

# CHAPTER 13 PARTIAL DERIVATIVES

## 13.1 Surfaces and Level Curves (page 475)

The graph of  $z = f(x, y)$  is a surface in three-dimensional space. The level curve  $f(x, y) = 7$  lies down in the base plane. Above this level curve are all points at height 7 in the surface. The plane  $z = 7$  cuts through the surface at those points. The level curves  $f(x, y) = c$  are drawn in the  $xy$  plane and labeled by  $c$ . The family of labeled curves is a contour map.

For  $z = f(x, y) = x^2 - y^2$ , the equation for a level curve is  $x^2 - y^2 = c$ . This curve is a hyperbola. For  $z = x - y$  the curves are straight lines. *Level curves never cross because  $f(x, y)$  cannot equal two numbers  $c$  and  $c'$ .* They crowd together when the surface is steep. The curves tighten to a point when  $f$  reaches a maximum or minimum. The steepest direction on a mountain is perpendicular to the level curve.

- 3  $x$  derivatives  $\infty, -1, -2, -4e^{-4}$  (flattest)    5 Straight lines    7 Logarithm curves  
 9 Parabolas    11 No:  $f = (x + y)^n$  or  $(ax + by)^n$  or any function of  $ax + by$     13  $f(x, y) = 1 - x^2 - y^2$   
 15 Saddle    17 Ellipses  $4x^2 + y^2 = c^2$     19 Ellipses  $5x^2 + y^2 = c^2 + 4cx + x^2$   
 21 Straight lines not reaching (1,2)    23 Center (1,1);  $f = x^2 + y^2 - 1$     25 Four, three, planes, spheres  
 27 Less than 1, equal to 1, greater than 1    29 Parallel lines, hyperbolas, parabolas  
 31  $\frac{d}{dx} : 48x - 3x^2 = 0, x = 16$  hours    33 Plane; planes; 4 left and 3 right (3 pairs)

- 2 Level curves are circles for any function of  $x^2 + y^2$ ; the maximum is at (0,0); the functions equal 1 when  $x^2 + y^2 = 3, 1, 2, \infty$  (radius is square root: increasing order  $f_2, f_3, f_1, f_4$ ).
- 4  $\frac{d}{dx} \sqrt{3-x^2} = \frac{-x}{\sqrt{3-x^2}} = \frac{-1}{\sqrt{2}}$  at  $x = 1$ ;  $\frac{df_2}{dx} = \frac{-x}{\sqrt{x^2+1}} = \frac{-1}{\sqrt{2}}$ ;  $\frac{df_3}{dx} = -x = -1$ ;  $\frac{df_4}{dx} = -2xe^{-x^2-1} = -2e^{-2}$ .
- 6  $(x + y)^2 = 0$  gives the line  $y = -x$ ;  $(x + y)^2 = 1$  gives the pair of lines  $x + y = 1$  and  $x + y = -1$ ; similarly  $x + y = \sqrt{2}$  and  $x + y = -\sqrt{2}$ ; no level curve  $(x + y)^2 = -4$ .
- 8  $\sin(x - y) = 0$  on an infinite set of parallel lines  $x - y = 0, \pm\pi, \pm2\pi, \dots$ ; for  $c = 1$  the level curves  $\sin(x - y) = 1$  are parallel lines  $x - y = \frac{\pi}{2} + 2\pi n$ ; no level curves for  $c = 2$  and  $c = -4$ .
- 10 The curve  $\frac{y}{x^2} = 0$  is the axis  $y = 0$  excluding (0,0);  $\frac{y}{x^2} = 1$  or 2 or  $-4$  is a parabola.
- 12  $f(x, y) = xy - 1$  has level curve  $f = 0$  as two pieces of a hyperbola.
- 14  $f(x, y) = \sin(x + y)$  is zero on infinitely many lines  $x + y = 0, \pm\pi, \pm2\pi, \dots$
- 16  $f(x, y) = \{ \text{maximum of } x^2 + y^2 - 1 \text{ and zero} \}$  is zero inside the unit circle.
- 18  $\sqrt{4x^2 + y^2} = c + 2x$  gives  $4x^2 + y^2 = c^2 + 4cx + 4x^2$  or  $y^2 = c^2 + 4cx$ . This is a parabola opening to the left or right.
- 20  $\sqrt{3x^2 + y^2} = c + 2x$  gives  $3x^2 + y^2 = c^2 + 4cx + 4x^2$  or  $y^2 - x^2 = c^2 + 4cx$ . This is a hyperbola.
- 26 Since  $x^2 + y^2$  is always  $\geq 0$ , the surface  $x^2 + y^2 = z^2 - 1$  has no points with  $z^2$  less than 1.
- 30 Direct approach:  $xy = \left(\frac{x_1+x_2}{2}\right)\left(\frac{y_1+y_2}{2}\right) = \frac{1}{4}(x_1y_1 + x_2y_2 + x_1y_2 + x_2y_1) = \frac{1}{4}(1 + 1 + \frac{x_1}{x_2} + \frac{x_2}{x_1})$   
 $= 1 + \frac{(x_1-x_2)^2}{4x_1x_2} \geq 1$ . Quicker approach:  $y = \frac{1}{x}$  is concave up (or convex) because  $y'' = \frac{2}{x^3}$  is positive.  
*Note for convex functions:* Tangent lines below curve, secant line segments above curve!
- 32  $y = \frac{2048}{x^3}$  has  $\frac{dy}{dx} = -\frac{4096}{x^4} = -1$  at  $x = 16$ . Also  $y'' = \frac{12,288}{x^5} \geq 0$  so the curve is concave up (or convex).  
 The line  $x + y = 24$  also goes through (16,8) with slope  $-1$ ; it must be the tangent line.

34 The function  $f(x, y)$  is the height above the ground. The level curve  $f = 0$  is the outline of the shoe.

## 13.2 Partial Derivatives (page 479)

The partial derivative  $\partial f / \partial y$  comes from fixing  $\mathbf{x}$  and moving  $\mathbf{y}$ . It is the limit of  $(f(\mathbf{x}, \mathbf{y} + \Delta \mathbf{y}) - f(\mathbf{x}, \mathbf{y})) / \Delta y$ . If  $f = e^{2x} \sin y$  then  $\partial f / \partial x = 2e^{2x} \sin y$  and  $\partial f / \partial y = e^{2x} \cos y$ . If  $f = (x^2 + y^2)^{1/2}$  then  $f_x = x / (x^2 + y^2)^{1/2}$  and  $f_y = y / (x^2 + y^2)^{1/2}$ . At  $(x_0, y_0)$  the partial derivative  $f_x$  is the ordinary derivative of the partial function  $f(x, y_0)$ . Similarly  $f_y$  comes from  $f(\mathbf{x}_0, \mathbf{y})$ . Those functions are cut out by vertical planes  $x = x_0$  and  $y = y_0$ , while the level curves are cut out by horizontal planes.

The four second derivatives are  $f_{xx}, f_{xy}, f_{yx}, f_{yy}$ . For  $f = xy$  they are  $0, 1, 1, 0$ . For  $f = \cos 2x \cos 3y$  they are  $-4 \cos 2x \cos 3y, 6 \sin 2x \sin 3y, -9 \cos 2x \cos 3y$ . In those examples the derivatives  $f_{xy}$  and  $f_{yx}$  are the same. That is always true when the second derivatives are continuous. At the origin,  $\cos 2x \cos 3y$  is curving down in the  $x$  and  $y$  directions, while  $xy$  goes up in the  $45^\circ$  direction and down in the  $-45^\circ$  direction.

- 1  $3 + 2xy^2; -1 + 2yx^2$     3  $3x^2y^2 - 2x; 2x^3y - e^y$     5  $\frac{-2y}{(x-y)^2}; \frac{2x}{(x-y)^2}$     7  $\frac{-2x}{(x^2+y^2)^2}; \frac{-2y}{(x^2+y^2)^2}$   
 9  $\frac{x}{x^2+y^2}; \frac{y}{x^2+y^2}$     11  $\frac{-y}{x^2+y^2}; \frac{x}{x^2+y^2}$     13 2, 3, 4    15  $6(x + iy), 6i(x + iy), -6(x + iy)$   
 17  $(f = \frac{1}{r}) f_{xx} = \frac{2x^2 - y^2}{r^5}; f_{xy} = \frac{3xy}{r^5}; f_{yy} = \frac{2y^2 - x^2}{r^5}$     19  $-a^2 \cos ax \cos by, ab \sin ax \sin by, -b^2 \cos ax \cos by$   
 21 Omit line  $x = y$ ; all positive numbers;  $f_x = -2(x - y)^{-3}, f_y = 2(x - y)^{-3}$   
 23 Omit  $z = t$ ; all numbers;  $\frac{-1}{z-t}, \frac{1}{z-t}, \frac{(x-y)}{(x-t)^2}, \frac{(y-x)}{(x-t)^2}$   
 25  $x > 0, t > 0$  and  $x = 0, t > 1$  and  $x = -1, -2, \dots, t = e, e^2, \dots$ ;  $f_x = (\ln t)x^{\ln t - 1}, f_t = (\ln x)t^{\ln x - 1}$   
 27  $y, x; f = G(x) + H(y)$     29  $\frac{\partial f}{\partial x} = \frac{\partial(xy)}{\partial x} v(xy) = yv(xy)$   
 31  $f_{xxx} = 6y^3, f_{yyy} = 6x^3, f_{xxy} = f_{xyx} = f_{yxx} = 18xy^2, f_{yyx} = f_{yxy} = f_{xyy} = 18x^2y$   
 33  $g(y) = Ae^{cy/7}$     35  $g(y) = Ae^{cy/2} + Be^{-cy/2}$   
 37  $f_t = -2f, f_{xx} = f_{yy} = -e^{-2t} \sin x \sin y; e^{-13t} \sin 2x \sin 3y$   
 39  $\sin(x + t)$  moves left    41  $\sin(x - ct), \cos(x + ct), e^{x-ct}$   
 43  $(B - A)h_y(C^*) = (B - A)[f_y(b, C^*) - f_y(a, C^*)] = (B - A)(b - a)f_{yx}(c^*, C^*)$ ; continuous  $f_{xy}$  and  $f_{yx}$   
 45  $y$  converges to  $b$ ; inside and stay inside;  $d_n = \sqrt{(x_n - a)^2 + (y_n - b)^2} \rightarrow \text{zero}; d_n < \epsilon$  for  $n > N$   
 47  $\epsilon$ , less than  $\delta$     49  $f(a, b); \frac{1}{x-1}$  or  $\frac{1}{(x-1)(y-2)}$     51  $f(0, 0) = 1; f(0, 0) = 1$ ; not defined for  $x < 0$

- 2  $\frac{\partial f}{\partial x} = 3 \cos(3x - y), \frac{\partial f}{\partial y} = -\cos(3x - y) + 1$     4  $\frac{\partial f}{\partial x} = e^{x+4} + xe^{x+4}, \frac{\partial f}{\partial y} = 0$   
 6  $\frac{\partial f}{\partial x} = -x(x^2 + y^2)^{-3/2}, \frac{\partial f}{\partial y} = -y(x^2 + y^2)^{-3/2}$     8  $\frac{\partial f}{\partial x} = \frac{1}{x+2y}, \frac{\partial f}{\partial y} = \frac{2}{x+2y}$     10  $\frac{\partial f}{\partial x} = y^x(\ln y), \frac{\partial f}{\partial y} = xy^{x-1}$   
 12  $\frac{\partial f}{\partial x} = \frac{1}{x}, \frac{\partial f}{\partial y} = \frac{1}{y}$     14  $f_{xx} = 2, f_{xy} = f_{yx} = 6, f_{yy} = 18$   
 16  $f_{xx} = a^2e^{ax+by}, f_{xy} = f_{yx} = abe^{ax+by}, f_{yy} = b^2e^{ax+by}$   
 18  $f_{xx} = n(n - 1)(x + y)^{n-2} = f_{xy} = f_{yx} = f_{yy}!$   
 20  $f_{xx} = \frac{2}{(x+iy)^3}, f_{xy} = f_{yx} = \frac{2i}{(x+iy)^3}, f_{yy} = \frac{2i^2}{(x+iy)^3} = \frac{-2}{(x+iy)^3}$  Note  $f_{xx} + f_{yy} = 0$ .  
 22 Domain: all  $(x, y, t)$  such that  $x^2 + y^2 \geq t^2$  (interior of cone  $x^2 + y^2 = t^2$ ); range: all values  $f \geq 0$ ;

$$\frac{\partial f}{\partial x} = \frac{x}{f}, \frac{\partial f}{\partial y} = \frac{y}{f}, \frac{\partial f}{\partial t} = -\frac{1}{f}.$$

- 24 Domain: halfplane where  $x + t > 0$ ; range: all real numbers;  $f_x = \frac{1}{x+t} = f_t$ .
- 26 Domain: all  $(x, y)$  with  $|y| \leq 1$ ; range: all numbers with absolute value  $|f(x, y)| \leq \pi$  (since  $-1 \leq \cos x \leq 1$  and  $0 \leq \cos^{-1} y \leq \pi$ );  $\frac{\partial f}{\partial x} = -\sin x \cos^{-1} y$ ,  $\frac{\partial f}{\partial y} = -\frac{\cos x}{\sqrt{1-y^2}}$ .
- 28  $\frac{\partial f}{\partial x} = -v(x)$  and  $\frac{\partial f}{\partial y} = v(y)$ .
- 30  $f(x, y) = \int_x^y \frac{dt}{t} = \ln y - \ln x = \ln \frac{y}{x}$ ;  $f_x = -\frac{1}{x}$  and  $f_y = \frac{1}{y}$  (confirming Problem 28).
- 32  $g(y) = e^{-cy}$  or any multiple  $Ae^{-cy}$ . 34  $g(y) = e^{c^2y}$  or any multiple  $Ae^{c^2y}$ .
- 36  $f_x = \frac{1}{\sqrt{t}} \left( \frac{-2x}{4t} \right) e^{-x^2/4t}$ . Then  $f_{xx} = f_t = \frac{-1}{2t^{3/2}} e^{-x^2/4t} + \frac{x^2}{4t^{5/2}} e^{-x^2/4t}$ .
- 38  $e^{-m^2t-n^2t} \sin mx \cos ny$  solves  $f_t = f_{xx} + f_{yy}$ . Also  $f = \frac{1}{t} e^{-(x^2+y^2)/4t}$  has  $f_t = f_{xx} + f_{yy} = \left( -\frac{1}{t^2} + \frac{x^2+y^2}{4t^3} \right) e^{-(x^2+y^2)/4t}$ .
- 40  $f = \sin(x-t)$  peaks when  $x-t = \frac{\pi}{2}$  and  $f = 1$ . If  $t$  increases by  $\Delta t = \frac{\pi}{4}$  then  $x$  increases by  $\Delta x = \frac{\pi}{4}$ . The wave velocity is  $\frac{\Delta x}{\Delta t} = 1$ . The other function  $\sin(x+t)$  has  $\frac{\Delta x}{\Delta t} = -1$  (velocity 1 to the left).
- 42  $\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}$  gives  $\frac{\partial^2 f}{\partial t^2} = \frac{\partial^2 f}{\partial t \partial x}$  and also  $\frac{\partial^2 f}{\partial x \partial t} = \frac{\partial^2 f}{\partial x^2}$ . Use  $f_{xt} = f_{tx}$  to find  $\frac{\partial^2 f}{\partial t^2} = \frac{\partial^2 f}{\partial x^2}$ .
- 44 (a)  $\frac{\partial f}{\partial x} = y$  and  $\frac{\partial f}{\partial y} = x$  when  $xy > 0$  (two quadrants);  $\frac{\partial f}{\partial x} = -y$  and  $\frac{\partial f}{\partial y} = -x$  when  $xy < 0$  (other two quadrants);  $\frac{\partial f}{\partial x}$  doesn't exist but  $\frac{\partial f}{\partial y} = 0$  up the  $y$  axis;  $\frac{\partial f}{\partial x} = 0$  but  $\frac{\partial f}{\partial y}$  doesn't exist on the  $x$  axis.  
 (b)  $\frac{\partial f}{\partial x} = 2x$  and  $\frac{\partial f}{\partial y} = 2y$  except  $f$  is not continuous when  $x = 0$  and  $y \neq 0$ . OK at  $(0,0)$ .
- 46 (a)  $(x_2, y_2) = (\frac{1}{3}, \frac{2}{3})$ ;  $(x_4, y_4) = (\frac{1}{5}, \frac{4}{5})$ ; approaching  $(0,1)$  (b)  $(x_2, y_2) = (x_4, y_4) = (1,0)$  approaching  $(1,0)$   
 (c)  $(x_2, y_2) = (x_4, y_4) = (1,0)$  but no limit (d)  $(x_2, y_2) = (2,0)$ ,  $(x_4, y_4) = (4,0)$  has no limit.
- 48 (a) The limit is  $\sqrt{a^2 + b^2}$  (continuous function) (b) The limit is  $\frac{a}{b}$  provided  $b \neq 0$  (c) The limit is  $\frac{1}{a+b}$  provided  $a + b \neq 0$  (d) The limit is  $\frac{ab}{a^2+b^2}$  except no limit at  $(0,0)$ .
- 50 Along  $y = mx$  the function is  $\frac{mx^3}{x^4+m^3x^3} \rightarrow 0$  (the ratio is near  $\frac{mx^3}{m^3x^3}$  for small  $x$ ). But on the parabola  $y = x^2$  the function is  $\frac{x^4}{2x^4} = \frac{1}{2}$ . So this function  $f(x, y)$  has no limit: not continuous at  $(0,0)$ .
- 52 (a)  $\frac{xy^2}{x^2+y^2} = y \left( \frac{xy}{x^2+y^2} \right) \rightarrow 0$  because always  $|\frac{xy}{x^2+y^2}| \leq \frac{1}{2}$ . (b)  $\frac{x^2y^2}{x^4+y^4}$  equals 0 on the axes but  $\frac{1}{2}$  on 45° lines; no limit; (c)  $\frac{x^m y^n}{x^m + y^n} = (x^{m/2} y^{n/2}) \left( \frac{x^{m/2} y^{n/2}}{x^m + y^n} \right) \rightarrow 0$  if  $m > 0, n > 0$ , because the second factor is  $\leq \frac{1}{2}$  as in (a). For negative  $x$  and  $y$ ,  $m$  and  $n$  should be positive integers. Further problem by same method:  
 $\frac{x^a y^b}{x^m + y^n} \rightarrow 0$  if  $a > \frac{m}{2}$  and  $b > \frac{n}{2}$ .

13.3 Tangent Planes and Linear Approximations (page 488)

The tangent line to  $y = f(x)$  is  $y - y_0 = f'(x_0)(x - x_0)$ . The tangent plane to  $w = f(x, y)$  is  $w - w_0 = (\partial f/\partial x)_0(x - x_0) + (\partial f/\partial y)_0(y - y_0)$ . The normal vector is  $N = (f_x, f_y, -1)$ . For  $w = x^3 + y^3$  the tangent equation at  $(1,1,2)$  is  $w - 2 = 3(x - 1) + 3(y - 1)$ . The normal vector is  $N = (3, 3, -1)$ . For a sphere, the direction of  $N$  is out from the origin.

The surface given implicitly by  $F(x, y, z) = c$  has tangent plane with equation  $(\partial F/\partial x)_0(x - x_0) + (\partial F/\partial y)_0(y - y_0) + (\partial F/\partial z)_0(z - z_0) = 0$ . For  $xyz = 6$  at  $(1,2,3)$  the tangent plane has the equation  $6(x - 1) + 3(y - 2) + 2(z - 3) = 0$ . On that plane the differentials satisfy  $6dx + 3dy + 2dz = 0$ . The differential of  $z = f(x, y)$  is  $dz = f_x dx + f_y dy$ . This holds exactly on the tangent plane, while  $\Delta z \approx f_x \Delta x + f_y \Delta y$  holds approximately on the surface. The height  $z = 3x + 7y$  is more sensitive to a change in  $y$  than in  $x$ ,

because the partial derivative  $\partial z/\partial y = 7$  is larger than  $\partial z/\partial x = 3$ .

The linear approximation to  $f(x, y)$  is  $f(x_0, y_0) + (\partial f/\partial x)_0(x - x_0) + (\partial f/\partial y)_0(y - y_0)$ . This is the same as  $\Delta f \approx (\partial f/\partial x)\Delta x + (\partial f/\partial y)\Delta y$ . The error is of order  $(\Delta x)^2 + (\Delta y)^2$ . For  $f = \sin xy$  the linear approximation around  $(0,0)$  is  $f_L = 0$ . We are moving along the tangent plane instead of the surface. When the equation is given as  $F(x, y, z) = c$ , the linear approximation is  $\mathbf{F}_x\Delta x + \mathbf{F}_y\Delta y + \mathbf{F}_z\Delta z = 0$ .

Newton's method solves  $g(x, y) = 0$  and  $h(x, y) = 0$  by a linear approximation. Starting from  $x_n, y_n$  the equations are replaced by  $\mathbf{g}_x\Delta x + \mathbf{g}_y\Delta y = -\mathbf{g}(x_n, y_n)$  and  $\mathbf{h}_x\Delta x + \mathbf{h}_y\Delta y = -\mathbf{h}(x_n, y_n)$ . The steps  $\Delta x$  and  $\Delta y$  go to the next point  $(x_{n+1}, y_{n+1})$ . Each solution has a basin of attraction. Those basins are likely to be fractals.

- 1  $z - 1 = y - 1; \mathbf{N} = \mathbf{j} - \mathbf{k}$       3  $z - 2 = \frac{1}{3}(x - 6) - \frac{2}{3}(y - 3); \mathbf{N} = \frac{1}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} - \mathbf{k}$
- 5  $2(x - 1) + 4(y - 2) + 2(z - 1) = 0; \mathbf{N} = 2\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$       7  $z - 1 = x - 1; \mathbf{N} = \mathbf{i} - \mathbf{k}$
- 9 Tangent plane  $2z_0(z - z_0) - 2x_0(x - x_0) - 2y_0(y - y_0) = 0; (0, 0, 0)$  satisfies this equation because  $z_0^2 - x_0^2 - y_0^2 = 0$  on the surface;  $\cos \theta = \frac{\mathbf{N} \cdot \mathbf{k}}{|\mathbf{N}||\mathbf{k}|} = \frac{-z_0}{\sqrt{x_0^2 + y_0^2 + z_0^2}} = \frac{-1}{\sqrt{2}}$  (surface is the  $45^\circ$  cone)
- 11  $dz = 3dx - 2dy$  for both;  $dz = 0$  for both;  $\Delta z = 0$  for  $3x - 2y, \Delta z = .00029$  for  $x^3/y^2$ ; tangent plane
- 13  $z = z_0 + F_x t$ ; plane  $6(x - 4) + 12(y - 2) + 8(z - 3) = 0$ ; normal line  $x = 4 + 6t, y = 2 + 12t, z = 3 + 8t$
- 15 Tangent plane  $4(x - 2) + 2(y - 1) + 4(z - 2) = 0$ ; normal line  $x = 2 + 4t, y = 1 + 2t, z = 2 + 4t; (0, 0, 0)$  at  $t = -\frac{1}{2}$
- 17  $dw = y_0 dx + x_0 dy$ ; product rule;  $\Delta w - dw = (x - x_0)(y - y_0)$
- 19  $dI = 4000dR + .08dP; dP = \$100; I = (.78)(4100) = \$319.80$
- 21 Increase =  $\frac{26}{101} - \frac{25}{100} = \frac{3}{404}$ , decrease =  $\frac{25}{100} - \frac{25}{101} = \frac{1}{404}; dA = \frac{1}{y}dx - \frac{x}{y^2}dy; 3$       23  $\Delta \theta \approx \frac{-y\Delta x + x\Delta y}{\sqrt{x^2 + y^2}}$
- 25  $Q$  increases;  $Q_s = -\frac{250}{3}, Q_t = \frac{-5}{3}, P_s = -.2Q_s = \frac{50}{3}, P_t = -.2Q_t = \frac{1}{3}; Q = 50 - \frac{250}{3}(s - .4) - \frac{5}{3}(t - 10)$
- 27  $s = 1, t = 10$  gives  $Q = 40$ :  $\begin{matrix} P_s & = & -Q_s & = & sQ_s & + & Q & = & Q_s & + & 40 \\ P_t & = & -Q_t & = & sQ_t & + & 1 & = & Q_t & + & 1 \end{matrix}$ ;  $Q_s = -20, Q_t = -\frac{1}{2}, P_s = 20, P_t = \frac{1}{2}$
- 29  $z - 2 = x - 2 + 2(y - 1)$  and  $z - 3 = 4(x - 2) - 2(y - 1); x = 1, y = \frac{1}{2}, z = 0$
- 31  $\Delta x = -\frac{1}{2}, \Delta y = \frac{1}{2}; x_1 = \frac{1}{2}, y_1 = -\frac{1}{2}$ ; line  $x + y = 0$
- 33  $3a^2\Delta x - \Delta y = -a - a^3$  gives  $\Delta y = -\Delta x = \frac{a+a^3}{1+3a^2}$ ; lemon starts at  $(1/\sqrt{3}, -1/\sqrt{3})$   
 $-\Delta x + 3a^2\Delta y = a + a^3$
- 35 If  $x^3 = y$  then  $y^3 = x^9$ . Then  $x^9 = x$  only if  $x = 0$  or  $1$  or  $-1$  (or complex number)
- 37  $\Delta x = -x_0 + 1, \Delta y = -y_0 + 2, (x_1, y_1) = (1, 2) = \text{solution}$
- 39  $G = H = \frac{x^2}{2x^n - 1}$       41  $J = \begin{bmatrix} e^x & 0 \\ 1 & e^y \end{bmatrix}, \Delta x = -1 + e^{-x_n}, \Delta y = -1 - (x_n - 1 + e^{-x_n})e^{-y_n}$
- 43  $(x_1, y_1) = (0, \frac{5}{4}), (-\frac{3}{4}, \frac{5}{4}), (\frac{3}{4}, 0)$

- 2  $\mathbf{N} = \mathbf{i} + \mathbf{j} + \mathbf{k}; (x - 3) + (y - 4) + (z - 10) = 0$       4  $\mathbf{N} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}; x + 2y = z - 1$
- 6  $\mathbf{N} = 2\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}; 2(x - 1) + 4(y - 2) + 4(z - 1) = 0$
- 8  $\mathbf{N} = 8\pi\mathbf{i} + 4\pi\mathbf{j} - \mathbf{k}; 8\pi(r - 2) + 4\pi(h - 2) = V - 8\pi$
- 10  $\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4 & -1 \\ 2 & 3 & -1 \end{vmatrix} = -\mathbf{i} - \mathbf{j} - 5\mathbf{k}$  (both planes go through  $(0,0,0)$  and so does the line!)

- 12  $\mathbf{N}_1 = 2\mathbf{i} + 4\mathbf{j} - \mathbf{k}$  and  $\mathbf{N}_2 = 2\mathbf{i} + 6\mathbf{j} - \mathbf{k}$  give  $\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 4 & -1 \\ 2 & 6 & -1 \end{vmatrix} = 2\mathbf{i} + 4\mathbf{k}$  tangent to both surfaces
- 14 The direction of  $\mathbf{N}$  is  $2xy^2\mathbf{i} + 2x^2y\mathbf{j} - \mathbf{k} = 8\mathbf{i} + 4\mathbf{j} - \mathbf{k}$ . So the line through  $(1, 2, 4)$  has  $x = 1 + 8t$ ,  $y = 2 + 4t$ ,  $z = 4 - t$ .
- 16 The normal line through  $(x_0, y_0, z_0)$  has direction  $\mathbf{N} = (F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k})_0$ . This is the radial line from the origin if  $(F_x)_0 = cx_0$ ,  $(F_y)_0 = cy_0$ ,  $(F_z)_0 = cz_0$ . Then  $F$  is a function of  $x^2 + y^2 + z^2$  and the surface is a sphere.
- 18  $df = yz dx + xz dy + xy dz$ .
- 20 Direct method:  $R = \frac{R_1 R_2}{R_1 + R_2}$  and  $\frac{\partial R}{\partial R_1} = \frac{R_2^2}{(R_1 + R_2)^2}$  and  $\frac{\partial R}{\partial R_2} = \frac{R_1^2}{(R_1 + R_2)^2}$ . If  $R_1 = 1$  and  $R_2 = 2$  then  $\frac{\partial R}{\partial R_1}$  is four times larger ( $\frac{4}{9}$  vs.  $\frac{1}{9}$ ; more sensitive to  $R_1$ ). By chain rule:  $-\frac{1}{R^2} \frac{\partial R}{\partial R_1} = -\frac{1}{R_1^2}$  and  $-\frac{1}{R^2} \frac{\partial R}{\partial R_2} = -\frac{1}{R_2^2}$ .
- 22 (a) Common sense: 2 hits in 5 at bats ( $\frac{x}{y} = .4$ ) raises an average that is below .4. Mathematics:  $A = \frac{x}{y}$  has  $dA = \frac{y dx - x dy}{y^2} > 0$  if  $y dx > x dy$ . This is again  $\frac{dx}{dy} > \frac{x}{y}$  or  $.4 > A$ . (b) The player has  $x = 200$  hits since  $\frac{200}{400} = .5$ . We want to choose  $\Delta x = \Delta y$  (all hits) so  $\Delta A$  reaches .005. But  $\Delta A \approx \frac{y \Delta x - x \Delta y}{y^2} = \frac{200}{(400)^2} \Delta x = .005$  when  $\Delta x = 4$  hits. Check:  $\frac{204}{404} = .505$  (to 3 decimals). If averages are rounded down we need  $\Delta x = 5$  hits.
- 24 (1)  $c$  is between  $\mathbf{x}_0$  and  $\mathbf{x}_0 + \Delta \mathbf{x}$  by the Mean Value Theorem (2)  $C$  is between  $\mathbf{y}_0$  and  $\mathbf{y}_0 + \Delta \mathbf{y}$  (3) the limit exists if  $f_{\mathbf{x}}$  is continuous (4) the limit exists if  $f_{\mathbf{y}}$  is continuous.
- 26  $P = \frac{40s + 2t}{s + 2}$  and  $Q = \frac{40 - t}{s + 2}$  so  $\frac{\partial P}{\partial s} = \frac{8 - 2t}{(s + 2)^2}$  and  $\frac{\partial P}{\partial t} = \frac{2}{s + 2}$ . At  $s = .4$ ,  $t = 10$  this gives  $\frac{\partial P}{\partial s} = \frac{6}{(.6)^2} = \frac{50}{3}$  and  $\frac{\partial P}{\partial t} = \frac{2}{.6} = \frac{1}{3}$ .
- 28 Take partial derivatives with respect to  $b$ :  $2x \frac{\partial x}{\partial b} + b \frac{\partial x}{\partial b} + x = 0$  or  $\frac{\partial x}{\partial b} = \frac{-x}{2x + b}$ . Similarly  $2x \frac{\partial x}{\partial c} + b \frac{\partial x}{\partial c} + 1 = 0$  gives  $\frac{\partial x}{\partial c} = \frac{-1}{2x + b}$ . Then  $\frac{\partial x}{\partial b}$  is larger (in magnitude) when  $x = 2$ .
- 30 (a) The third surface is  $z = 0$ . (b) Newton uses the tangent plane to the graph of  $g$ , the tangent plane to the graph of  $h$ , and  $z = 0$ .
- 32  $\frac{3}{4}\Delta x - \Delta y = \frac{3}{8}$  and  $-\Delta x + \frac{3}{4}\Delta y = \frac{3}{8}$  give  $\Delta x = \Delta y = -\frac{3}{2}$ . The new point is  $(-1, -1)$ , an exact solution. The point  $(\frac{1}{2}, \frac{1}{2})$  is in the gray band (upper right in Figure 13.11a) or the blue band on the front cover of the book.
- 34  $3a^2\Delta x - \Delta y = -a^3$  and  $-\Delta x + 0\Delta y = a$  give  $\Delta x = -a$  and  $\Delta y = -2a^3$ . The new point is  $(0, -2a^3)$  on the  $y$  axis. Then  $0\Delta x - \Delta y = -2a^3$  and  $-\Delta x + 3(4a^6)\Delta y = 8a^9$  give  $\Delta y = 2a^3$  and  $\Delta x = 16a^9$ . The new point  $(16a^9, 0)$  is the same as the start  $(a, 0)$  if  $16a^8 = 1$  or  $a = \pm \frac{1}{\sqrt[8]{2}}$ . In these cases Newton's method cycles. Question: Is this where the white basin ends along the  $x$  axis?
- 36 By Problem 34 Newton's method diverges if  $16a^8 > 1$ : for instance  $(x_0, y_0) = (1, 0)$  as in Example 9 in the text.
- 38 A famous fractal shows the three basins of attraction - see almost any book displaying fractals. Remarkable property of the boundary points between basins: they touch all three basins! Try to draw 3 regions with this property.
- 40 Problem 39 has  $2x\Delta x - \Delta y = y - x^2$  and  $\Delta x - \Delta y = y - x$ . Subtraction gives  $(2x - 1)\Delta x = x - x^2$ . Then  $x + \Delta x = x + \frac{x - x^2}{2x - 1} = \frac{x^2}{2x - 1}$ . By the second equation this is also  $y + \Delta y$ . Now find the basin: If  $x < 0$  then  $\Delta x > 0$  but  $x + \Delta x$  still  $< 0$ : moving toward 0. If  $0 \leq x < \frac{1}{2}$  then  $x + \Delta x < 0$ . So the basin for  $(0, 0)$  has all  $x < \frac{1}{2}$ . The line  $x = \frac{1}{2}$  gives blowup. If  $\frac{1}{2} < x \leq 1$  then  $\Delta x > 0$ . If  $x > 1$  then  $\Delta x < 0$  but  $x + \Delta x = \frac{x^2}{2x - 1} \geq 1$  (because  $x^2 - 2x + 1 \geq 0$ ). So the basin for  $(1, 1)$  has all  $x > \frac{1}{2}$ .

42  $J = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$  is singular;  $g$  and  $h$  have the same tangent planes. Newton's equations  $2\Delta x + 2\Delta y = -2$  and  $\Delta x + \Delta y = -1$  have infinitely many solutions.

### 13.4 Directional Derivatives and Gradients (page 495)

$D_{\mathbf{u}}f$  gives the rate of change of  $f(\mathbf{x}, \mathbf{y})$  in the direction  $\mathbf{u}$ . It can be computed from the two derivatives  $\partial f/\partial x$  and  $\partial f/\partial y$  in the special directions  $(1,0)$  and  $(0,1)$ . In terms of  $u_1, u_2$  the formula is  $D_{\mathbf{u}}f = f_x u_1 + f_y u_2$ . This is a dot product of  $\mathbf{u}$  with the vector  $(f_x, f_y)$ , which is called the gradient. For the linear function  $f = ax + by$ , the gradient is  $\text{grad } f = (a, b)$  and the directional derivative is  $D_{\mathbf{u}}f = (a, b) \cdot \mathbf{u}$ .

The gradient  $\nabla f = (f_x, f_y)$  is not a vector in three dimensions, it is a vector in the base plane. It is perpendicular to the level lines. It points in the direction of steepest climb. Its magnitude  $|\text{grad } f|$  is the steepness  $\sqrt{f_x^2 + f_y^2}$ . For  $f = x^2 + y^2$  the gradient points out from the origin and the slope in that steepest direction is  $|(2x, 2y)| = 2r$ .

The gradient of  $f(x, y, z)$  is  $(f_x, f_y, f_z)$ . This is different from the gradient on the surface  $F(x, y, z) = 0$ , which is  $-(F_x/F_z)\mathbf{i} - (F_y/F_z)\mathbf{j}$ . Traveling with velocity  $\mathbf{v}$  on a curved path, the rate of change of  $f$  is  $df/dt = (\text{grad } f) \cdot \mathbf{v}$ . When the tangent direction is  $\mathbf{T}$ , the slope of  $f$  is  $df/ds = (\text{grad } f) \cdot \mathbf{T}$ . In a straight direction  $\mathbf{u}$ ,  $df/ds$  is the same as the directional derivative  $D_{\mathbf{u}}f$ .

- 1  $\text{grad } f = 2x\mathbf{i} - 2y\mathbf{j}, D_{\mathbf{u}}f = \sqrt{3}x - y, D_{\mathbf{u}}f(P) = \sqrt{3}$
- 3  $\text{grad } f = e^x \cos y \mathbf{i} - e^x \sin y \mathbf{j}, D_{\mathbf{u}}f = -e^x \sin y, D_{\mathbf{u}}f(P) = -1$
- 5  $f = \sqrt{x^2 + (y-3)^2}, \text{grad } f = \frac{x}{f} \mathbf{i} + \frac{y-3}{f} \mathbf{j}, D_{\mathbf{u}}f = \frac{x}{f}, D_{\mathbf{u}}f(P) = \frac{1}{\sqrt{5}}$       7  $\text{grad } f = \frac{2x}{x^2+y^2} \mathbf{i} + \frac{2y}{x^2+y^2} \mathbf{j}$
- 9  $\text{grad } f = 6x\mathbf{i} + 4y\mathbf{j} = 6\mathbf{i} + 8\mathbf{j} = \text{steepest direction at } P; \text{ level direction } -8\mathbf{i} + 6\mathbf{j} \text{ is perpendicular}; 10, 0$
- 11  $\mathbf{T}; \mathbf{F} (\text{grad } f \text{ is a vector}); \mathbf{F}; \mathbf{T}$       13  $\mathbf{u} = (\frac{a}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}}), D_{\mathbf{u}}f = \sqrt{a^2 + b^2}$
- 15  $\text{grad } f = (e^{x-y}, -e^{x-y}) = (e^{-1}, -e^{-1}) \text{ at } P; \mathbf{u} = (\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}), D_{\mathbf{u}}f = \sqrt{2}e^{-1}$
- 17  $\text{grad } f = \mathbf{O} \text{ at maximum; level curve is one point}$       19  $\mathbf{N} = (-1, 1, -1), \mathbf{U} = (-1, 1, 2), \mathbf{L} = (1, 1, 0)$
- 21 Direction  $-\mathbf{U} = (-2, 0, -4)$       23  $-\mathbf{U} = (\frac{x}{\sqrt{1-x^2-y^2}}, \frac{y}{\sqrt{1-x^2-y^2}}, \frac{-x^2-y^2}{1-x^2-y^2})$
- 25  $f = (x + 2y) \text{ and } (x + 2y)^2; \mathbf{i} + 2\mathbf{j}; \text{ straight lines } x + 2y = \text{constant (perpendicular to } \mathbf{i} + 2\mathbf{j})$
- 27  $\text{grad } f = \pm(\frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}}); \text{grad } g = \pm(2\sqrt{5}, \sqrt{5}), f = \pm(\frac{x}{\sqrt{5}} - \frac{2y}{\sqrt{5}}) + C, g = \pm(2\sqrt{5}x + \sqrt{5}y) + C$
- 29  $\theta = \text{constant along ray in direction } \mathbf{u} = \frac{3\mathbf{i}+4\mathbf{j}}{5}; \text{grad } \theta = \frac{-y\mathbf{i}+x\mathbf{j}}{x^2+y^2} = \frac{-4\mathbf{i}+3\mathbf{j}}{25}; \mathbf{u} \cdot \text{grad } \theta = 0$
- 31  $\mathbf{U} = (f_x, f_y, f_x^2 + f_y^2) = (-1, -2, 5); -\mathbf{U} = (-1, -2, 5); \text{tangent at the point } (2, 1, 6)$
- 33  $\text{grad } f \text{ toward } 2\mathbf{i} + \mathbf{j} \text{ at } P, \mathbf{j} \text{ at } Q, -2\mathbf{i} + \mathbf{j} \text{ at } R; (2, \frac{1}{2}) \text{ and } (2\frac{1}{2}, 2); \text{largest upper left, smallest lower right;}$   
 $z_{\max} > 9; z \text{ goes from } 2 \text{ to } 8 \text{ and back to } 6$
- 35  $f = \frac{1}{2}\sqrt{(x-1)^2 + (y-1)^2}; (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})_{0,0} = (\frac{-1}{2\sqrt{2}}, \frac{-1}{2\sqrt{2}})$
- 37 Figure C now shows level curves;  $|\text{grad } f|$  is varying;  $f$  could be  $xy$
- 39  $x^2 + xy; e^{x-y}$ ; no function has  $\frac{\partial f}{\partial x} = y$  and  $\frac{\partial f}{\partial y} = -x$  because then  $f_{xy} \neq f_{yx}$
- 41  $\mathbf{v} = (1, 2t); \mathbf{T} = \mathbf{v}/\sqrt{1+4t^2}; \frac{d\mathbf{f}}{dt} = \mathbf{v} \cdot (2t, 2t^2) = 2t + 4t^3; \frac{df}{do} = (2t + 4t^3)/\sqrt{1+4t^2}$

- 43  $\mathbf{v} = (2, 3)$ ;  $\mathbf{T} = \frac{\mathbf{v}}{\sqrt{13}}$ ;  $\frac{df}{dt} = \mathbf{v} \cdot (2x_0 + 4t, -2y_0 - 6t) = 4x_0 - 6y_0 - 10t$ ;  $\frac{df}{ds} = \frac{df/dt}{\sqrt{13}}$
- 45  $\mathbf{v} = (e^t, 2e^{2t}, -e^{-t})$ ;  $\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$ ;  $\text{grad } f = (\frac{1}{x}, \frac{1}{y}, \frac{1}{z}) = (e^{-t}, e^{-2t}, e^t)$ ,  $\frac{df}{dt} = 1 + 2 - 1$ ,  $\frac{df}{ds} = \frac{2}{|\mathbf{v}|}$
- 47  $\mathbf{v} = (-2 \sin 2t, 2 \cos 2t)$ ,  $\mathbf{T} = (-\sin 2t, \cos 2t)$ ;  $\text{grad } f = (y, x)$ ,  $\frac{df}{ds} = -2 \sin^2 2t + 2 \cos^2 2t$ ,  $\frac{df}{dt} = \frac{1}{2} \frac{df}{ds}$ ;  
 zero slope because  $f = 1$  on this path
- 49  $z - 1 = 2(x - 4) + 3(y - 5)$ ;  $f = 1 + 2(x - 4) + 3(y - 5)$       51  $\text{grad } f \cdot \mathbf{T} = 0$ ;  $\mathbf{T}$
- 2  $\text{grad } f = f_x \mathbf{i} + f_y \mathbf{j} = 3\mathbf{i} + 4\mathbf{j}$ ;  $D_{\mathbf{u}} f = 3(\frac{3}{5}) + 4(\frac{4}{5}) = 5$  at every point  $P$ .
- 4  $\text{grad } f = 10y^2 \mathbf{j}$ ;  $D_{\mathbf{u}} f = -10y^2$ ;  $D_{\mathbf{u}} f(P) = 10$ .      6  $\text{grad } f = \frac{-x\mathbf{i} - y\mathbf{j} - z\mathbf{k}}{(x^2 + y^2 + z^2)^{3/2}}$
- 8  $\text{grad } f = -\frac{(x-1)\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{((x-1)^2 + y^2 + z^2)^{3/2}} - \frac{(x+1)\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{((x+1)^2 + y^2 + z^2)^{3/2}}$
- 10  $\frac{d}{du_1}(3u_1 + \sqrt{1 - u_1^2}) = 3 - \frac{u_1}{\sqrt{1 - u_1^2}} = 0$  if  $u_1 = 3\sqrt{1 - u_1^2}$ . Then  $u_1^2 = 9(1 - u_1^2)$  or  $10u_1^2 = 9$  or  $u_1 = \frac{3}{\sqrt{10}}$ ,  
 which makes the slope equal to  $\frac{3-3}{\sqrt{10}} + \sqrt{\frac{1}{10}} = \sqrt{10}$ .
- 12 In one dimension the gradient of  $f(x)$  is  $\frac{df}{dx}\mathbf{i}$ . The two possible directions are  $\mathbf{u} = \mathbf{i}$  and  $\mathbf{u} = -\mathbf{i}$ . The two directional derivatives are  $+\frac{df}{dx}$  and  $-\frac{df}{dx}$ . The normal vector  $\mathbf{N}$  is  $\frac{df}{dx}\mathbf{i} - \mathbf{j}$ .
- 14 Here  $f = 2x$  above the line  $y = 2x$  and  $f = y$  below that line. The two pieces agree on the line. Then  $\text{grad } f = 2\mathbf{i}$  above and  $\text{grad } f = \mathbf{j}$  below. Surprisingly  $f$  increases fastest *along* the line, which is the direction  $\mathbf{u} = \frac{1}{\sqrt{5}}(\mathbf{i} + 2\mathbf{j})$  and gives  $D_{\mathbf{u}} f = \frac{2}{\sqrt{5}}$ .
- 16  $\text{grad } f = \frac{-x\mathbf{i} - y\mathbf{j}}{\sqrt{5 - x^2 - y^2}} = \frac{-\mathbf{i} - 2\mathbf{j}}{0}$  and  $P$  is a *rough point!* The rate of increase is infinite (provided  $x^2 + y^2$  stays below 5; the direction must point into this circle).
- 18 (a)  $\mathbf{N} \cdot \mathbf{U} = \mathbf{N} \cdot \mathbf{L} = \mathbf{U} \cdot \mathbf{L} = 0$  (b)  $\mathbf{N}$  is perpendicular to the tangent plane,  $\mathbf{U}$  and  $\mathbf{L}$  are parallel to the tangent plane. (c) The gradient is the  $xy$  projection of  $\mathbf{N}$  and also of  $\mathbf{U}$ . The projection of  $\mathbf{L}$  points along the level curve.
- 20  $\mathbf{N} = (\frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}}, -1)$ ,  $\mathbf{U} = (\frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}}, 1)$ , and  $\mathbf{L} = (\frac{y}{\sqrt{x^2 + y^2}}, \frac{-x}{\sqrt{x^2 + y^2}}, 0)$ .  $\mathbf{U}$  goes up the side of the cone.
- 22  $-\mathbf{U} = (-4, 3, -25)$ . The  $xy$  direction of flow is  $-\text{grad } z = -4\mathbf{i} + 3\mathbf{j}$ .
- 24  $-\mathbf{U} = (\frac{-x}{\sqrt{x^2 + y^2}}, \frac{-y}{\sqrt{x^2 + y^2}}, -1)$ . The  $xy$  direction of flow is radially inward.
- 26  $f = \frac{y}{x} = \frac{1}{1}$  is a straight level curve  $y = x$ . The direction of the gradient is perpendicular to that level curve: gradient along  $-\mathbf{i} + \mathbf{j}$ . Check:  $\text{grad } f = \frac{-y}{x^2}\mathbf{i} + \frac{1}{x}\mathbf{j} = -\mathbf{i} + \mathbf{j}$ .
- 28 (a) False because  $f + C$  has the same gradient as  $f$  (b) True because the line direction  $(1, 1, -1)$  is also the normal direction  $\mathbf{N}$  (c) False because the gradient is in 2 dimensions.
- 30  $\theta = \tan^{-1} \frac{y}{x}$  has  $\text{grad } \theta = (\frac{-y/x^2}{1 + (y/x)^2}, \frac{1/x}{1 + (y/x)^2}) = (\frac{-y, x}{x^2 + y^2})$ . The unit vector in this direction is  $\mathbf{T} = (\frac{-y}{\sqrt{x^2 + y^2}}, \frac{x}{\sqrt{x^2 + y^2}})$ . Then  $\text{grad } \theta \cdot \mathbf{T} = \frac{y^2 + x^2}{(x^2 + y^2)^{3/2}} = \frac{1}{r}$ .
- 32  $T = e^{-x^2 - y^2}$  has  $\Delta T \approx \frac{\partial T}{\partial x} \Delta x + \frac{\partial T}{\partial y} \Delta y = (-2x\Delta x - 2y\Delta y)e^{-x^2 - y^2} = (-2\Delta x + 4\Delta y)e^{-5}$ . This is largest going in toward  $(0, 0)$ , in the direction  $\mathbf{u} = (-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}})$ .
- 34 The gradient is  $(2ax + c)\mathbf{i} + (2by + d)\mathbf{j}$ . The figure shows  $c = 0$  and  $d \approx \frac{1}{3}$  at the origin. Then  $b \approx \frac{1}{3}$  from the gradient at  $(0, 1)$ . Then  $a \approx -\frac{1}{4}$  from the gradient at  $(2, 0)$ . The function  $-\frac{1}{4}x^2 + \frac{1}{3}y^2 + \frac{1}{3}y$  has hyperbolas opening upwards as level curves.
- 36  $\text{grad } f$  is tangent to  $xy = c$  and therefore perpendicular to  $y\mathbf{i} + x\mathbf{j}$ . So  $\text{grad } f$  is a multiple of  $x\mathbf{i} - y\mathbf{j}$ .  $|\text{grad } f|$  is larger at  $\mathbf{Q}$  than  $\mathbf{P}$ . It is *not* constant on the hyperbolas. The function could be  $f = x^2 - y^2$ . Its level curves are also hyperbolas, perpendicular to those in the figure.
- 38  $f(0, 1) = B + C = 0$ ,  $f(1, 0) = A + C = 1$ , and  $f(2, 1) = 2A + B + C = 2$ . Solution  $A = 1, B = C = 0$ .  
 So  $\text{grad } f = \mathbf{i}$ .

40 The function is  $xy + C$  so its level curves are standard hyperbolas.

42  $\mathbf{v} = (\frac{dx}{dt}, \frac{dy}{dt}) = (-2 \sin 2t, 2 \cos 2t)$ ;  $\mathbf{T} = (-\sin 2t, \cos 2t)$ ;  $\text{grad } f = (1, 0)$  so  $\frac{df}{dt} = -2 \sin 2t$  and  $\frac{df}{ds} = -\sin 2t$ .

44  $\mathbf{v} = (2t, 0)$  and  $\mathbf{T} = (1, 0)$ ;  $\text{grad } f = (y, x)$  so  $\frac{df}{dt} = 2ty = 6t$  and  $\frac{df}{ds} = y = 3$ .

46  $\mathbf{v} = (1, 2t, 3t^2)$  and  $\mathbf{T} = \mathbf{v}/\sqrt{1+4t^2+9t^4}$ ;  $\text{grad } f = (4x, 6y, 2z) = (4t, 6t^2, 2t^3)$  so  $\frac{df}{dt} = 4t + 12t^3 + 6t^5$   
and  $\frac{df}{ds} = \frac{4t+12t^3+6t^5}{\sqrt{1+4t^2+9t^4}}$ .

48  $D^2 = (x-1)^2 + (y-2)^2$  has  $2D \frac{\partial D}{\partial x} = 2(x-1)$  or  $\frac{\partial D}{\partial x} = \frac{x-1}{D}$ . Similarly  $2D \frac{\partial D}{\partial y} = 2(y-2)$  and  $\frac{\partial D}{\partial y} = \frac{y-2}{D}$ .

Then  $|\text{grad } D| = (\frac{x-1}{D})^2 + (\frac{y-2}{D})^2 = 1$ . The graph of  $D$  is a  $45^\circ$  cone with its vertex at  $(1, 2)$ .

50 The directional derivative at  $P$  is the limit as  $\Delta s \rightarrow 0$  of  $\frac{\Delta f}{\Delta s} = \frac{f(x_0+u_1\Delta s, y_0+u_2\Delta s) - f(x_0, y_0)}{\Delta s}$ . Then

$$\Delta f \approx \frac{\partial f}{\partial x} \Delta x + \frac{\partial f}{\partial y} \Delta y = D_{\mathbf{u}}f(P) \text{ times } \Delta s \text{ and } D_{\mathbf{u}}f(P) = u_1 \frac{\partial f}{\partial x}(P) + u_2 \frac{\partial f}{\partial y}(P).$$

### 13.5 The Chain Rule (page 503)

The chain rule applies to a function of a function. The  $x$  derivative of  $f(g(x, y))$  is  $\partial f/\partial x = (\partial f/\partial g)(\partial g/\partial x)$ . The  $y$  derivative is  $\partial f/\partial y = (\partial f/\partial g)(\partial g/\partial y)$ . The example  $f = (x+y)^n$  has  $g = x+y$ . Because  $\partial g/\partial x = \partial g/\partial y$  we know that  $\partial f/\partial x = \partial f/\partial y$ . This partial differential equation is satisfied by any function of  $x+y$ .

Along a path, the derivative of  $f(x(t), y(t))$  is  $df/dt = (\partial f/\partial x)(dx/dt) + (\partial f/\partial y)(dy/dt)$ . The derivative of  $f(x(t), y(t), z(t))$  is  $f_x x_t + f_y y_t + f_z z_t$ . If  $f = xy$  then the chain rule gives  $df/dt = y dx/dt + x dy/dt$ . That is the same as the product rule! When  $x = u_1 t$  and  $y = u_2 t$  the path is a straight line. The chain rule for  $f(x, y)$  gives  $df/dt = f_x u_1 + f_y u_2$ . That is the directional derivative  $D_{\mathbf{u}}f$ .

The chain rule for  $f(x(t, u), y(t, u))$  is  $\partial f/\partial t = (\partial f/\partial x)(\partial x/\partial t) + (\partial f/\partial y)(\partial y/\partial t)$ . We don't write  $df/dt$  because  $f$  also depends on  $u$ . If  $x = r \cos \theta$  and  $y = r \sin \theta$ , the variables  $t, u$  change to  $r$  and  $\theta$ . In this case  $\partial f/\partial r = (\partial f/\partial x) \cos \theta + (\partial f/\partial y) \sin \theta$  and  $\partial f/\partial \theta = (\partial f/\partial x)(-r \sin \theta) + (\partial f/\partial y)(r \cos \theta)$ . That connects the derivatives in rectangular and polar coordinates. The difference between  $\partial r/\partial x = x/r$  and  $\partial r/\partial x = 1/\cos \theta$  is because  $y$  is constant in the first and  $\theta$  is constant in the second.

With a relation like  $xyz = 1$ , the three variables are not independent. The derivatives  $(\partial f/\partial x)_y$  and  $(\partial f/\partial x)_z$  and  $(\partial f/\partial x)$  mean that  $y$  is held constant, and  $z$  is constant, and both are constant. For  $f = x^2 + y^2 + z^2$  with  $xyz = 1$ , we compute  $(\partial f/\partial x)_z$  from the chain rule  $\partial f/\partial x + (\partial f/\partial y)(\partial y/\partial x)$ . In that rule  $\partial z/\partial x = -1/x^2 y$  from the relation  $xyz = 1$ .

- 1  $f_x = f_y = \cos(x+y)$       3  $f_y = c f_x = c \cos(x+cy)$       5  $3g^2 \frac{\partial g}{\partial x} \frac{dx}{dt} + 3g^2 \frac{\partial g}{\partial y} \frac{dy}{dt}$       7 Moves left at speed 2
- 9  $\frac{dx}{dt} = 1$  (wave moves at speed 1)
- 11  $\frac{\partial^2}{\partial x^2} f(x+iy) = f''(x+iy)$ ,  $\frac{\partial^2}{\partial y^2} f(x+iy) = i^2 f''(x+iy)$   
so  $f_{xx} + f_{yy} = 0$ ;  $(x+iy)^2 = (x^2 - y^2) + i(2xy)$
- 13  $\frac{df}{dt} = 2x(1) + 2y(2t) = 2t + 4t^3$       15  $\frac{df}{dt} = y \frac{dx}{dt} + x \frac{dy}{dt} = -1$       17  $\frac{df}{dt} = \frac{1}{x+y} \frac{dx}{dt} + \frac{1}{x+y} \frac{dy}{dt} = 1$
- 19  $V = \frac{1}{3} \pi r^2 h$ ,  $\frac{dV}{dt} = \frac{2\pi r h}{3} \frac{dr}{dt} + \frac{\pi r^2}{3} \frac{dh}{dt} = 36\pi$
- 21  $\frac{dD}{dt} = \frac{90}{\sqrt{90^2+90^2}}(60) + \frac{90}{\sqrt{90^2+90^2}}(45) = \frac{105}{\sqrt{2}}$  mph;  $\frac{dD}{dt} = \frac{60}{\sqrt{45^2+60^2}}(60) + \frac{45}{\sqrt{45^2+60^2}}(45) \approx 74$  mph



- 23  $\frac{df}{dt} = u_1 \frac{\partial f}{\partial x} + u_2 \frac{\partial f}{\partial y} + u_3 \frac{\partial f}{\partial z}$       25  $\frac{\partial f}{\partial t} = 1$  with  $x$  and  $y$  fixed;  $\frac{df}{dt} = 6$
- 27  $f_t = f_x t + f_y(2t)$ ;  $f_{tt} = f_{xt}t + f_x + 2f_{yt}t + 2f_y = (f_{xx}t + f_{yx}(2t))t + f_x + 2(f_{xy}t + f_{yy}(2t))t + 2f_y$
- 29  $\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial r} = \frac{\partial f}{\partial x} \cos \theta + \frac{\partial f}{\partial y} \sin \theta$ ,  $\theta$  is fixed
- 31  $r_{xx} = \frac{1}{\sqrt{x^2+y^2}} - \frac{x^2}{(x^2+y^2)^{3/2}} = \frac{y^2}{(x^2+y^2)^{3/2}}$ ;  $\frac{\partial}{\partial x}(\frac{x}{r}) = \frac{1}{r} - xr^{-2} \frac{\partial r}{\partial x} = \frac{1}{r} - \frac{x^2}{r^3} = \frac{y^2}{r^3}$
- 33  $(\frac{\partial z}{\partial x})_y = \frac{1}{\sqrt{1-(x+y)^2}}$ ;  $(\cos z)(\frac{\partial z}{\partial x})_y = 1$ ; first answer is also  $\frac{1}{\sqrt{1-\sin^2 z}} = \frac{1}{\cos z}$
- 35  $f_r = f_x \cos \theta + f_y \sin \theta$ ,  $f_{r\theta} = -f_x \sin \theta + f_y \cos \theta + f_{xx}(-r \sin \theta \cos \theta) + f_{xy}(-r \sin^2 \theta + r \cos^2 \theta) + f_{yy}(r \cos \theta \sin \theta)$
- 37 Yes (with  $y$  constant):  $\frac{\partial z}{\partial x} = ye^{xy}$ ,  $\frac{\partial z}{\partial x} = \frac{1}{xy} = \frac{1}{ye^{xy}}$       39  $f_t = f_x x_t + f_y y_t$ ;  $f_{tt} = f_{xx}x_t^2 + 2f_{xy}x_t y_t + f_{yy}y_t^2$
- 41  $(\frac{\partial f}{\partial x})_x = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial x} = a - \frac{3}{5}b$ ;  $(\frac{\partial f}{\partial x})_y = a$ ;  $(\frac{\partial f}{\partial x})_x = \frac{\partial f}{\partial y} \frac{\partial y}{\partial x} = \frac{1}{5}b$
- 43 1      45  $f = y^2$  so  $f_x = 0$ ,  $f_y = 2y = 2r \sin \theta$ ;  $f = r^2$  so  $f_r = 2r = 2\sqrt{x^2+y^2}$ ,  $f_\theta = 0$
- 47  $g_u = f_x x_u + f_y y_u = f_x + f_y$ ;  $g_v = f_x x_v + f_y y_v = f_x - f_y$ ;  $g_{uu} = f_{xx}x_u + f_{xy}y_u + f_{yx}x_u + f_{yy}y_u = f_{xx} + 2f_{xy} + f_{yy}$ ;  $g_{vv} = f_{xx}x_v + f_{xy}y_v - f_{yx}x_v - f_{yy}y_v = f_{xx} - 2f_{xy} + f_{yy}$ . Add  $g_{uu} + g_{vv}$       49 True
- 2  $f_x = 10a(ax+by)^9$  and  $f_y = 10b(ax+by)^9$ ;  $b f_x = a f_y$ .      4  $f_x = \frac{1}{x+7y}$  and  $f_y = \frac{7}{x+7y}$ ;  $7 f_x = f_y$ .
- 6  $\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$  is the product rule  $y \frac{dx}{dt} + x \frac{dy}{dt}$ . In terms of  $u$  and  $v$  this is  $\frac{d}{dt}(uv) = v \frac{du}{dt} + u \frac{dv}{dt}$ .
- 8  $f_{tt} = c^2 n(n-1)(x+ct)^{n-2}$  which equals  $c^2 f_{xx}$ . Choose  $C = -c$ :  $f = (x-ct)^n$  also has  $f_{tt} = c^2 f_{xx}$ .
- 10 Since  $\sin(0-t)$  is decreasing (it is  $-\sin t$ ), you go down. At  $t = 4$ , your height is  $-\sin 4$  and your velocity is  $-\cos(-4) = -\cos 4$ .
- 12 (a)  $f_r = 2re^{2i\theta}$ ,  $f_{rr} = 2e^{2i\theta}$ ,  $f_{\theta\theta} = r^2(2i)^2 e^{2i\theta}$  and  $f_{rr} + \frac{f_r}{r} + \frac{f_{\theta\theta}}{r^2} = 0$ . Take real parts *throughout* to find the same for  $r^2 \cos 2\theta$  (and imaginary parts for  $r^2 \sin 2\theta$ ). (b) Any function  $f(re^{i\theta})$  has  $f_r = e^{i\theta} f'(re^{i\theta})$  and  $f_{rr} = (e^{i\theta})^2 f''(re^{i\theta})$  and  $f_\theta = ire^{i\theta} f'(re^{i\theta})$  and  $f_{\theta\theta} = i^2 r e^{i\theta} f' + (ire^{i\theta})^2 f''$ . Any  $f(re^{i\theta})$  or any  $f(x+iy)$  will satisfy the polar or rectangular form of Laplace's equation.
- 14  $\frac{df}{dt} = \frac{x}{\sqrt{x^2+y^2}}(1) + \frac{y}{\sqrt{x^2+y^2}}(2t) = \frac{t+2t^3}{\sqrt{t^2+t^4}} = \frac{1+2t^2}{\sqrt{1+t^2}}$ .
- 16 Since  $\frac{x}{y} = \frac{1}{2}$  we must find  $\frac{df}{dt} = 0$ . The chain rule gives  $\frac{1}{y} \frac{dx}{dt} - \frac{x}{y^2} \frac{dy}{dt} = \frac{1}{2e^t}(e^t) - \frac{e^t}{4e^{2t}}(2e^t) = 0$ .
- 18  $\frac{df}{dt} = (4t^3)(1) + (0)(1) = 4t^3$ .
- 20 The rocket's position is  $x = 6t$ ,  $y = t^2$ . Its speed from  $(0,0)$  is  $\frac{d}{dt} \sqrt{(6t)^2 + (t^2)^2} = \frac{36t+2t^3}{\sqrt{(6t)^2+(t^2)^2}}$ . At  $t = 0$  this speed is  $\frac{36}{6} = 6$ . The rate of change of  $\theta = \tan^{-1} \frac{t^2}{6t} = \tan^{-1} \frac{t}{6}$  is  $\frac{\frac{1}{6}}{1+(\frac{t}{6})^2}$ . At  $t = 0$  this is  $\frac{1}{6}$ .
- 22 Driving south  $\frac{dT}{dt} = (.05)(70) = 3.5$  degrees per hour. Southeast now gives  $\frac{dT}{dt} = (.05) \frac{80}{\sqrt{2}} + (.01) \frac{80}{\sqrt{2}} \approx 3.4$  degrees per hour.  $\frac{dT}{dt}$  is larger going south.
- 24  $\frac{d^2 f}{dt^2} = \frac{\partial f_t}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f_t}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial f_t}{\partial z} \frac{\partial z}{\partial t} = (f_x u_1 + f_y u_2 + f_z u_3)_x u_1 + (f_x u_1 + f_y u_2 + f_z u_3)_y u_2 + (f_x u_1 + f_y u_2 + f_z u_3)_z u_3 = f_{xx} u_1^2 + 2f_{xy} u_1 u_2 + f_{yy} u_2^2 + 2f_{xz} u_1 u_3 + 2f_{yz} u_2 u_3 + f_{zz} u_3^2$ . For  $f = xyz$  this is  $2zu_1 u_2 + 2yu_1 u_3 + 2xu_2 u_3 = 6tu_1 u_2 u_3$ . Check:  $f = u_1 u_2 u_3 t^3$  and  $f_{tt} = 6u_1 u_2 u_3 t$ .
- 26  $\frac{\partial z}{\partial x} = 2(x+y)$  and  $\frac{\partial z}{\partial y} = \frac{1}{2\sqrt{x}} = \frac{1}{2(x+y)}$ . Yes: The product is 1 because  $y$  is constant.
- 28  $\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} = 3(x+y)^2(t+t^2)$  and  $\frac{d^2 f}{dt^2} = 3(x+y)^2(1+2t) + 6(x+y)(t+t^2)^2$ .
- 30  $f_{xx} = 90a^2(ax+by+c)^8$  and  $(at+bt+c)^{10}$  has  $f_{tt} = 90(a+b)^2(at+bt+c)^8$ . It is false that  $\frac{\partial f}{\partial x} \frac{\partial z}{\partial t} = \frac{\partial f}{\partial t}$  (we also need the term  $\frac{\partial f}{\partial y} \frac{\partial y}{\partial t}$ ).
- 32  $\frac{\partial r}{\partial x} = \frac{x}{r}$  and then  $\frac{\partial^2 r}{\partial y \partial x} = -\frac{x}{r^3} \frac{\partial r}{\partial y} = -\frac{x}{r^3} \frac{y}{r} = -\frac{xy}{r^4}$ .
- 34  $f_x = \frac{x}{\sqrt{x^2+y^2}}$ ;  $f_{xx} = \frac{\sqrt{x^2+y^2}-x^2/\sqrt{x^2+y^2}}{x^2+y^2} = \frac{y^2}{(x^2+y^2)^{3/2}}$ ;  $f_{xy} = \frac{-xy}{(x^2+y^2)^{3/2}}$ .
- 36 Yes, if  $y$  is simply held constant then the old rule continues to apply.

- 38  $\frac{\partial f}{\partial t}(x, y, z) = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial t}$ .
- 40 (a)  $\frac{\partial f}{\partial x} = 2x$  (b)  $f = x^2 + y^2 + (x^2 + y^2)^2$  so  $\frac{\partial f}{\partial x} = 2x + 4x(x^2 + y^2)$   
 (c)  $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial x} \frac{\partial x}{\partial x} = 2x + 2x(2x) = 2x + 4x(x^2 + y^2)$  (d)  $y$  is constant for  $(\frac{\partial f}{\partial x})_y$ .
- 42  $(\frac{\partial P}{\partial V})_T = -\frac{\partial F}{\partial V} / \frac{\partial F}{\partial P}$  and similarly  $(\frac{\partial V}{\partial T})_P = -\frac{\partial F}{\partial T} / \frac{\partial F}{\partial V}$  and  $(\frac{\partial T}{\partial P})_V = -\frac{\partial F}{\partial P} / \frac{\partial F}{\partial T}$ . Multiply these three equations: the right hand sides produce  $-1$ .
- 44  $\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}(1) + \frac{\partial f}{\partial y}(u)$  and  $\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}(1) + \frac{\partial f}{\partial y}(t)$ . For  $f = x^2 - 2y$  these become  $\frac{\partial f}{\partial t} = 2x(1) - 2(u) = 2(t+u) - 2u = 2t$  and similarly  $\frac{\partial f}{\partial u} = 2u$ . Check:  $f = (t+u)^2 - 2tu = t^2 + u^2$  has  $f_t = 2t$  and  $f_u = 2u$ .
- 46  $\sin x + \sin y = 0$  gives  $\cos x + \cos y \frac{dy}{dx} = 0$  and  $-\sin x - \sin y (\frac{dy}{dx})^2 + \cos y \frac{d^2y}{dx^2} = 0$ . Then  $\frac{dy}{dx} = -\frac{\cos x}{\cos y}$  and  $\frac{d^2y}{dx^2} = \frac{\sin x + \sin y \frac{\cos^2 x}{\cos^2 y}}{\cos y}$ .
- 48 The  $c$  derivative of  $f(cx, cy) = cf(x, y)$  is  $x \frac{\partial f}{\partial x}(cx, cy) + y \frac{\partial f}{\partial y}(cx, cy) = f(x, y)$ . At  $c = 1$  this becomes  $x \frac{\partial f}{\partial x}(x, y) + y \frac{\partial f}{\partial y}(x, y) = f(x, y)$ . Test on  $f = \sqrt{x^2 + y^2}$ :  $x \frac{x}{\sqrt{x^2 + y^2}} + y \frac{y}{\sqrt{x^2 + y^2}} = \sqrt{x^2 + y^2}$ .  
 Test on  $f = \sqrt{xy}$ :  $x(\frac{1}{2} \frac{y}{\sqrt{x}}) + y(\frac{1}{2} \frac{x}{\sqrt{y}}) = \sqrt{xy}$ . Other examples:  $f(x, y) = \sqrt{ax^2 + bxy + cy^2}$  or  $f = Ax + By$  or  $f = x^{1/4}y^{3/4}$ .

## 13.6 Maxima, Minima, and Saddle Points (page 512)

A minimum occurs at a stationary point (where  $f_x = f_y = 0$ ) or a rough point (no derivative) or a boundary point. Since  $f = x^2 - xy + 2y$  has  $f_x = 2x - y$  and  $f_y = 2 - x$ , the stationary point is  $x = 2, y = 4$ . This is not a minimum, because  $f$  decreases when  $y = 2x$  increases.

The minimum of  $d^2 = (x - x_1)^2 + (y - y_1)^2$  occurs at the rough point  $(x_1, y_1)$ . The graph of  $d$  is a cone and  $\text{grad } d$  is a unit vector that points out from  $(x_1, y_1)$ . The graph of  $f = |xy|$  touches bottom along the lines  $x = 0$  and  $y = 0$ . Those are "rough lines" because the derivative does not exist. The maximum of  $d$  and  $f$  must occur on the boundary of the allowed region because it doesn't occur inside.

When the boundary curve is  $x = x(t), y = y(t)$ , the derivative of  $f(x, y)$  along the boundary is  $f_x x_t + f_y y_t$  (chain rule). If  $f = x^2 + 2y^2$  and the boundary is  $x = \cos t, y = \sin t$ , then  $df/dt = 2 \sin t \cos t$ . It is zero at the points  $t = 0, \pi/2, \pi, 3\pi/2$ . The maximum is at  $(0, \pm 1)$  and the minimum is at  $(\pm 1, 0)$ . Inside the circle  $f$  has an absolute minimum at  $(0, 0)$ .

To separate maximum from minimum from saddle point, compute the second derivatives at a stationary point. The tests for a minimum are  $f_{xx} > 0$  and  $f_{xx}f_{yy} > f_{xy}^2$ . The tests for a maximum are  $f_{xx} < 0$  and  $f_{xx}f_{yy} > f_{xy}^2$ . In case  $ac < b^2$  or  $f_{xx}f_{yy} < f_{xy}^2$ , we have a saddle point. At all points these tests decide between concave up and concave down and "indefinite". For  $f = 8x^2 - 6xy + y^2$ , the origin is a saddle point. The signs of  $f$  at  $(1, 0)$  and  $(1, 3)$  are  $+$  and  $-$ .

The Taylor series for  $f(x, y)$  begins with the terms  $f(0, 0) + xf_x + yf_y + \frac{1}{2}x^2f_{xx} + xyf_{xy} + \frac{1}{2}y^2f_{yy}$ . The coefficient of  $x^n y^m$  is  $\partial^{n+m} f / \partial x^n \partial y^m (0, 0)$  divided by  $n!m!$  To find a stationary point numerically, use New-

ton's method or steepest descent.

- 1 (0,0) is a minimum      3 (3,0) is a saddle point      5 No stationary points      7 (0,0) is a maximum  
 9 (0,0,2) is a minimum      11 All points on the line  $x = y$  are minima      13 (0,0) is a saddle point  
 15 (0,0) is a saddle point; (2,0) is a minimum; (0,-2) is a maximum; (2,-2) is a saddle point  
 17 Maximum of area  $(12 - 3y)y$  is 12  
 19  $2(x + y) + 2(x + 2y - 5) + 2(x + 3y - 4) = 0$  gives  $x = 2$ ;  
 $2(x + y) + 4(x + 2y - 5) + 6(x + 3y - 4) = 0$  gives  $y = -1$  min because  $E_{xx}E_{yy} = (6)(28) > E_{xy}^2 = 12^2$   
 21 Minimum at  $(0, \frac{1}{2})$ ; (0, 1); (0, 1)  
 23  $\frac{d}{dt} = 0$  when  $\tan t = \sqrt{3}$ ;  $f_{\max} = 2$  at  $(\frac{1}{2}, \frac{\sqrt{3}}{2})$ ,  $f_{\min} = -2$  at  $(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$   
 25  $(ax + by)_{\max} = \sqrt{a^2 + b^2}$ ;  $(x^2 + y^2)_{\min} = \frac{1}{a^2 + b^2}$       27  $0 < c < \frac{1}{4}$   
 29 The vectors head-to-tail form a 60-60-60 triangle. The outer angle is  $120^\circ$       31  $2 + \sqrt{3}$ ;  $1 + \sqrt{3}$ ;  $1 + \frac{\sqrt{3}}{10}$   
 35 Steiner point where the arcs meet      39 Best point for  $p = \infty$  is equidistant from corners  
 41  $\text{grad } f = (\sqrt{2} \frac{x-x_1}{d_1} + \frac{x-x_2}{d_2} + \frac{x-x_3}{d_3}, \sqrt{2} \frac{y-y_1}{d_1} + \frac{y-y_2}{d_2} + \frac{y-y_3}{d_3})$ ; angles are 90-135-135  
 43 Third derivatives all 6;  $f = \frac{6}{3!}x^3 + \frac{6}{2!}x^2y + \frac{6}{2!}xy^2 + \frac{6}{3!}y^3$   
 45  $(\frac{\partial}{\partial x})^n (\frac{\partial}{\partial y})^m \ln(1 - xy)|_{0,0} = n!(n-1)!$  for  $m = n > 0$ , other derivatives zero;  $f = -xy - \frac{x^2y^2}{2} - \frac{x^3y^3}{3} - \dots$   
 47 All derivatives are  $e^2$  at (1,1);  $f \approx e^2[1 + (x-1) + (y-1) + \frac{1}{2}(x-1)^2 + (x-1)(y-1) + \frac{1}{2}(y-1)^2]$   
 49  $x = 1, y = -1$ :  $f_x = 2, f_y = -2, f_{xx} = 2, f_{xy} = 0, f_{yy} = 2$ ; series must recover  $x^2 + y^2$   
 51 Line  $x - 2y = \text{constant}$ ;  $x + y = \text{constant}$   
 53  $\frac{x^2}{2}f_{xx} + xyf_{xy} + \frac{y^2}{2}f_{yy}|_{0,0}$ ;  $f_{xx} > 0$  and  $f_{xx}f_{yy} > f_{xy}^2$  at (0,0);  $f_x = f_y = 0$       55  $\Delta x = -1, \Delta y = -1$   
 57  $f = x^2(12 - 4x)$  has  $f_{\max} = 16$  at (2,4); line has slope  $-4, y = \frac{16}{x^2}$  has slope  $-\frac{32}{x^3} = -4$   
 59 If the fence were not perpendicular, a point to the left or right would be closer
- 2  $f_x = y - 1, f_y = x - 1; b^2 - ac = 1$ ; (1, 1) is a saddle point  
 4  $f_x = 2x, f_y = -2y + 4; b^2 - ac = 1$ ; (0, 2) is a saddle point  
 6  $f_x = e^y - e^x, f_y = xe^y$ ; (0, 0) is the stationary point;  $f_{xx} = -e^x = -1, f_{xy} = e^y = 1, f_{yy} = xe^y = 0$  so  
 $b^2 - ac = 1$ : saddle point  
 8  $f_x = 2(x + y) + 2(x + 2y - 6), f_y = 2(x + y) + 4(x + 2y - 6)$ ; (-6, 6) is the stationary point:  $f_{xx} = 4,$   
 $f_{xy} = 6, f_{yy} = 10$  give  $b^2 - ac = -4$ : minimum  
 10  $f_x = x + 2y - 6 + x + y$  and  $f_y = x + 2y - 6 + 2(x + y)$ ; (-6, 6) is the stationary point;  $f_{xx} = 2, f_{xy} = 3, f_{yy} = 4$   
 give  $b^2 - ac = 9 - 8 = 1$ : saddle point  
 12  $f_x = \frac{2x}{1+y^2}$  and  $f_y = \frac{-2y(1+x^2)}{(1+y^2)^2}$ ; (0, 0) is the stationary point;  $f_{xx} = \frac{2}{1+y^2} = 2, f_{xy} = \frac{-4xy}{(1+y^2)^2} = 0,$   
 $f_{yy} = \frac{-2(1+x^2)}{(1+y^2)^2} + \frac{8y^2(1+x^2)}{(1+y^2)^3} = -2; b^2 - ac = 4$ : saddle point  
 14  $f_x = \cos x$  and  $f_y = \sin y$ ; stationary points have  $x = \frac{\pi}{2} + n\pi$  and  $y = m\pi$ ; maximum when  $f = 2,$   
 saddle point when  $f = 0,$  minimum when  $f = -2$   
 16  $f_x = 8y - 4x^3$  and  $f_y = 8x - 4y^3$ ; stationary points are (0,0) = saddle point,  $(\sqrt{2}, \sqrt{2}) = \text{maximum},$   
 $(-\sqrt{2}, -\sqrt{2}) = \text{minimum}.$   
 18 Volume =  $xyz = xy(1 - 3x - 2y) = xy - 3x^2y - 2xy^2$ ;  $V_x = y - 6x - 2y^2$  and  $V_y = x - 4xy$ ; at  $(0, \frac{1}{2}, 0)$  and  
 $(\frac{1}{3}, 0, 0)$  and (0,0,1) the volume is  $V = 0$  (minimum); at  $(\frac{1}{48}, \frac{12}{48}, \frac{21}{48})$  the volume is  $V = \frac{7}{3072}$  (maximum)  
 20 Minimize  $f(x, y) = (x - y - 1)^2 + (2x + y + 1)^2 + (x + 2y - 1)^2$ :  $\frac{\partial f}{\partial x} = 2(x - y - 1) + 4(2x + y + 1) + 2(x + 2y - 1) = 0$   
 and  $\frac{\partial f}{\partial y} = -2(x - y - 1) + 2(2x + y + 1) + 4(x + 2y - 1) = 0$ . Solution:  $x = y = 0!$   
 22  $\frac{\partial f}{\partial x} = 2x + 2$  and  $\frac{\partial f}{\partial y} = 2y + 4$ . (a) Stationary point (-1, -2) yields  $f_{\min} = -5$ . (b) On the boundary  $y = 0$   
 the minimum of  $x^2 + 2x$  is  $-1$  at (-1, 0) (c) On the boundary  $x \geq 0, y \geq 0$  the minimum is 0 at (0,0).

- 24  $f(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = \frac{5}{2} - \sqrt{2}$ ;  $f(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) = \frac{5}{2} + \sqrt{2} = f_{\max}$ ;  $f(1, 0) = f(0, 1) = 1 = f_{\min}$
- 26  $f_x = x^3 - y = 0$  and  $f_y = y^3 - x = 0$  combine into  $y = x^3 = y^9$ . Then  $y = y^9$  gives  $y = 1$  or  $-1$  or  $0$ .  
At those points  $f_{\min} = -\frac{1}{2}$  and  $f = 0$  (relative maximum). These equations  $x^3 = y, y^3 = x$  are solved by Newton's method in Section 13.3 (the basins are on the front cover).
- 28  $d_1 = x, d_2 = d_3 = \sqrt{(1-x)^2 + 1}, \frac{d}{dx}(x + 2\sqrt{(1-x)^2 + 1}) = 1 + \frac{2(x-1)}{\sqrt{(1-x)^2 + 1}} = 0$  when  $(1-x)^2 + 1 = 4(x-1)^2$  or  $1-x = \frac{1}{\sqrt{3}}$  or  $x = 1 - \frac{1}{\sqrt{3}}$ . From that point to  $(1, 1)$  the line goes up 1 and across  $\frac{1}{\sqrt{3}}$ , a  $60^\circ$  angle with the horizontal that confirms three  $120^\circ$  angles.
- 30  $d_1 = \sqrt{(x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2}$  and then  $\text{grad } d_1 = (\frac{x-x_1}{d_1}, \frac{y-y_1}{d_1}, \frac{z-z_1}{d_1})$  has length  $|\text{grad } d_1| = 1$ . This gradient of  $d_1$  points directly away from  $(x_1, y_1, z_1)$ . The gradient of  $f = d_1 + d_2 + d_3 + d_4$  is a sum of 4 unit vectors. The sum is zero for  $\frac{1}{\sqrt{3}}(1, -1, -1), \frac{1}{\sqrt{3}}(-1, -1, 1), \frac{1}{\sqrt{3}}(-1, 1, -1), \frac{1}{\sqrt{3}}(1, 1, 1)$ . The equal angles have  $\cos \theta = -\frac{1}{3}$  by Problem 45 of Section 11.1.
- 32 From an outside point the lines to the three vertices give two angles that add to less than  $180^\circ$ . So they cannot both be  $120^\circ$  as a Steiner point requires.
- 34 From the point  $C = (0, -\sqrt{3})$  the lines to  $(-1, 0)$  and  $(1, 0)$  make a  $60^\circ$  angle.  $C$  is the center of the circle  $x^2 + (y - \sqrt{3})^2 = 4$  through those two points. From any point on that circle, the lines to  $(-1, 0)$  and  $(1, 0)$  make an angle of  $2 \times 60^\circ = 120^\circ$ . Theorem from geometry: angle from circle =  $2 \times$  angle from center.
- 40 The vertices are  $(0, 0), (1, 0),$  and  $(0, 1)$ . The point  $(\frac{1}{2}, \frac{1}{2})$  is an equal distance ( $\frac{\sqrt{2}}{2}$ ) from all three vertices. Note: In any triangle the intersection of the altitudes (perpendicular to edges at their midpoints) is equally distant from the vertices. If it is in the triangle, it is the best point with  $p = \infty$ : it minimizes the largest distance.
- 42 For two points,  $d_1 + d_2$  is a minimum at all points on the line between them. (Note equal  $180^\circ$  angles from the vertices!) For three points, the corner with largest angle is the best corner.
- 44  $\frac{\partial^{n+m}}{\partial x^n \partial y^m}(xe^y) = xe^y$  for  $n = 0, e^y$  for  $n = 1$ , zero for  $n > 1$ . Taylor series  $xe^y = x + xy + \frac{1}{2!}xy^2 + \frac{1}{3!}xy^3 + \dots$
- 46 All derivatives equal 1 at  $(0, 0)$ . Quadratic =  $1 + x + y + \frac{1}{2}x^2 + xy + \frac{1}{2}y^2$ .
- 48  $\frac{\partial}{\partial x}(\sin x \cos y) = 1$  at  $(0, 0)$  but  $f = f_y = f_{xx} = f_{xy} = f_{yy} = 0$ . Quadratic =  $x$ .  
Check:  $\sin x \cos y \approx (x - \frac{x^3}{6} + \dots)(1 - \frac{y^2}{2} + \dots) = x$  to quadratic accuracy.
- 50  $f(x+h, y+k) \approx f(x, y) + h\frac{\partial f}{\partial x}(x, y) + k\frac{\partial f}{\partial y}(x, y) + \frac{h^2}{2}\frac{\partial^2 f}{\partial x^2}(x, y) + hk\frac{\partial^2 f}{\partial x \partial y}(x, y) + \frac{k^2}{2}\frac{\partial^2 f}{\partial y^2}(x, y)$
- 52  $(x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z})(0, 0, 0)$ ; then  $(\frac{x^2}{2}f_{xx} + \frac{y^2}{2}f_{yy} + \frac{z^2}{2}f_{zz} + xyf_{xy} + xzf_{xz} + yzf_{yz})(0, 0, 0)$
- 54  $f = (1-2s)^2 + 2(1-4s)^2$  has  $\frac{df}{ds} = 0$  at  $s = \frac{3}{10}$ . Step ends at  $x = 1-2s = \frac{4}{10}, y = 1-4s = -\frac{2}{10}$ .
- 56 A maximum has  $f_{xx} < 0$  and  $f_{yy} < 0$ , so they cannot add to zero. A minimum has  $f_{xx} > 0$  and  $f_{yy} > 0$ .  
The functions  $xy$  and  $x^2 - y^2$  solve  $f_{xx} + f_{yy} = 0$  and have saddle points.
- 58 A house costs  $p$ , a yacht costs  $q$ :  $\frac{d}{dx}f(x, \frac{k-px}{q}) = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}(-\frac{p}{q}) = 0$  gives  $-\frac{\partial f}{\partial x} / \frac{\partial f}{\partial y} = -\frac{p}{q}$ .

13.7 Constraints and Lagrange Multipliers (page 519)

A restriction  $g(x, y) = k$  is called a constraint. The minimizing equations for  $f(x, y)$  subject to  $g = k$  are  $\partial f / \partial x = \lambda \partial g / \partial x, \partial f / \partial y = \lambda \partial g / \partial y$ , and  $g = k$ . The number  $\lambda$  is the Lagrange multiplier. Geometrically,  $\text{grad } f$  is parallel to  $\text{grad } g$  at the minimum. That is because the level curve  $f = f_{\min}$  is tangent to the constraint curve  $g = k$ . The number  $\lambda$  turns out to be the derivative of  $f_{\min}$  with respect to  $k$ . The Lagrange function is

$L = f(x, y) - \lambda(g(x, y) - k)$  and the three equations for  $x, y, \lambda$  are  $\partial L/\partial x = 0$  and  $\partial L/\partial y = 0$  and  $\partial L/\partial \lambda = 0$ .

To minimize  $f = x^2 - y$  subject to  $g = x - y = 0$ , the three equations for  $x, y, \lambda$  are  $2x = \lambda, -1 = -\lambda, x - y = 0$ . The solution is  $x = \frac{1}{2}, y = \frac{1}{2}, \lambda = 1$ . In this example the curve  $f(x, y) = f_{\min} = -\frac{1}{4}$  is a parabola which is tangent to the line  $g = 0$  at  $(x_{\min}, y_{\min})$ .

With two constraints  $g(x, y, z) = k_1$  and  $h(x, y, z) = k_2$  there are two multipliers  $\lambda_1$  and  $\lambda_2$ . The five unknowns are  $x, y, z, \lambda_1$ , and  $\lambda_2$ . The five equations are  $f_x = \lambda_1 g_x + \lambda_2 h_x, f_y = \lambda_1 g_y + \lambda_2 h_y, f_z = \lambda_1 g_z + \lambda_2 h_z, g = 0$ , and  $h = 0$ . The level surface  $f = f_{\min}$  is tangent to the curve where  $g = k_1$  and  $h = k_2$ . Then  $\text{grad } f$  is perpendicular to this curve, and so are  $\text{grad } g$  and  $\text{grad } h$ . With nine variables and six constraints, there will be six multipliers and eventually 15 equations. If a constraint is an inequality  $g \leq k$ , then its multiplier must satisfy  $\lambda \leq 0$  at a minimum.

- 1  $f = x^2 + (k - 2x)^2; \frac{df}{dx} = 2x - 4(k - 2x) = 0; (\frac{2k}{5}, \frac{k}{5}), \frac{k^2}{5}$     3  $\lambda = -4, x_{\min} = 2, y_{\min} = 2$
- 5  $\lambda = \frac{1}{3(4)^{1/3}} : (x, y) = (\pm 2^{1/6}, 0)$  or  $(0, \pm 2^{1/6}), f_{\min} = 2^{1/3}; \lambda = \frac{1}{3} : (x, y) = (\pm 1, \pm 1), f_{\max} = 2$
- 7  $\lambda = \frac{1}{2}, (x, y) = (2, -3);$  tangent line is  $2x - 3y = 13$
- 9  $(1 - c)^2 + (-a - c)^2 + (2 - a - b - c)^2 + (2 - b - c)^2$  is minimized at  $a = -\frac{1}{2}, b = \frac{3}{2}, c = \frac{3}{4}$
- 11  $(1, -1)$  and  $(-1, 1); \lambda = -\frac{1}{2}$
- 13  $f$  is not a minimum when  $C$  crosses to lower level curve; stationary point when  $C$  is tangent to level curve
- 15 Substituting  $\frac{\partial L}{\partial x} = \frac{\partial L}{\partial y} = \frac{\partial L}{\partial \lambda} = 0$  and  $L = f_{\min}$  leaves  $\frac{df_{\min}}{dk} = \lambda$
- 17  $x^2$  is never negative;  $(0, 0); 1 = \lambda(-3y^2)$  but  $y = 0; g = 0$  has a cusp at  $(0, 0)$
- 19  $2x = \lambda_1 + \lambda_2, 4y = \lambda_1, 2z = \lambda_1 - \lambda_2, x + y + z = 0, x - z = 1$  gives  $\lambda_1 = 0, \lambda_2 = 1, f_{\min} = \frac{1}{2}$  at  $(\frac{1}{2}, 0, -\frac{1}{2})$
- 21  $(1, 0, 0); (0, 1, 0); (\lambda_1, \lambda_2, 0); x = y = 0$     23  $\frac{\partial L}{\partial x}$  and  $\frac{\partial L}{\partial y}; \lambda = 0$
- 25  $(1, 0, 0), (0, 1, 0), (0, 0, 1);$  at these points  $f = 4$  and  $-2$  (min) and  $5$  (max)
- 27 By increasing  $k$ , more points are available so  $f_{\max}$  goes up. Then  $\lambda = \frac{df_{\min}}{dk} \geq 0$
- 29  $(0, 0); \lambda = 0; f_{\min}$  stays at 0
- 31  $5 = \lambda_1 + \lambda_2, 6 = \lambda_1 + \lambda_3, \lambda_2 \geq 0, \lambda_3 \leq 0;$  subtraction  $5 - 6 = \lambda_2 - \lambda_3$  or  $-1 \geq 0$  (impossible);  
 $x = 2004, y = -2000$  gives  $5x + 6y = -1980$
- 33  $2x = 4\lambda_1 + \lambda_2, 2y = 4\lambda_1 + \lambda_3, \lambda_2 \geq 0, \lambda_3 \geq 0, 4x + 4y = 40;$  max area 100 at  $(10, 0)(0, 10);$  min 25 at  $(5, 5)$

- 2  $x^2 + y^2 = 1$  and  $2xy = \lambda(2x)$  and  $x^2 = \lambda(2y)$  yield  $2\lambda^2 + \lambda^2 = 1$ . Then  $\lambda = \frac{1}{\sqrt{3}}$  gives  $x_{\max} = \pm \frac{\sqrt{6}}{3},$   
 $y_{\max} = \frac{\sqrt{3}}{3}, f_{\max} = \frac{2\sqrt{3}}{9}$ . Also  $\lambda = -\frac{1}{\sqrt{3}}$  gives  $f_{\min} = -\frac{2\sqrt{3}}{9}$ .
- 4  $x^2 + 9y^2 = 1$  and  $3 = \lambda(2x)$  and  $1 = \lambda(18y)$  give  $\frac{1}{\lambda^2}(\frac{9}{4} + 9 \cdot \frac{1}{18^2}) = 1$  or  $\lambda^2 = \frac{41}{18}$ . Then  $x_{\max} = \frac{9}{\sqrt{82}},$   
 $y_{\max} = \frac{1}{3\sqrt{82}}, f_{\max} = \frac{\sqrt{82}}{3}$ . Change signs for  $(x, y, f)_{\min}$ . Second approach: Fix  $3x + y$  and maximize  $x^2 + 9y^2$ .
- 6  $1 = \frac{\lambda}{3}(\frac{x}{y})^{2/3}$  and  $1 = \frac{2\lambda}{3}(\frac{x}{y})^{1/3}$  yield  $1 = \frac{\lambda}{3}(\frac{2\lambda}{3})^2$  or  $\frac{\lambda}{3} = (4)^{-1/3}$ . Then  $\frac{x}{y} = (\frac{3}{2\lambda})^3 = \frac{4}{8}$  so  $y = 2x$ . The constraint gives  $x^{1/3}(2x)^{2/3} = k$  or  $x = k(4)^{-1/3}$  and then  $y = 2k(4)^{-1/3}$ . Then  $f = x + y = 3k(4)^{-1/3}$ .
- 8  $a = \lambda(2x), b = \lambda(2y), c = \lambda(2z)$  give  $\frac{1}{4\lambda^2}(a^2 + b^2 + c^2) = k^2$  and  $\lambda = \sqrt{a^2 + b^2 + c^2}/2k$ . Then  $x_{\max} = ak/\sqrt{a^2 + b^2 + c^2}, y_{\max} = bk/\sqrt{a^2 + b^2 + c^2},$  and  $z_{\max} = ck/\sqrt{a^2 + b^2 + c^2}$ .  
Thus  $(a, b, c) \cdot (x, y, z) \leq f_{\max} = \sqrt{a^2 + b^2 + c^2} k$  is the Schwarz inequality.
- 10 The base is  $b$ , the rectangle height is  $a$ , the triangle height is  $h$ , the area is  $ab + \frac{1}{2}bh = 1$ .  
Minimize  $f = b + 2a + 2\sqrt{b^2/4 + h^2}$ . The  $\frac{\partial}{\partial a}, \frac{\partial}{\partial h}, \frac{\partial}{\partial b}$  equations are  $2 = \lambda b, \frac{2h}{\sqrt{b^2/4 + h^2}} = \lambda(\frac{1}{2}b),$

$$1 + \frac{b/2}{\sqrt{b^2/4+h^2}} = \lambda(a + \frac{1}{2}h). \text{ Put } \lambda b = 2 \text{ in the second equation and square: } (2h)^2 = \frac{b^2}{4} + h^2 \text{ or } h^2 = b^2/12.$$

The third equation becomes  $1 + \frac{1/2}{\sqrt{1/3}} = \lambda(a + \frac{b}{2\sqrt{12}}) = \lambda a + \frac{1}{\sqrt{12}}$ . Then  $\lambda a = 1 + \frac{\sqrt{3}}{3}$ .

The area is  $\frac{1}{\lambda^2}[(1 + \frac{\sqrt{3}}{3})(2) + \frac{1}{2}(2)\frac{1}{\sqrt{3}}] = 1$  so  $\lambda^2 = 2 + \sqrt{3}$ . This gives  $b, h$ , and  $a$ . (Not an easy problem!)

12  $y = \lambda(2x)$  and  $x = \lambda(2y)$  require  $2\lambda = 1$  or  $2\lambda = -1$ . Then  $y = \pm x$ . The equation  $x^2 + y^2 = 2$  gives  $x^2 = 1$ .

The maximum is at  $x = 1, y = 1$  or  $x = -1, y = -1$ .

14 (a)  $y - 1 = \lambda(2x), x - 1 = \lambda(2y), x^2 + y^2 = 1$  (b)  $x = y = \frac{1}{1-2\lambda}$  (c) At  $\lambda = -\frac{1}{2}$  both equations become  $x + y = 1$  and we find the minimum points  $(1,0)$  and  $(0,1)$  where  $x \neq y$ .

16 Those equations come from the chain rule:  $\frac{dg}{dx} = 0$  along the curve because  $g = \text{constant}$ . Together the two equations give  $\frac{\partial f}{\partial x} - \lambda \frac{\partial g}{\partial x} = 0$  (the Lagrange equation).

18  $f = 2x + y = 1001$  at the point  $x = 1000, y = -999$ . The Lagrange equations are  $2 = \lambda$  and  $1 = \lambda$  (no solution). Linear functions with linear constraints generally have no maximum.

20 (a)  $yz = \lambda, xz = \lambda, xy = \lambda$ , and  $x + y + z = k$  give  $x = y = z = \frac{k}{3}$  and  $\lambda = \frac{k^2}{9}$  (b)  $V_{\max} = (\frac{k}{3})^3$  so  $\partial V_{\max}/\partial k = k^2/9$  (which is  $\lambda$ ) (c) Approximate  $\Delta V = \lambda$  times  $\Delta k = \frac{108^2}{9}(111 - 108) = 3888 \text{ in}^3$ . Exact  $\Delta V = (\frac{111}{3})^3 - (\frac{108}{3})^3 = 3677 \text{ in}^3$ .

22  $2x = \lambda a, 2y = \lambda b, 2z = \lambda c$  and  $ax + by + cz = d$  give  $\lambda(a^2 + b^2 + c^2) = 2d$  and  $x = \frac{a\lambda}{2} = \frac{ad}{a^2+b^2+c^2}$ . Similarly  $y = \frac{bd}{a^2+b^2+c^2}$  and  $z = \frac{cd}{a^2+b^2+c^2}$ . Then  $f_{\min} = \frac{d^2}{a^2+b^2+c^2}$  is the square of the minimum distance.

24  $3 = \lambda_1 + \lambda_2$  and  $5 = 2\lambda_1 + \lambda_3$  with  $\lambda_2 \geq 0$  and  $\lambda_3 \geq 0$ .  $\lambda_2 = 0$  is impossible because then  $\lambda_1 = 3, \lambda_3 = -1$ . So  $\lambda_3 = 0, \lambda_1 = \frac{5}{2}, \lambda_2 = \frac{1}{2}$ . The minimum is  $f = 10$  at  $x = 0, y = 2$ . (Note  $\lambda = 0$  goes with  $y \neq 0$ .)

26 Reasoning: By increasing  $k$ , more points satisfy the constraints. More points are available to minimize  $f$ . Therefore  $f_{\min}$  goes down.

28  $\lambda = 0$  when  $h > k$  (not  $h = k$ ) at the minimum. Reasoning: An increase in  $k$  leaves the same minimum. Therefore  $f_{\min}$  is unchanged. Therefore  $\lambda = df_{\min}/dk$  is zero.

30  $f = x^2 + y^2, x + y \geq 4$  has minimum at  $x = y = 2$ . From  $2x = \lambda(1)$  and  $2y = \lambda(1)$ , the multiplier is  $\lambda = 4$  and  $f_{\min} = 8$ . Change to  $x + y \geq 4 + dk$ . Then  $f_{\min} = 8 + \lambda dk = 8 + 4dk$ . Check:  $x_{\min} = y_{\min} = \frac{1}{2}(4 + dk)$  give  $f_{\min} = (\frac{1}{2})^2(4 + dk)^2(2) = 8 + 4dk + \frac{1}{2}(dk)^2$ .

32 Lagrange equations:  $2 = \lambda_1 + \lambda_2, 3 = \lambda_1 + \lambda_3, 4 = \lambda_1 + \lambda_4$ . Then  $\lambda_4 > \lambda_3 > \lambda_2 \geq 0$ . We need  $\lambda_4 > 0$  and  $\lambda_3 > 0$  (correction: not = 0). Zero multiplier goes with nonzero  $x = 1$ . Nonzero multipliers go with  $y = z = 0$ . Then  $f_{\min} = 2$ . (We can see directly that  $f_{\min} = 2$ .)

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