# CHAPTER 9     POLAR COORDINATES AND COMPLEX NUMBERS

## 9.1   Polar Coordinates          (page 350)

Polar coordinates $r$ and $\theta$ correspond to $x = r\cos\theta$ and $y = r\sin\theta$. The points with $r > 0$ and $\theta = \pi$ are located on **the negative x axis**. The points with $r = 1$ and $0 \leq \theta \leq \pi$ are located on a **semicircle**. Reversing the sign of $\theta$ moves the point $(x, y)$ to $(\mathbf{x}, -\mathbf{y})$.

Given $x$ and $y$, the polar distance is $r = \sqrt{\mathbf{x^2 + y^2}}$. The tangent of $\theta$ is $\mathbf{y/x}$. The point $(6,8)$ has $r = \mathbf{10}$ and $\theta = \tan^{-1}\frac{\mathbf{8}}{\mathbf{6}}$. Another point with the same $\theta$ is $(\mathbf{3,4})$. Another point with the same $r$ is $(\mathbf{10,0})$. Another point with the same $r$ and $\tan\theta$ is $(-\mathbf{6}, -\mathbf{8})$.

The polar equation $r = \cos\theta$ produces a shifted circle. The top point is at $\theta = \pi/\mathbf{4}$, which gives $r = \sqrt{\mathbf{2}}/\mathbf{2}$. When $\theta$ goes from 0 to $2\pi$, we go **two** times around the graph. Rewriting as $r^2 = r\cos\theta$ leads to the $xy$ equation $\mathbf{x^2 + y^2 = x}$. Substituting $r = \cos\theta$ into $x = r\cos\theta$ yields $x = \cos^2\theta$ and similarly $y = \cos\theta\sin\theta$. In this form $x$ and $y$ are functions of the **parameter** $\theta$.

**1** $(1, \frac{\pi}{2})$   **3** $(2, \frac{\pi}{4})$   **5** $(\sqrt{2}, \frac{5\pi}{4})$   **7** $(0,2)$   **9** $(\sqrt{10}, \sqrt{10})$   **11** $(\sqrt{3}, -1)$   **13** $2\sqrt{2}$

**15** $\sqrt{r^2 + R^2 - 2rR\cos(\theta - \phi)}$

**17** $0 < r < \infty, -\frac{\pi}{2} < \theta < \frac{\pi}{2}; 0 \leq r < \infty, \pi \leq \theta \leq 2\pi; \sqrt{4} < r < \sqrt{5}, 0 \leq \theta < 2\pi; 0 \leq r < \infty, -\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$

**19** $y = x\tan\theta, r = x\sec\theta$   **21** $\theta = \frac{\pi}{4}$, all $r; r = \frac{1}{\sin\theta + \cos\theta}; r = \cos\theta + \sin\theta$

**23** $x^2 + y^2 = y$   **25** $x = r\sin\theta\cos\theta, y = r\sin^2\theta, x^2 + y^2 = y$

**27** $x^2 + y^2 = x + y, (x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 = (\frac{\sqrt{2}}{2})^2$   **29** $x = \frac{\cos\theta}{\cos\theta + \sin\theta}, y = \frac{\sin\theta}{\cos\theta + \sin\theta}$   **31** $(x^2 + y^2)^3 = x^4$

**2** $x = -4, y = 0$ has polar coordinates $\mathbf{r = 4}, \theta = \pi$

**4** $x = -1, y = \sqrt{3}$ has polar coordinates $\mathbf{r = 2}, \theta = \frac{2\pi}{3}$.

**6** $x = 3, y = 4$ has polar coordinates $\mathbf{r = 5}, \theta = \tan^{-1}(\frac{4}{3}) = .925$.

**8** $r = 1, \theta = \frac{3\pi}{2}$ has rectangular coordinates $\mathbf{x = 0}, \mathbf{y = -1}$.

**10** $r = 3\pi, \theta = 3\pi$ has rectangular coordinates $\mathbf{x = -3\pi}, \mathbf{y = 0}$

**12** $r = 2, \theta = \frac{5\pi}{6}$ has rectangular coordinates $\mathbf{x = -\sqrt{3}}, \mathbf{y = 1}$

**14** The distance is **5**. Better question with same answer: how far is $(3, \frac{\pi}{3})$ from $(4, \frac{2\pi}{3})$?

**16** (a) $(-1, \frac{\pi}{2})$ is the same point as $(1, \frac{3\pi}{2})$ or $(-1, \frac{5\pi}{2})$ or $\cdots$ (b) $(-1, \frac{3\pi}{4})$ is the same point as $(1, \frac{7\pi}{4})$ or $(-1, -\frac{\pi}{4})$ or $\cdots$ (c) $(1, -\frac{\pi}{2})$ is the same point as $(-1, \frac{\pi}{2})$ or $(1, \frac{3\pi}{2})$ or $\cdots$ (d) $r = 0, \theta = 0$ is the same point as $r = 0, \theta = $ **any angle**.

**18** (a) **False** ($r = 1, \theta = \frac{\pi}{4}$ is a different point from $r = -1, \theta = -\frac{\pi}{4}$) (b) **False** (for fixed $r$ we can add any multiple of $2\pi$ to $\theta$) (c) **True** ($r\sin\theta = 1$ is the horizontal line $y = 1$).

**20** $x = \sqrt{3}, y = 1$ yields $r = 2, \tan\theta = \frac{1}{\sqrt{3}}$. So does $\mathbf{x = -\sqrt{3}}, \mathbf{y = -1}$.

**22** Take the line from $(0,0)$ to $(r_1, \theta_1)$ as the base (its length is $r_1$). The height of the third point $(r_2, \theta_2)$, measured perpendicular to this base, is $r_2$ times $\sin(\theta_2 - \theta_1)$.

**24** The 13 values $\theta = 0°, 30°, \cdots, 360°$ give **six** different points with $r = \sin\theta$. To go once around the circle take $0 \leq \theta < \pi$.

**26** From $x = \cos^2\theta$ and $y = \sin\theta\cos\theta$, square and add to find $\mathbf{x^2 + y^2} = \cos^2\theta(\cos^2\theta + \sin^2\theta) = \cos^2\theta = \mathbf{x}$.

**28** Multiply $r = a\cos\theta + b\sin\theta$ by $r$ to find $x^2 + y^2 = ax + by$. Complete squares in $x^2 - ax = (x - \frac{a}{2})^2 - (\frac{a}{2})^2$ and similarly in $y^2 - by$ to find $(x - \frac{a}{2})^2 + (y - \frac{b}{2})^2 = (\frac{a}{2})^2 + (\frac{b}{2})^2$. This is a circle centered at $(\frac{a}{2}, \frac{b}{2})$ with radius $r = \sqrt{(\frac{a}{2})^2 + (\frac{b}{2})^2} = \frac{1}{2}\sqrt{\mathbf{a^2 + b^2}}$.

**30** The point $x = \cos^2\theta, y = \sin^2\theta$ is generally not at the polar angle $\theta$. For example let $\theta = \frac{\pi}{6} = 30°$ : then $x = \frac{3}{4}$ and $y = \frac{1}{4}$. The polar angle for this point has tangent $= \frac{1}{3}$, but the tangent of $\frac{\pi}{6}$ is $\frac{1}{\sqrt{3}}$. Conclusion: an angle named $\theta$ is not automatically the polar angle. **See Problem 9.3.40.**

**32** The second figure is not a closed curve as it stands. As the parameter $t$ keeps going, the spaces around the circle fill up and the curve eventually closes (but the figure becomes less beautiful).

# 9.2   Polar Equations and Graphs       (page 355)

The circle of radius 3 around the origin has polar equation $\mathbf{r = 3}$. The 45° line has polar equation $\theta = \pi/4$. Those graphs meet at an angle of **90°**. Multiplying $r = 4\cos\theta$ by $r$ yields the $xy$ equation $\mathbf{x^2 + y^2 = 4x}$. Its graph is a **circle** with center at $(2,0)$. The graph of $r = 4/\cos\theta$ is the line $x = 4$. The equation $r^2 = \cos 2\theta$ is not changed when $\theta \to -\theta$ (symmetric across the **x axis**) and when $\theta \to \pi + \theta$ (or $r \to -r$). The graph of $r = 1 + \cos\theta$ is a **cardioid**.

The graph of $r = A/(1 + e\cos\theta)$ is a conic section with one focus at $(0,0)$. It is an ellipse if $e < 1$ and a hyperbola if $e > 1$. The equation $r = 1/(1 + \cos\theta)$ leads to $r + x = 1$ which gives a **parabola**. Then $r = $ distance from origin equals $1 - x = $ distance from **directrix y = 1**. The equations $r = 3(1 - x)$ and $r = \frac{1}{3}(1 - x)$ represent a **hyperbola** and an **ellipse**. Including a shift and rotation, conics are determined by **five** numbers.

**1** Line $y = 1$    **3** Circle $x^2 + y^2 = 2x$    **5** Ellipse $3x^2 + 4y^2 = 1 - 2x$    **7** $x, y, r$ symmetries

**9** $x$ symmetry only    **11** No symmetry    **13** $x, y, r$ symmetries!

**15** $x^2 + y^2 = 6y + 8x \to (x-4)^2 + (y-3)^2 = 5^2$, center $(4,3)$    **17** $(2,0), (0,0)$

**19** $r = 1 - \frac{\sqrt{2}}{2}, \theta = \frac{3\pi}{4}; r = 1 + \frac{\sqrt{2}}{2}, \theta = \frac{7\pi}{4}; (0,0)$    **21** $r = 2, \theta = \pm\frac{\pi}{12}, \pm\frac{5\pi}{12}, \pm\frac{7\pi}{12}, \pm\frac{11\pi}{12}$

**23** $(x,y) = (1,1)$    **25** $r = \cos 5\theta$ has 5 petals    **27** $(x^2 + y^2 - x)^2 = x^2 + y^2$

**29** $(x^2 + y^2)^3 = (x^2 - y^2)^2$    **31** $\cos\theta = -\frac{1}{2}, \sin\theta = \frac{\sqrt{3}}{2} \to y = \frac{2\sqrt{3}}{3}, x = -\frac{2}{3}$    **33** $x = \frac{4}{3}, r = -\frac{5}{3}$    **35** .967

**2** $r\cos\theta - r\sin\theta = 2$ is the straight line $\mathbf{x - y = 2}$.

**4** $r = -2\sin\theta$ is the circle $r^2 = -2r\sin\theta$ or $x^2 + y^2 = -2y$ or $\mathbf{x^2 + (y+1)^2 = 1}$; below the origin with center at $(0, -1)$ and radius 1.

**6** $r = \frac{1}{1 + 2\cos\theta}$ is the hyperbola of Example 7 and Figure 9.5c: $r + 2r\cos\theta = 1$ is $r = 1 - 2x$ or $x^2 + y^2 = 1 - 4x + 4x^2$. The figure should show $r = -1$ and $\theta = \pi$ on the right branch.

**8** $r^2 = 4\sin 2\theta$ has loops in the first and third quadrants. It possesses $r$ symmetry (change $r$ to $-r$ and the equation is unchanged). Changing $\theta$ to $-\theta$ or $\pi - \theta$ or $2\pi - \theta$ reverses the sign of the right hand side.

**10** $r^2 = 10 + 6\cos 4\theta$ has $x, y$, and $r$ symmetry. It comes in from $r = 4$ at $\theta = 0$ to $r = 2$ at $\theta = \frac{\pi}{4}$ and back out to $r = 4$ at $\theta = \frac{\pi}{2}$. Repeat in each quadrant to form a "star-fish".

**12** $r = 1/\theta$ has $y$ symmetry. Change $\theta$ to $-\theta$ and $r$ to $-r$: same equation ($\theta$ to $\pi - \theta$ gives a different equation:

must try both tests.) Note that the maximum of $y = r \sin \theta = \frac{\sin \theta}{\theta}$ is $y = 1$ as $\theta \to 0$ : the line $y = 1$ is a *horizontal asymptote!* As negative $\theta$ approach zero, the spiral goes left toward the same asymptote $y = 1$.

**14** $r = 1 - 2 \sin 3\theta$ has **y axis symmetry**: change $\theta$ to $\pi - \theta$, then $\sin 3(\pi - \theta) = \sin(\pi - 3\theta) = \sin 3\theta$.

**16** This is another case where the **parameter t is not the polar angle.** (The Earth completed a circle at $t = 1$.)

**18** $r^2 = \sin 2\theta$ and $r^2 = \cos 2\theta$ are lemniscates (or "spectacles"). They meet when $\sin 2\theta = \cos 2\theta$ or $2\theta = \frac{\pi}{4}$ or $\frac{5\pi}{4}$ or $\theta = \frac{\pi}{8}$ or $\frac{5\pi}{8}$ ($r$ can be positive or negative). They also meet at the origin $\mathbf{r} = \mathbf{0}$.

**20** $r = 1 + \cos \theta$ and $r = 1 - \cos \theta$ are cardioids (reaching right to $r = 2, \theta = 0$ and left to $r = 2, \theta = \pi$). They meet when $\cos \theta = 0$ at $\mathbf{r} = 1, \theta = \frac{\pi}{2}$ and $\mathbf{r} = 1, \theta = \frac{3\pi}{2}$. They also meet at the origin $\mathbf{r} = \mathbf{0}$.

**22** If $\cos \theta = \frac{r^2}{4}$ and $\cos \theta = 1 - r$ then $\frac{r^2}{4} = 1 - r$ and $r^2 + 4r - 4 = 0$. This gives $r = -2 - \sqrt{8}$ and $\mathbf{r} = -2 + \sqrt{8}$. The first $r$ is negative and cannot equal $1 - \cos \theta$. The second gives $\cos \theta = 1 - r = 3 - \sqrt{8}$ and $\theta \approx 80°$ or $\theta \approx -80°$. The curves also meet at **the origin $\mathbf{r} = \mathbf{0}$** and at the point $\mathbf{r} = -2, \theta = 0$ which is also $\mathbf{r} = +2, \theta = \pi$.

**24** The limacon $r = 1 + b \cos \theta$ has $x = r \cos \theta = \cos \theta + b \cos^2 \theta$ and $\frac{dx}{d\theta} = -\sin \theta - 2b \cos \theta \sin \theta$. Then $\frac{d^2 x}{d\theta^2} = -\cos \theta - 2b \cos^2 \theta + 2b \sin^2 \theta$ which equals $1 - 2b$ at $\theta = \pi$. The dimple begins at $\mathbf{b} = \frac{1}{2}$. At $b = 1$ it becomes the cusp in the cardioid.

**26** The other 101 petals in $r = \cos 101\theta$ are **duplicates of the first 101.** For example $\theta = \pi$ gives $r = \cos 101\pi = -1$ which is also $\theta = 0, r = +1$. (Note that $\cos 100\pi = +1$ gives a new point.)

**28** (a) Yes, $x$ and $y$ symmetry imply $r$ symmetry. Reflections across the $x$ axis and then the $y$ axis take $(x, y)$ to $(x, -y)$ to $(-x, -y)$ which is reflection through the origin. (b) The point $r = -1, \theta = \frac{3\pi}{2}$ satisfies the equation $r = \cos 2\theta$ and it is the same point as $r = 1, \theta = \frac{\pi}{2}$.

**30** (a) $r^2 = \theta$ (b) $x + y = 1$ or $r \cos \theta + r \sin \theta = 1$ or $r = \frac{1}{\cos \theta + \sin \theta}$ (c) ellipse $x^2 + 2y^2 = 1$ or $r^2(\cos^2 \theta + 2 \sin^2 \theta) = 1$

**32** (a) $\theta = \frac{\pi}{2}$ gives $r = 1$; this is $x = 0, y = 1$ (b) The graph crosses the $x$ axis at $\theta = 0$ and $\pi$ where $x = \frac{1}{1+e}$ and $x = \frac{-1}{1-e}$. The center of the graph is halfway between at $x = \frac{1}{2}(\frac{1}{1+e} - \frac{1}{1-e}) = \frac{-e}{1-e^2}$. The second focus is twice as far from the origin at $\frac{-2e}{1-e^2}$. (Check: $e = 0$ gives center of circle, $e = 1$ gives second focus of parabola at infinity.)

**34** $r = \frac{A}{1 + e \cos \theta}$ and $r = \frac{1}{C + D \cos \theta}$ are the same if $\mathbf{C} = \frac{1}{A}$ and $\mathbf{D} = \frac{e}{A}$. For the mirror image across the $y$ axis, $\theta$ becomes $\pi - \theta$ and $\cos \theta$ changes sign.

**36** Maximize $y = \frac{A \sin \theta}{1 + e \cos \theta}$ where $\frac{dy}{d\theta} = \frac{(1 + e \cos \theta)A \cos \theta + (A \sin \theta)e \sin \theta}{(1 + e \cos \theta)^2} = 0$. Then $A \cos \theta + Ae = 0$ or $\cos \theta = -e$ and $y_{max} = \frac{A\sqrt{1-e^2}}{1-e^2} = \frac{A}{\sqrt{1-e^2}}$ (which equals $b$ in $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$).

# 9.3 Slope, Length, and Area for Polar Curves (page 359)

A circular wedge with angle $\Delta \theta$ is a fraction $\Delta \theta / 2\pi$ of a whole circle. If the radius is $r$, the wedge area is $\frac{1}{2} r^2 \Delta \theta$. Then the area inside $r = F(\theta)$ is $\int \frac{1}{2} r^2 d\theta = \int \frac{1}{2}(F(\theta))^2 d\theta$. The area inside $r = \theta^2$ from 0 to $\pi$ is $\pi^5/10$. That spiral meets the circle $r = 1$ at $\theta = 1$. The area inside the circle and outside the spiral is $\frac{1}{2} - \frac{1}{10}$. A chopped wedge of angle $\Delta \theta$ between $r_1$ and $r_2$ has area $\frac{1}{2} r_2^2 \Delta \theta - \frac{1}{2} r_1^2 \Delta \theta$.

The curve $r = F(\theta)$ has $x = r \cos \theta = F(\theta) \cos \theta$ and $y = F(\theta) \sin \theta$. The slope $dy/dx$ is $dy/d\theta$ divided by $dx/d\theta$. For length $(ds)^2 = (dx)^2 + (dy)^2 = (dr)^2 + (rd\theta)^2$. The length of the spiral $r = \theta$ to $\theta = \pi$ is

$\int \sqrt{1+\theta^2}d\theta$. The surface area when $r = \theta$ is revolved around the $x$ axis is $\int 2\pi y\, ds = \int 2\pi\theta \sin\theta\sqrt{1+\theta^2}d\theta$. The volume of that solid is $\int \pi y^2 dx = \int \pi\theta^2 \sin^2\theta\ (\cos\theta - \theta\sin\theta)d\theta$.

**1** Area $\frac{3\pi}{2}$  **3** Area $\frac{9\pi}{2}$  **5** Area $\frac{\pi}{8}$  **7** Area $\frac{\pi}{8} - \frac{1}{4}$  **9** $\int_{-\pi/3}^{\pi/3}(\frac{9}{2}\cos^2\theta - \frac{(1+\cos\theta)^2}{2})d\theta = \pi$

**11** Area $8\pi$  **13** Only allow $r^2 > 0$, then $4\int_0^{\pi/4}\frac{1}{2}\cos 2\theta\, d\theta = 1$  **15** $2 + \frac{\pi}{4}$

**17** $\theta = 0$; left points $r = \frac{1}{2}, \theta = \pm\frac{2\pi}{3}, x = -\frac{1}{4}, y = \pm\frac{\sqrt{3}}{4}$

**19** $\frac{r^2}{2c}]_6^{14} = 40,000;\ \frac{1}{2c}[r\sqrt{r^2+c^2}+c^2\ln(r+\sqrt{r^2+c^2})]_6^{14} = 40,000.001$

**21** $\tan\psi = \tan\theta$  **23** $x = 0, y = 1$ is on limacon but not circle  **25** $\frac{1}{2}\ln(2\pi + \sqrt{1+4\pi^2}) + \pi\sqrt{1+4\pi^2}$

**27** $\frac{3\pi}{2}$  **29** $\frac{1}{2}$ (base)(height) $\approx \frac{1}{2}(r\Delta\theta)r$  **31** $\frac{4\pi}{5}\sqrt{2}$  **33** $2\pi(2-\sqrt{2})$  **35** $\frac{8\pi}{3}$  **39** $\sec\theta$

**2** $A = \int \frac{1}{2}r^2 d\theta = \int_0^\pi \frac{1}{2}(\sin\theta + \cos\theta)^2 d\theta = \int_0^\pi \frac{1}{2}(\sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta)d\theta = \int_0^\pi \frac{1}{2}(1+\sin 2\theta)d\theta$
$= [\frac{\theta}{2} - \frac{\cos 2\theta}{4}]_0^\pi = \frac{\pi}{2}$. This is the area of the circle $r^2 = x + y$ or $(x-\frac{1}{2})^2 + (y-\frac{1}{2})^2 = \frac{1}{2}$

**4** The inner loop is where $r < 0$ or $\cos\theta < -\frac{1}{2}$ or $\frac{2\pi}{3} < \theta < \frac{4\pi}{3}$. Its area is $\int \frac{r^2}{2}d\theta = \int \frac{1}{2}(1+4\cos\theta+4\cos^2\theta)d\theta =$
$[\frac{\theta}{2} + 2\sin\theta + \theta + \cos\theta\ \sin\theta]_{2\pi/3}^{4\pi/3} = \frac{\pi}{3} - 2(\sqrt{3}) + \frac{2\pi}{3} + \frac{1}{2}\sqrt{3} = \pi - \frac{3}{2}\sqrt{3}$.

**6** A petal begins and ends at $r = 0$. For $r = \cos 3\theta$ this is from $\theta = -\frac{\pi}{6}$ to $\frac{\pi}{6}$. The area is
$\int \frac{1}{2}\cos^2 3\theta\, d\theta = \frac{1}{4}\int(1+\cos 6\theta)d\theta = [\frac{\theta}{4} + \frac{\sin 6\theta}{24}]_{-\pi/6}^{\pi/6} = \frac{\pi}{12}$.

**8** The $y$ axis is $\theta = \frac{\pi}{2}$. The area is $\int \frac{1}{2}r^2 d\theta = \int_0^{\pi/2}\frac{1}{2}\theta^2 d\theta = [\frac{\theta^3}{6}]_0^{\pi/2} = \frac{\pi^3}{48}$.

**10** $r^2 = 4\cos 2\theta$ meets $r^2 = 2$ when $\cos 2\theta = \frac{1}{2}$ or $2\theta = \pm\frac{\pi}{3}$ and $\pm(\frac{\pi}{3}+2\pi)$. Then $\theta = \pm\frac{\pi}{6}$ and $\pm\frac{5\pi}{6}$. By symmetry,
integrate from 0 to $\frac{\pi}{6}$ and multiply by 4. Area $= 4\int_0^{\pi/6}\frac{1}{2}(r_1^2 - r_2^2)d\theta = 2\int_0^{\pi/6}(4\cos 2\theta - 2)d\theta =$
$[4\sin 2\theta - 4\theta]_0^{\pi/6} = 4(\frac{\sqrt{3}}{2}) - 4(\frac{\pi}{6}) = 2\sqrt{3} - \frac{2\pi}{3}$.

**12** Intersection when $10\cos\theta = 6$ or $\cos\theta = .6$. Area $\int \frac{1}{2}(r_1^2 - r_2^2)d\theta = \int_0^{\cos^{-1}.6}(100 - \frac{36}{\cos^2\theta})d\theta =$
$[100\theta - 36\tan\theta]_0^{\cos^{-1}.6} = 100\cos^{-1}.6 - 36(\frac{4}{3})$.

**14** From (3) the area is $\int_{-\pi/3}^{\pi/3}\frac{1}{2}(\cos^2\theta - \frac{1}{4})d\theta = [\frac{\theta}{4} + \frac{\sin 2\theta}{8} - \frac{\theta}{8}]_{-\pi/3}^{\pi/3} = \frac{1}{8}(\frac{2\pi}{3}) + \frac{1}{8}(\frac{\sqrt{3}}{2} - (-\frac{\sqrt{3}}{2})) = \frac{\pi}{12} + \frac{\sqrt{3}}{8}$.

**16** The spiral $r = e^{-\theta}$ starts at $r = 1$ and returns to the $x$ axis at $r = e^{-2\pi}$. Then it goes inside itself (no
new area). So area $= \int_0^{2\pi}\frac{1}{2}e^{-2\theta}d\theta = [-\frac{1}{4}e^{-2\theta}]_0^{2\pi} = \frac{1}{4}(1 - e^{-4\pi})$.

**18** $\frac{dx}{d\theta} = -\sin\theta F(\theta) + \cos\theta\frac{dF}{d\theta}$ and similarly for $\frac{dy}{d\theta}$. Then $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cos\theta\ F(\theta)+\sin\theta\ dF/d\theta}{-\sin\theta\ F(\theta)+\cos\theta\ dF/d\theta}$. Divide top
and bottom by $\cos\theta$ to reach $\frac{F(\theta)+\tan\theta\ dF/d\theta}{-\tan\theta F(\theta)+dF/d\theta}$.

**20** Simplify $\frac{\tan\phi-\tan\theta}{1+\tan\phi\ \tan\theta} = \frac{\frac{F+\tan\theta F'}{-\tan\theta F+F'}-\tan\theta}{1+\frac{F+\tan\theta F'}{-\tan\theta F+F'}\tan\theta} = \frac{F+\tan\theta F'-\tan\theta(-\tan\theta F+F')}{-\tan\theta F+F'+\tan\theta(F+\tan\theta F')} = \frac{(1+\tan^2\theta)F}{(1+\tan^2\theta)F'} = \frac{F}{F'}$.

**22** $r = 1 - \cos\theta$ is the mirror image of Figure 9.4c across the $y$ axis. By Problem 20, $\tan\psi = \frac{F}{F'} = \frac{1-\cos\theta}{\sin\theta}$.
This is $\frac{\frac{1}{2}\sin^2\frac{\theta}{2}}{\frac{1}{2}\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = \tan\frac{\theta}{2}$. So $\psi = \frac{\theta}{2}$ (check at $\theta = \pi$ where $\psi = \frac{\pi}{2}$).

**24** By Problem 18 $\frac{dy}{dx} = \frac{\cos\theta+\tan\theta(-\sin\theta)}{-\cos\theta\ \tan\theta-\sin\theta} = \frac{\cos^2\theta-\sin^2\theta}{\cos\theta(-2\sin\theta)} = -\frac{\cos 2\theta}{\sin 2\theta} = -\frac{1}{\sqrt{3}}$ at $\theta = \frac{\pi}{6}$. At that point $x = r\cos\theta =$
$\cos^2\frac{\pi}{6} = (\frac{\sqrt{3}}{2})^2$ and $y = r\sin\theta = \cos\frac{\pi}{6}\sin\frac{\pi}{6} = \frac{1}{2}(\frac{\sqrt{3}}{2})$. The tangent line is $y - \frac{\sqrt{3}}{4} = -\frac{1}{\sqrt{3}}(x - \frac{3}{4})$.

**26** $r = \sec\theta$ has $\frac{dr}{d\theta} = \sec\theta\tan\theta$ and $\frac{ds}{d\theta} = \sqrt{\sec^2\theta + \sec^2\theta\ \tan^2\theta} = \sqrt{\sec^4\theta} = \sec^2\theta$. Then arc length
$= \int_0^{\pi/4}\sec^2\theta\, d\theta = \tan\frac{\pi}{4} = 1$. Note: $r = \sec\theta$ is the line $r\cos\theta = 1$ or $x = 1$ from $y = 0$ up to $y = 1$.

**28** $r = \theta^2$ has $\frac{dr}{d\theta} = 2\theta$ and $\frac{ds}{d\theta} = \sqrt{\theta^4 + 4\theta^2}$. Then arc length $= \int_0^\pi \theta\sqrt{\theta^2 + 4}d\theta = [\frac{1}{3}(\theta^2 + 4)^{3/2}]_0^\pi$
$= \frac{1}{3}[(\pi^2 + 4)^{3/2} - 4^{3/2}]$.

**30** $ds = \sqrt{\cos^2\theta + \sin^2\theta}d\theta = d\theta$ and surface area $= \int 2\pi y\, ds = \int 2\pi\, r\sin\theta\, ds = \int_0^{\pi/2}2\pi\cos\theta\sin\theta\, d\theta = \pi$.

**32** $r = 1 + \cos\theta$ has $\frac{ds}{d\theta} = \sqrt{(1+2\cos\theta+\cos^2\theta)+\sin^2\theta} = \sqrt{2+2\cos\theta}$. Also $y = r\sin\theta = (1+\cos\theta)\sin\theta$.
Surface area $\int 2\pi y\, ds = 2\pi\sqrt{2}\int_0^\pi(1+\cos\theta)^{3/2}\sin\theta\, d\theta = [2\pi\sqrt{2}(-\frac{2}{5})(1+\cos\theta)^{5/2}]_0^\pi =$
$2\pi\sqrt{2}(\frac{2}{5})2^{5/2} = \frac{32\pi}{5}$.

**34** $y = r \sin \theta = \sin \theta \cos \theta$ and $x = \cos^2 \theta$ (which moves *left*). Volume $= \int \pi y^2 dx =$
$\int_0^{\pi/2} \pi \sin^2 \theta \cos^2 \theta (2 \cos \theta \sin \theta) d\theta = 2\pi \int_0^{\pi/2} (\sin^3 \theta - \sin^5 \theta) \cos \theta \, d\theta = 2\pi [\frac{\sin^4 \theta}{4} - \frac{\sin^6 \theta}{6}]_0^{\pi/2} = 2\pi[\frac{1}{4} - \frac{1}{6}] = \frac{\pi}{6}$.
Check: The sphere has radius $\frac{1}{2}$ and volume $\frac{4\pi}{3}(\frac{1}{2})^3 = \frac{\pi}{6}$.

**36** $r = \sec \theta$ has $x = r \cos \theta = 1$ and $ds = \sec^2 \theta \, d\theta$ as in Problem 26. Surface area $=$
$\int 2\pi x ds = \int_0^{\pi/4} 2\pi(1) \sec^2 \theta \, d\theta = [2\pi \tan \theta]_0^{\pi/4} = 2\pi$. The surface is a cylinder.

**38** The triangle connecting the three centers has $60°$ angles and base 2. Its area is $\frac{1}{2}(2)(2 \sin \frac{\pi}{3}) = \sqrt{3}$.
Subtract the area inside the circles and triangle: 3 times $\frac{\pi}{6}$. Remaining area $= \sqrt{3} - \frac{\pi}{2}$.

**40** The parameter $\theta$ along the ellipse $x = 4 \cos \theta, y = 3 \sin \theta$ is *not* the angle from the origin. For example
at $\theta = \frac{\pi}{4}$ the point $(x, y)$ is *not* on the $45°$ line. So the area formula $\int \frac{1}{2} r^2 d\theta$ does not apply. The
correct area is $12\pi$.

# 9.4 Complex Numbers (page 364)

The complex number $3 + 4i$ has real part **3** and imaginary part **4**. Its absolute value is $r = \mathbf{5}$ and its complex conjugate is $\mathbf{3} - 4i$. Its position in the complex plane is at **(3,4)**. Its polar form is $r \cos \theta + i r \sin \theta = r e^{i\theta}$ (or $5 e^{i\theta}$). Its square is $-\mathbf{7} - 14i$. Its $n$th power is $\mathbf{r^n} e^{in\theta}$.

The sum of $1 + i$ and $1 - i$ is **2**. The product of $1 + i$ and $1 - i$ is **2**. In polar form this is $\sqrt{2} e^{i\pi/4}$ times $\sqrt{2} e^{-i\pi/4}$. The quotient $(1+i)/(1-i)$ equals the imaginary number **i**. The number $(1+i)^8$ equals **16**. An eighth root of 1 is $w = (1 + i)/\sqrt{2}$. The other eighth roots are $\mathbf{w^2, w^3, \cdots, w^7, w^8 = 1}$.

To solve $d^8 y/dt^8 = y$, look for a solution of the form $y = e^{ct}$. Substituting and canceling $e^{ct}$ leads to the equation $c^8 = 1$. There are eight choices for $c$, one of which is $(-1 + i)/\sqrt{2}$. With that choice $| e^{ct} | = e^{-t/\sqrt{2}}$. The real solutions are Re $e^{ct} = e^{-t/\sqrt{2}} \cos \frac{t}{\sqrt{2}}$ and Im $e^{ct} = e^{-t/\sqrt{2}} \sin \frac{t}{\sqrt{2}}$.

**1** Sum $= 4$, product $= 5$    **5** Angles $\frac{3\pi}{4}, \frac{3\pi}{2}, \frac{9\pi}{4}$    **7** Real axis; imaginary axis; $\frac{1}{2}$ axis $x \geq 0$; unit circle
**9** $cd = 5 + 10i, \frac{c}{d} = \frac{11-2i}{25}$   **11** $2 \cos \theta, 1; -1, 1$   **13** Sum $= 0$, product $= -1$   **15** $r^4 e^{4i\theta}, \frac{1}{r} e^{-i\theta}, \frac{1}{r^4} e^{-4i\theta}$
**17** Evenly spaced on circle around origin    **19** $e^{it}, e^{-it}$    **21** $e^t, e^{-t}, e^0$    **23** $\cos 7t, \sin 7t$
**29** $t = -\frac{2\pi}{\sqrt{3}}, y = -e^{\pi/\sqrt{3}}$    **31** F; T; at most 2; Re $c < 0$    **33** $\frac{1}{r} e^{-i\theta}, x = \frac{1}{r} \cos \theta, y = -\frac{1}{r} \sin \theta; \pm \frac{1}{\sqrt{r}} e^{-i\theta/2}$

**2** $1 + i$ has $r = \sqrt{2}$ and $\theta = \frac{\pi}{4}$; $(1 + i)^2 = 2i$ has $r = 2$ and $\theta = \frac{\pi}{2}$; $\frac{1}{1+i} = \frac{1-i}{1-i^2} = \frac{1-i}{2}$ has $r = \frac{\sqrt{2}}{2}$ and $\theta = -\frac{\pi}{4}$.
**4** The powers of $e^{2\pi i/6}$ are on the unit circle at equally spaced angles $\frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}, 2\pi$.
**6** $4e^{i\pi/3} = 4(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}) = 4(\frac{1}{2} + i\frac{\sqrt{3}}{2}) = 2 + 2\sqrt{3}i$. The square roots are $2e^{i\pi/6}$ and $-2e^{i\pi/6} = 2e^{7\pi i/6}$.
**8** $x + iy = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$ at $\theta = 45°$, $x + iy = i$ at $\theta = 90°$, $x + iy = -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$ at $\theta = 135°$. Verify
$(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2})^2 = \frac{1}{2} + i + i^2(\frac{1}{2}) = i$ and then $i(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}) = -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$.
**10** $e^{ix} = i$ yields $x = \frac{\pi}{2}$ (note that $\frac{i\pi}{2}$ becomes $\ln i$); $e^{ix} = e^{-1}$ yields $x = i$, second solutions are $\frac{\pi}{2} + 2\pi$ and $i + 2\pi$.
**12** $e^{i\theta} + e^{i\phi}$ is at the middle angle $\frac{\theta + \phi}{2}$ with length $2 \cos \frac{\theta - \phi}{2}$; $e^{i\theta}$ times $e^{i\phi}$ equals $e^{i(\theta + \phi)}$; $e^{2\pi i/3} + e^{4\pi i/3} = e^{2\pi i/3} + e^{-2\pi i/3} = 2 \cos \frac{2\pi}{3} = -1$; $e^{2\pi i/3}$ times $e^{4\pi i/3}$ equals $e^{6\pi i/3} = 1$.
**14** The roots of $c^2 - 4c + 5 = 0$ must multiply to give **5**. Check: The roots are $\frac{4 \pm \sqrt{16-20}}{2} = 2 \pm i$. Their product

is $(2+i)(2-i) = 4 - i^2 = 5$.

**16** $(\cos\theta + i\sin\theta)^3 = (\cos^3\theta + 3i^2\cos\theta\sin^2\theta) + (3i\cos^2\theta\sin\theta + i^3\sin^3\theta)$. Match with $e^{3i\theta}$ to find real part $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$ (or $4\cos^3\theta - 3\cos\theta$). The imaginary part is $\sin 3\theta = 3\cos^2\theta\sin\theta$ $-\sin^3\theta$ (or $3\sin\theta - 4\sin^3\theta$).

**18** The fourth roots of $re^{i\theta}$ are $r^{1/4}$ times $e^{i\theta/4}, e^{i(\theta+2\pi)/4}, e^{i(\theta+4\pi)/4}, e^{i(\theta+6\pi)/4}$. Multiply $(r^{1/4})^4$ to get $r$. Add angles to get $(4\theta + 12\pi)/4 = \theta + 3\pi$. The product of the 4 roots is $re^{i(\theta+3\pi)} = -re^{i\theta}$.

**20** $(e^{ct})''' + e^{ct} = 0$ gives $(c^3 + 1)e^{ct} = 0$. Then $c^3 = -1 = e^{i\pi}$ and $c = e^{i\pi/3}, e^{i\pi}, e^{i5\pi/3}$. The root $c = e^{i\pi} = -1$ gives $y = e^{-t}$. The other roots give $y = e^{(1+\sqrt{3}i)t/2}$ and $y = e^{(1-\sqrt{3}i)t/2}$. (Note: Real solutions are: $y = e^{t/2}\cos\frac{\sqrt{3}}{2}t$ and $y = e^{t/2}\sin\frac{\sqrt{3}}{2}t$.)

**22** $(e^{ct})'' + 6(e^{ct})' + 5e^{ct} = 0$ gives $c^2 + 6c + 5 = 0$ or $(c+5)(c+1) = 0$. Then $c = -5$ yields $y = e^{-5t}$ and $c = -1$ yields $y = e^{-t}$.

**24** $c^2 - 2c + 2 = 0$ gives $c = 1 \pm i$. Then the real part of $e^{(1+i)t}$ is $y = e^t\cos t$ and the imaginary part is $e^t\sin t$.

**26** $e^{(-1+i)t} = e^{-t}\cos t + ie^{-t}\sin t$ spirals in to $e^c = e^{-1}\cos 1 + ie^{-1}\sin 1 \approx .2 + .3i$ at $t = 1$.

**28** $\frac{dy}{dt} = iy$ leads to $y = e^{it} = \cos t + i\sin t$. Matching real and imaginary parts of $\frac{d}{dt}(\cos t + i\sin t) = i(\cos t + i\sin t)$ yields $\frac{d}{dt}\cos t = -\sin t$ and $\frac{d}{dt}\sin t = \cos t$.

**30** $\frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}(\cos\theta + i\sin\theta + \cos(-\theta) + i\sin(-\theta)) = \frac{1}{2}(2\cos\theta) = \cos\theta$. Similarly $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$.

**32** $re^{i\theta}$ times $Re^{i\phi}$ equals $(rR)e^{i(\theta+\phi)}$. The rectangular form is $rR\cos(\theta+\phi) + irR\sin(\theta+\phi)$. This equals $(r\cos\theta + ir\sin\theta)(R\cos\phi + iR\sin\phi) = rR(\cos\theta\cos\phi - \sin\theta\sin\phi) + irR(\cos\theta\sin\phi + \sin\theta\cos\phi)$.

**34** Problem 30 yields $\cos ix = \frac{1}{2}(e^{i(ix)} + e^{-i(ix)}) = \frac{1}{2}(e^{-x} + e^x) = \cosh x$; similarly $\sin ix = \frac{1}{2i}(e^{i(ix)} - e^{-i(ix)}) = \frac{i}{2i}(e^{-x} - e^x) = i\sinh x$. With $x = 1$ the cosine of $i$ equals $\frac{1}{2}(e^{-1} + e^1) = $ **3.086**. The cosine of $i$ is larger than 1!

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