pi}{4})\) or \((1, -\frac{\pi}{4})\) or \((1, \frac{\pi}{4})\) or \((-1, \frac{\pi}{4}) \) or \((1, -\frac{\pi}{4})\) or \((-1, \frac{\pi}{4})\) or \((-1, -\frac{\pi}{4})\) or \((-3, \frac{\pi}{4})\) or \((-3, -\frac{\pi}{4})\) or \((1, -\frac{\pi}{4})\) or \((1, \frac{\pi}{4})\).

9. \( r = 1, \theta = \frac{\pi}{4} \) is a different point from \((3, \frac{\pi}{4})\) or \((-1, \frac{\pi}{4})\) or \((-3, \frac{\pi}{4})\) or \((-3, -\frac{\pi}{4})\) or \((1, -\frac{\pi}{4})\) or \((1, \frac{\pi}{4})\) or \((-1, \frac{\pi}{4}) \) or \((1, -\frac{\pi}{4})\) or \((-1, \frac{\pi}{4})\) or \((-1, -\frac{\pi}{4})\) or \((-3, \frac{\pi}{4})\) or \((-3, -\frac{\pi}{4})\) or \((1, -\frac{\pi}{4})\) or \((1, \frac{\pi}{4})\) or \((-1, \frac{\pi}{4}) \) or \((1, -\frac{\pi}{4})\) or \((-1, \frac{\pi}{4})\) or \((-1, -\frac{\pi}{4})\) or \((-3, \frac{\pi}{4})\) or \((-3, -\frac{\pi}{4})\) or \((1, -\frac{\pi}{4})\) or \((1, \frac{\pi}{4})\).

10. \( r = 0, \theta = 0 \) is the same point as \((r, \theta)\) for \( r \neq 0, \theta \neq 0 \) (b) False (for fixed \( r \) we can add any multiple of \( 2\pi \) to \( \theta \)) (c) True (r sin \( \theta = 1 \) is the horizontal line \( y = 1 \)).

11. \( x = \sqrt{3}, y = 1 \) yields \( r = 2, \tan \theta = \frac{\sqrt{3}}{3} \). So does \( x = -\sqrt{3}, y = -1 \).

12. Take the line from \((0,0)\) to \((r_1, \theta_1)\) as the base (its length is \( r_1 \)). The height of the third point \((r_2, \theta_2)\), measured perpendicular to this base, is \( r_2 \) times \( \sin(\theta_2 - \theta_1) \).

13. The 13 values \( \theta = 0^\circ, 30^\circ, \cdots, 360^\circ \) give six different points with \( r = \sin \theta \). To go once around the circle take \( 0 \leq \theta < \pi \).
26 From \( x = \cos^2 \theta \) and \( y = \sin \theta \cos \theta \), square and add to find \( x^2 + y^2 = \cos^2 \theta (\cos^2 \theta + \sin^2 \theta) = \cos^2 \theta = x \).

28 Multiply \( r = a \cos \theta + b \sin \theta \) by \( r \) to find \( x^2 + y^2 = ax + by \). Complete squares in \( x^2 - ax = (x - \frac{a}{2})^2 - \left(\frac{a}{2}\right)^2 \) and similarly in \( y^2 - by \) to find \( (x - \frac{a}{2})^2 + (y - \frac{b}{2})^2 = \left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2 \). This is a circle centered at \( \left(\frac{a}{2}, \frac{b}{2}\right) \) with radius \( r = \sqrt{\left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2} = \frac{1}{2}\sqrt{a^2 + b^2} \).

30 The point \( z = \cos^2 \theta, y = \sin^2 \theta \) is generally not at the polar angle \( \theta \). For example let \( \theta = \frac{\pi}{6} = 30' \): then \( x = \frac{3}{4} \) and \( y = \frac{1}{4} \). The polar angle for this point has tangent \( \tan \theta \), but the tangent of \( \theta - r \theta \) is \( \tan \left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{3} \).

Conclusion: an angle named \( \theta \) is not automatically the polar angle. See Problem 9.5.40.

32 The second figure is not a closed curve as it stands. As the parameter \( t \) keeps going, the spaces around the circle fill up and the curve eventually closes (but the figure becomes less beautiful).

9.2 Polar Equations and Graphs (page 355)

The circle of radius 3 around the origin has polar equation \( r = 3 \). The 45° line has polar equation \( \theta = \frac{\pi}{4} \).

Those graphs meet at an angle of 90°. Multiplying \( r = 4 \cos \theta \) by \( r \) yields the \( xy \) equation \( x^2 + y^2 = 4x \). Its graph is a circle with center at (2,0). The graph of \( r = 4 \cos \theta \) is the line \( x = 4 \). The equation \( f = \cos^2 \theta \) is not changed when \( \theta \rightarrow -\theta \) (symmetric across the \( x \) axis) and when \( \theta \rightarrow \pi + \theta \) (or \( r \rightarrow -r \)). The graph of \( r = 1 + \cos \theta \) is a cardioid.

The graph of \( r = A/(1 + e \cos \theta) \) is a conic section with one focus at \((0, 0)\). It is an ellipse if \( e < 1 \) and a hyperbola if \( e > 1 \). The equation \( r = 1/(1 + \cos \theta) \) leads to \( r + x = 1 \) which gives a parabola. Then \( r \) is distance from origin \( 1 - x = \) distance from directrix \( y = 1 \). The equations \( r = 3(1 - x) \) and \( r = \frac{1}{3}(1 - x) \) represent a hyperbola and an ellipse. Including a shift and rotation, conics are determined by five numbers.

1 Line \( y = 1 \) 3 Circle \( x^2 + y^2 = 2x \) 5 Ellipse \( 3x^2 + 4y^2 = 1 - 2x \) 7 \( x, y, r \) symmetries
9 \( x \) symmetry only 11 No symmetry 13 \( x, y, r \) symmetries!
16 \( x^2 + y^2 = 6y + 8x \rightarrow (x - 4)^2 + (y - 3)^2 = 5^2 \), center \((4, 3)\) 17 \((2, 0), (0, 0)\)
19 \( r = 1 - \sqrt{2}/2, \theta = 3\pi/4 \); \( r = 1 + \sqrt{2}/2, \theta = 7\pi/4 \); \( (0, 0) \) 21 \( r = 2, \theta = \pm \pi/12, \pm 5\pi/12, \pm 7\pi/12, \pm 11\pi/12 \)
23 \((x, y) = (1, 1)\) 25 \( r = \cos 5\theta \) has 5 petals 27 \((x^2 + y^2 - x)^2 = x^2 + y^2 \)
29 \((x^2 + y^2)^3 = (x^2 - y^2)^2 \) 31 \( \cos \theta = -\frac{1}{2}, \sin \theta = \sqrt{3}/2 \rightarrow y = 2\sqrt{3}, x = -\frac{3}{2} \) 33 \( x = \frac{4}{3}, r = -\frac{5}{3} \) 35 .967

2 \( r \cos \theta - r \sin \theta = 2 \) is the straight line \( x - y = 2 \).
4 \( r = -2 \sin \theta \) is the circle \( r^2 = -2r \sin \theta \) or \( x^2 + y^2 = -2y \) or \( x^2 + (y + 1)^2 = 1 \); below the origin with center at \((0, -1)\) and radius 1.
6 \( r = \frac{1}{1 + 2 \cos \theta} \) is the hyperbola of Example 7 and Figure 9.5c: \( r + 2r \cos \theta = 1 \) is \( r = 1 - 2x \) or \( x^2 + y^2 = 1 - 4x + 4x^2 \).

The figure should show \( r = -1 \) and \( \theta = \pi \) on the right branch.
8 \( r^2 = 4 \sin 2\theta \) has loops in the first and third quadrants. It possesses \( r \) symmetry (change \( r \) to \(-r \) and the equation is unchanged). Changing \( \theta \rightarrow -\theta \) or \( \pi - \theta \) or \( 2\pi - \theta \) reverses the sign of the right hand side.
10 \( r^2 = 10 + 6 \cos \theta \) has \( x, y, \) and \( r \) symmetry. It comes in from \( r = 4 \) at \( \theta = 0 \) to \( r = 2 \) at \( \theta = \pi/4 \) and back out to \( r = 4 \) at \( \theta = 5\pi/4 \). Repeat in each quadrant to form a “star-fish”.
12 \( r = 1/\theta \) has \( y \) symmetry. Change \( \theta \) to \(-\theta \) and \( r \) to \(-r \); same equation \( (\theta \) to \( \pi - \theta \) gives a different equation:

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9.3 Slope, Length, and Area for Polar Curves (page 359)

must try both tests.) Note that the maximum of \( y = r \sin \theta = \frac{\sin \theta}{2} \) is \( y = 1 \) as \( \theta \to 0 \): the line \( y = 1 \) is a horizontal asymptote! As negative \( \theta \) approach zero, the spiral goes left toward the same asymptote \( y = 1 \).

14 \( r = 1 - 2 \sin 3 \theta \) has y axis symmetry: change \( \theta \) to \( \pi - \theta \), then \( \sin 3(\pi - \theta) = \sin(3\pi - 3\theta) = -\sin 3\theta \).

16 This is another case where the parameter \( t \) is not the polar angle. (The Earth completed a circle at \( t = 1 \).)

18 \( r^2 = \sin 2 \theta \) and \( r^2 = \cos 2 \theta \) are lemniscates (or \( \text{"spectacles"} \)). They meet when \( \sin 2 \theta = \cos 2 \theta \) or \( \theta = \frac{\pi}{4} \) or \( \frac{5\pi}{4} \) (\( r \) can be positive or negative). They also meet at the origin \( r = 0 \).

20 \( r = 1 + \cos \theta \) and \( r = 1 - \cos \theta \) are cardioids (reaching right to \( r = 2, \theta = 0 \) and left to \( r = 2, \theta = \pi \)).

They meet when \( \cos \theta = 0 \) at \( r = 1, \theta = \frac{\pi}{2} \) and \( r = 1, \theta = \frac{3\pi}{2} \). They also meet at the origin \( r = 0 \).

22 If \( \cos \theta = \frac{1}{4} \) and \( \cos \theta = 1 - r \) then \( \frac{\pi}{4} = 1 - r \) and \( \cos \theta = 2r - 4r - 4 = 0 \). This gives \( r = -2 \) and \( \theta = \pi \). The first \( r \) is negative and cannot equal \( 1 - \cos \theta \). The second gives \( \cos \theta = 1 - r \) and \( \theta = \frac{\pi}{2} \) or \( \theta = \pi \) \( r \) can be positive or negative). They also meet at the origin \( r = 0 \).

24 The limacon \( r = 1 + b \cos \theta \) has \( z = r \cos \theta = \cos \theta + b \cos^2 \theta \) and \( \frac{dr}{d\theta} = -\sin \theta - 2b \cos \theta \sin \theta \). Then \( \frac{d^2r}{d\theta^2} = -\cos \theta - 2b \cos^2 \theta + 2b \sin^2 \theta \) which equals \( 1 - 2b \) at \( \theta = \pi \). The dimple begins at \( b = \frac{1}{2} \).

At \( b = 1 \) it becomes the cusp in the cardioid.

26 The other 101 petals in \( r = \cos 101 \theta \) are duplicates of the first 101. For example \( \theta = r \) gives \( r = \cos 101 \% \), which is also \( \theta = 0, r = -1 \). (Note that \( \cos 100 \pi + 1 \) gives a new point.)

28 (a) Yes, \( z \) and \( y \) symmetry imply \( r \) symmetry. Reflections across the \( z \) axis and then the \( y \) axis take \( (z, y) \) to \( (-z, y) \) which is reflection through the origin. (b) The point \( r = -1, \theta = \$ \) satisfies the equation \( r = \cos 2 \theta \) and it is the same point as \( r = 1, \theta = \frac{\pi}{2} \).

30 (a) \( r^2 = \theta \) (b) \( x + y = 1 \) or \( r \cos \theta + \sin \theta = 1 \) or \( r = \frac{1}{\cos \theta + \sin \theta} \) (c) ellipse \( x^2 + 2y^2 = 1 \) or \( r^2(\cos^2 \theta + \sin^2 \theta) = 1 \)

32 (a) \( \theta = \frac{\pi}{2} \) gives \( r = 1 \); this is \( z = 0, y = 1 \) (b) The graph crosses the \( x \) axis at \( \theta = 0 \) and \( \pi \) where \( z = \frac{1}{1+\varepsilon} \) and \( z = \frac{1}{1+\varepsilon} \). The center of the graph is halfway between at \( x = \frac{1}{2}(1+\varepsilon) - \frac{1}{2} = \frac{1}{2} \). The second focus is twice as far from the origin at \( \frac{-2\varepsilon}{1-\varepsilon^2} \). (Check: \( \varepsilon = 0 \) gives center of circle, \( \varepsilon = 1 \) gives second focus of parabola at infinity.)

34 \( r = 1 + \frac{A}{\cos \theta} \) and \( r = \frac{B+D}{\cos \theta} \) are the same if \( C = \frac{A}{B} \) and \( D = \frac{C}{A} \). For the mirror image across the \( y \) axis, \( \theta \) becomes \( -\theta \) and \( \cos \theta \) changes sign.

36 Maximize \( y = \frac{A \sin \theta}{1+\cos \theta} \) where \( \frac{dy}{d\theta} = \frac{(1+\cos \theta)A \cos \theta + (A \sin \theta) \sin \theta}{(1+\cos \theta)^2} = 0 \). Then \( A \cos \theta + A \varepsilon = 0 \) or \( \cos \theta = -\varepsilon \) and \( y_{max} = \frac{A \sqrt{1-\varepsilon^2}}{\sqrt{1-\varepsilon^2}} = A (\frac{\sqrt{1-\varepsilon^2}}{\sqrt{1-\varepsilon^2}}) \) (which equals \( b \) in \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \)).

9.3 Slope, Length, and Area for Polar Curves (page 359)

A circular wedge with angle \( \Delta \theta \) is a fraction \( \Delta \theta / 2\pi \) of a whole circle. If the radius is \( r \), the wedge area is \( \frac{1}{2} r^2 \Delta \theta \). Then the area inside \( r = F(\theta) \) is \( \int\frac{1}{2} r^2 \, d\theta = \int \frac{1}{2} (F(\theta))^2 \, d\theta \). The area inside \( r = \theta^2 \) from 0 to \( \pi \) is \( \pi \). That spiral meets the circle \( r = 1 \) at \( \theta = 1 \). The area inside the circle and outside the spiral is \( \frac{\pi}{2} - \frac{1}{10} \). A chopped wedge of angle \( \Delta \theta \) between \( r_1 \) and \( r_2 \) has area \( \frac{1}{2} r_2^2 \Delta \theta - \frac{1}{2} r_1^2 \Delta \theta \).

The curve \( r = F(\theta) \) has \( z = r \cos \theta = F(\theta) \cos \theta \) and \( y = F(\theta) \sin \theta \). The slope \( dy/dx \) is \( dy/d\theta \) divided by \( dx/d\theta \). For length \( (ds)^2 = (dx)^2 + (dy)^2 = (dx)^2 + (rd\theta)^2 \). The length of the spiral \( r = \theta \) to \( \theta = \pi \) is
\[ \int \sqrt{1 + \theta^2} \, d\theta. \] The surface area when \( r = \theta \) is revolved around the x-axis is \( \int 2\pi y \, ds = \int 2\pi \theta \sin \theta \sqrt{1 + \theta^2} \, d\theta. \) The volume of that solid is \( \int \pi y^2 \, ds = \int \pi \theta^2 \sin^2 \theta (\cos \theta - \theta \sin \theta) \, d\theta. \)

1. Area \( \frac{3\pi}{2} \)  
5. Area \( \frac{5\pi}{6} \)  
7. Area \( \frac{\pi}{3} - \frac{1}{4} \)  
9. \( \int_{-\pi/3}^{\pi/3} \left( \frac{1}{2} \cos^2 \theta - \frac{(1 + \cos \theta)^2}{2} \right) \, d\theta = \pi \)

11. \( \theta = 0; \) left points \( r = \frac{1}{2}, \theta = \pm \frac{3\pi}{2}, x = -\frac{1}{4}, y = \pm \frac{\sqrt{3}}{4} \)

17. \( \theta = 0; \) left points \( r = \frac{1}{2}, \theta = \pm \frac{3\pi}{2}, x = -\frac{1}{4}, y = \pm \frac{\sqrt{3}}{4} \)

19. \( \frac{1}{2} \ln(\pi) = 40,000; \frac{1}{2} \ln(r\sqrt{r^2 + c^2} + c^2 l_n(r + \sqrt{r^2 + c^2})) \approx 40,000.001 \)

23. \( \theta = 0, y = 1 \) is on limacon but not circle \( \frac{1}{2} \ln(2\pi + \sqrt{1 + 4x^2}) + \pi \sqrt{1 + 4x^2} \)

27. \( \frac{3\pi}{2} \)  
29. \( \frac{1}{2} \) (base)(height) \( \approx \frac{1}{2}(r\Delta \theta)r \)

2 A = \( \int \frac{1}{2} r^2 \, d\theta = \int_{0}^{\pi} \frac{1}{2} (\sin^2 \theta + 2 \sin \theta \cos \theta + \cos^2 \theta) \, d\theta = \int_{0}^{\pi} \frac{1}{2} (1 + \sin 2\theta) \, d\theta \)

= \( \left[ \frac{1}{2} \theta + \frac{1}{2} \cos 2\theta \right]_{0}^{\pi} = \frac{\pi}{2} \)

The spiral \( r = \theta \) has \( \text{Length} = \int_{0}^{\pi} \sqrt{\theta^2 + \theta^2} \, d\theta = \frac{1}{2} \sec^2 \theta \tan \theta \)

As \( \theta \to \infty \), \( \sec \theta \to \infty \), \( \text{Area} = \int_{0}^{\pi} (0) \, d\theta = 0 \)
34 y = r \sin \theta = \sin \theta \cos \theta and x = \cos^2 \theta \) (which moves left). Volume = $\int \pi y^2 \, dx = \int_{0}^{\pi/2} \pi \sin^2 \theta \cos^2 \theta \, d\theta = 2\pi \int_{0}^{\pi/2} (\sin^2 \theta - \sin^4 \theta) \, d\theta = 2\pi \left[ \frac{1}{4} - \frac{1}{6} \right] = \frac{\pi}{3}.

Check: The sphere has radius \( \frac{1}{2} \) and volume $\frac{4}{3} \pi \left( \frac{1}{2} \right)^3 = \frac{\pi}{3}.

36 r = \sec \theta \) has $x = \sec \theta - 1$ and $d\theta = \sec^2 \theta \, d\theta$ as in Problem 26. Surface area = $\int 2\pi x \, dx = \int \pi (1) \sec^2 \theta \, d\theta = 2\pi \tan \theta = \pi (\sec \theta - 1)$, the surface is a cylinder.

38 The triangle connecting the three centers has 60° angles and base 2. Its area is $\frac{1}{3} \times 2 \times \sin 60^\circ = \frac{\sqrt{3}}{3}$. Subtract the area inside the circles and triangle: $3 \times \frac{\sqrt{3}}{3}$, remaining area = 4.

40 The parameter $\theta$ along the ellipse $x = 4 \cos \theta$, $y = 3 \sin \theta$ is not the angle from the origin. For example at $\theta = \frac{\pi}{4}$ the point $(x, y)$ is not on the 45° line. So the area formula $\int \frac{1}{2} r^2 \, d\theta$ does not apply. The correct area is $12\pi$.

9.4 Complex Numbers (page 364)

The complex number $3 + 4i$ has real part 3 and imaginary part 4. Its absolute value is $r = 5$ and its complex conjugate is $3 - 4i$. Its position in the complex plane is at $(3, 4)$. Its polar form is $r \cos \theta + ir \sin \theta = 5 \cos \theta + 5i \sin \theta$, or $5 e^{i \theta}$. Its square is $-7 - 14i$. Its $n$th power is $P^n e^{i \theta n}$.

The sum of $1 + i$ and $1 - i$ is 2. The product of $1 + i$ and $1 - i$ is 2. In polar form this is $\sqrt{2}e^{i\pi/4}$ times $\sqrt{2}e^{-i\pi/4}$. The quotient $(1 + i)/(1 - i)$ equals the imaginary number $i$. The number $1 + i$ equals $\sqrt{2}e^{i\pi/4}$. An eighth root of 1 is $\frac{1}{2} + \frac{\sqrt{3}}{2}i$. The other eighth roots are $w^2, w^3, \ldots, w^7, w^8 = 1$.

To solve $d^2 y/dt^2 = y$, look for a solution of the form $y = e^{ct}$. Substituting and canceling $e^{ct}$ leads to the equation $e^{2ct} = 1$. There are eight choices for $c$, one of which is $(-1 + i)/\sqrt{2}$. With that choice $e^{ct} = e^{-t/\sqrt{2}}$. The real solutions are $\text{Re } e^{ct} = e^{-t/\sqrt{2}}\cos \frac{t}{\sqrt{2}}$ and $\text{Im } e^{ct} = e^{-t/\sqrt{2}}\sin \frac{t}{\sqrt{2}}$.

1 Sum = 4, product = 5  5 Angles $\pm \frac{2\pi}{4}, \pm \frac{3\pi}{4}$  7 Real axis; imaginary axis; $\frac{1}{2}$ axis $x \geq 0$; unit circle
9 cd = 5 + 10i, $\frac{\pi}{2} = \frac{11\pi}{12} 11.2 \cos \theta, 1, -1, 1 13 Sum = 0, product = -1 15 $r^2 e^{i\theta} = e^{-i\theta} = \frac{1}{r} e^{-i\theta}$
17 Evenly spaced on circle around origin 19 $e^{it}, e^{-it}$ 21 $e^t, e^{-t}, e^0$ 23 cos 7t, sin 7t
29 $t = -2\pi^{-1/2}, y = -e^0/\sqrt{2}$ 31 $F$; $T$; at most 2; Re $c < 0$ 33 $\frac{1}{r} e^{-i\theta}, x = \frac{1}{r} \cos \theta, y = -\frac{1}{r} \sin \theta; \pm \frac{1}{r} e^{-i\theta/2}$

2 $1 + i$ has $r = \sqrt{2}$ and $\theta = \frac{\pi}{4}$; $(1 + i)^2 = 2i$ has $r = 2$ and $\theta = \frac{\pi}{2}$; $\frac{1}{1+i} = \frac{1}{2} - \frac{1}{2}i$ has $r = \frac{\sqrt{2}}{2}$ and $\theta = -\frac{\pi}{4}$.

4 The powers of $e^{2\pi i/3}$ are on the unit circle at equally spaced angles $\frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}, 2\pi$.
6 $4e^{i\pi/3} = (4 \cos \frac{\pi}{3} + 4 \sin \frac{\pi}{3}) = 2 + 2\sqrt{3}$. The square roots are $2e^{i\pi/6}$ and $-2e^{i\pi/6} = 2e^{i\pi/6}$.

8 $x + iy = \frac{\sqrt{2}}{2} + 1 + \frac{\sqrt{2}}{2}$ at $\theta = 45^\circ$, $x + iy = 1$ at $\theta = 90^\circ$, $x + iy = -\sqrt{2} + 1 + \frac{\sqrt{2}}{2}$ at $\theta = 135^\circ$. Verify $(\frac{\pi}{4} + i\frac{\pi}{4})^2 = 1 + i + i^2 \left(\frac{1}{4}\right) = i$, and then $i \left(\frac{\pi}{4} + i\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$.

10 $i^2 = i$ yields $x = \frac{\sqrt{2}}{2}$ (note that $\frac{\pi}{2}$ becomes $\ln i$); $i^3 = i^{-1}$ yields $x = -i$, second solutions are $\frac{\pi}{2} + 2\pi$ and $i + 2\pi$.

12 $e^{i\theta} + e^{i\phi}$ is at the middle angle $\frac{\theta + \phi}{2}$ with length $2 \cos \frac{\theta - \phi}{2}$; $e^{i\theta} \times e^{i\phi}$ times $e^{i\theta}$ equals $e^{i(\theta + \phi)}$; $e^{2\pi i/3} + e^{4\pi i/3} = e^{2\pi i/3} + e^{-2\pi i/3} = 2 \cos \frac{2\pi}{3} = -1$, $e^{2\pi i/3}$ equals $e^{2\pi i/3} = 1$.

14 The roots of $c^2 - 4c + 5 = 0$ must multiply to give 5. Check: The roots are $\pm \frac{\sqrt{20} + 2}{2} = 2 \pm i$. Their product
is \((2+i)(2-i) = 4-i^2 = 5\).

16 \((\cos \theta + i \sin \theta)^3 = (\cos^3 \theta + 3i \cos \theta \sin^2 \theta + (3i \cos^2 \theta \sin \theta + i^3 \sin^3 \theta). Match with \(e^{3i\theta}\) to find real part \(\cos 3\theta = \cos^3 \theta - 3 \cos \theta \sin^2 \theta\) (or \(4 \cos^3 \theta - 3 \cos \theta\)). The imaginary part is \(\sin 3\theta = 3 \cos^2 \theta \sin \theta - \sin^3 \theta\) (or \(3 \sin \theta - 4 \sin^3 \theta\)).

18 The fourth roots of \(re^{i\theta}\) are \(r^{1/4}\) times \(e^{i\theta/4}, e^{i(\theta+2\pi)/4}, e^{i(\theta+4\pi)/4}, e^{i(\theta+6\pi)/4}\). Multiply \((r^{1/4})^4\) to get \(r\).

Add angles to get \((4\theta + 12\pi)/4 = \theta + 3\pi\). The product of the 4 roots is \(re^{i(\theta+3\pi)} = -re^{i\theta}\).

20 \((e^{it})^n + e^{it} = 0\) gives \((c^3 + 1)e^{it} = 0\). Then \(c^3 = -1 = e^{i\pi}\) and \(c = e^{i\pi/3}, e^{i\pi}, e^{i5\pi/3}\). The root \(c = e^{i\pi} = -1\) gives \(y = e^{-t}\). The other roots give \(y = e^{(1+i\sqrt{3})t/2}\) and \(y = e^{(1-i\sqrt{3})t/2}\). (Note: Real solutions are: \(y = e^{t/2 \cos \sqrt{3} t}\) and \(y = e^{t/2 \sin \sqrt{3} t}\)).

22 \((e^{it})^2 + 6(e^{it}) + 5e^{it} = 0\) gives \(c^2 + 6c + 5 = 0\) or \((c + 5)(c + 1) = 0\). Then \(c = -5\) yields \(y = e^{-5t}\) and \(c = -1\) yields \(y = e^{-t}\).

24 \(c^2 - 2c + 2 = 0\) gives \(c = 1 \pm i\). Then the real part of \(e^{(1+i)t}\) is \(y = e^t \cos t\) and the imaginary part is \(e^t \sin t\).

26 \(e^{(-1+i)t} = e^{-t} \cos t + ie^{-t} \sin t\) spirals in to \(e^t = e^{-1} \cos 1 + ie^{-1} \sin 1 \approx .2 + .3i\) at \(t = 1\).

28 \(\frac{dy}{dt} = iy\) leads to \(y = e^{it} = \cos t + i \sin t\). Matching real and imaginary parts of \(\frac{dy}{dt} (\cos t + i \sin t) = i(\cos t + i \sin t)\) yields \(\frac{\partial}{\partial t} \cos t = -\sin t\) and \(\frac{\partial}{\partial t} \sin t = \cos t\).

30 \(\frac{1}{2}(e^{it} + e^{-it}) = \frac{1}{2}(\cos \theta + i \sin \theta + \cos(-\theta) + i \sin(-\theta)) = \frac{1}{2}(2 \cos \theta) = \cos \theta\). Similarly \(\sin \theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})\).

32 \(re^{i\theta}\) times \(Re^{i\phi}\) equals \(rR(\cos(\theta + \phi) + iR \sin(\theta + \phi))\). The rectangular form is \(rR \cos(\theta + \phi) + iR \sin(\theta + \phi)\). This equals \((r \cos \theta + iR \sin \phi)(R \cos \phi + iR \sin \phi) = rR(\cos \theta \cos \phi - \sin \theta \sin \phi) + iR(\cos \phi \cos \theta + \sin \phi \sin \theta)\).

34 Problem 30 yields \(\cos ix = \frac{1}{2}(e^{i(ix)} + e^{-i(ix)}) = \frac{1}{2}(e^{-x} + e^x) = \cos x\); similarly \(\sin ix = \frac{1}{2i}(e^{i(ix)} - e^{-i(ix)}) = \frac{1}{2i}(e^{-x} - e^{x}) = i \sinh x\). With \(x = 1\) the cosine of \(i\) equals \(\frac{1}{2}(e^{-1} + e^{1}) = 3.086\). The cosine of \(i\) is larger than 1!
Resource: Calculus
Gilbert Strang

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