CHAPTER 14  MULTIPLE INTEGRALS

14.1 Double Integrals  (page 526)

The double integral \( \iint_R f(x,y) \,dx \,dy \) gives the volume between \( R \) and the surface \( z = f(x,y) \). The base is first cut into small squares of area \( \Delta A \). The volume above the \( i \)th piece is approximately \( f(x_i, y_i) \Delta A \). The limit of the sum \( \sum f(x_i, y_i) \Delta A \) is the volume integral. Three properties of double integrals are \( \int \int (f + g) \,dA = \int \int f \,dA + \int \int g \,dA \) and \( \int \int cf \,dA = c \int \int f \,dA \) and \( \int \int_R f \,dA = \int \int_S f \,dA + \int \int_T f \,dA \) if \( R \) splits into \( S \) and \( T \).

If \( R \) is the rectangle \( 0 \leq x \leq 4, 4 \leq y \leq 6 \), the integral \( \int \int x \,dx \,dy \) can be computed two ways. One is \( \int \int xy \,dx \,dy \), when the inner integral is \( xy \). The outer integral gives \( x^2 y^4 \). When the \( x \) integral comes first it equals \( \int x \,dx = \frac{1}{2}x^2 \). Then the \( y \) integral equals \( 8y^4 \). This is the volume between the base rectangle and the plane \( z = x \).

The area \( R \) is \( \int \int 1 \,dy \,dx \). When \( R \) is the triangle between \( x = 0, y = 2x \), and \( y = 1 \), the inner limits on \( y \) are \( 2x \) and \( 1 \). This is the length of a thin vertical strip. The \( (\text{outer}) \) limits on \( x \) are \( 0 \) and \( \frac{1}{2} \). The area is \( \frac{1}{4} \). In the opposite order, the \( (\text{inner}) \) limits on \( x \) are \( 0 \) and \( \frac{1}{2} \). Now the strip is horizontal and the outer integral is \( \int \int \frac{1}{2}y \,dy \,dx = \frac{1}{4} \). When the density is \( \rho(x,y) \), the total mass in the region \( R \) is \( \int \int \rho \,dx \,dy \). The moments are \( M_y = \int \int xy \,dx \,dy \) and \( M_x = \int \int \rho \,dy \,dx \). The centroid has \( \bar{x} = \frac{M_y}{M} \).

1. \( \frac{8}{3}, \frac{29}{3}, 3, \ln \frac{3}{2}, 52, \frac{7}{3}, \frac{9}{4}, 11 \)
2. \( \int_0^1 \int_x^1 \frac{x}{y} \,dx \,dy + \int_0^1 \int_2^y \frac{1}{x} \,dx \,dy \)
3. \( \frac{1}{5} \int_0^1 \int_x^1 \frac{1}{y} \,dx \,dy = \frac{1}{5} \ln \frac{3}{2} \)
4. \( \frac{1}{5} \int_0^1 \int_y^1 \frac{1}{x} \,dx \,dy = \ln \frac{3}{2} \)
5. \( \frac{1}{5} \int_0^1 \int_y^1 \frac{1}{x} \,dx \,dy = \ln \frac{3}{2} \)
6. \( \frac{1}{5} \int_0^1 \int_y^1 \frac{1}{x} \,dy \,dx = \ln \frac{3}{2} \)
7. \( \frac{1}{5} \int_0^1 \int_y^1 \frac{1}{x} \,dx \,dy = \ln \frac{3}{2} \)
8. \( \frac{1}{5} \int_0^1 \int_y^1 \frac{1}{x} \,dy \,dx = \ln \frac{3}{2} \)

The region is above \( y = x^3 \) and below \( y = x \) (from 0 to 1). Area = \( \int_0^1 \int_0^{x^3} \,dy \,dx = \left[ \frac{x^4}{4} - \frac{x^4}{16} \right]_0^1 = \frac{1}{4} \).

The region is below the parabola \( y = 1 - x^2 \) and above its mirror image \( y = x^2 - 1 \).
Area = \( \int_0^1 \int_0^{x^2} \,dy \,dx = \left[ \frac{x^4}{4} - \frac{x^4}{16} \right]_0^1 = \frac{3}{8} \).
10 The area is all below the axis $y = 0$, where horizontal strips cross from $x = y$ to $z = |y|$ (which is $-y$). Note
that the $y$ integral stops at $y = 0$. Area is $\int_{-1}^{1} \int_{y}^{1-y} dx \ dy = \int_{-1}^{0} -2y \ dy = \left[-y^2\right]_{-1}^{0} = 1.$
12 The strips in Problem 6 from $y = x^3$ up to $x$ are changed to strips from $x = y$ across to $x = y^{1/3}$. The outer 
integral on $y$ is by chance also from $0$ to $1$. Area is $\int_{0}^{1} (y^{1/3} - y) dy = \left[\frac{3}{4} y^{4/3} - \frac{1}{2} y^2\right]_{0}^{1} = \frac{1}{4}.$
14 Between the upper parabola $y = 1 - x^2$ in Problem 8 and the $x$ axis, the strips cross from the 
left side $x = -\sqrt{1-y}$ to the right side $x = +\sqrt{1-y}$. This half of the area is $\int_{-1}^{1} \int_{y}^{1+y} dx \ dy = 
\int_{0}^{1} 2\sqrt{1-y} \ dy = -\frac{3}{2} \left[(1-y)^{3/2}\right]_{0}^{1} = \frac{3}{4}$. The other half has strips from left side to right side of $y = x^2 - 1$ 
or $x = \pm\sqrt{1+y}$. This area is $\int_{-1}^{1} \int_{y}^{1+y} dx \ dy$ (also $\frac{3}{4}$).
16 The triangle in Problem 10 had sides $x = y$, $x = -y$, and $y = -1$. Now the strips are vertical. They go 
from $y = -1$ up to $y = x$ on the left side: area is $\int_{0}^{1} \int_{y}^{1-x} dx \ dy = \int_{0}^{-1} \left[(x+1)^2\right]_{0}^{1} = \frac{1}{2}$. The strips go from $-1$ up to $y = -x$ on the right side: area is $\int_{0}^{1} \int_{y}^{1-x} dy \ dx = \int_{0}^{1} (x-1)^2 dx = \frac{1}{2}$.
Check: $\frac{1}{2} + \frac{1}{2} = 1.$
18 The triangle has corners at $(0,0)$ and $(-1,0)$ and $(-1,-1)$. Its area is $\int_{0}^{1} \int_{y}^{1-y} dx \ dy = \int_{0}^{1} \int_{y}^{1-y} dx \ dy = \frac{1}{3}.$
20 The triangle has corners at $(0,0)$ and $(2,4)$ and $(4,4)$. Horizontal strips go from $x = \frac{y}{2}$ to $x = y$:
area is $\int_{0}^{2} \int_{y/2}^{y} dx \ dy = 4$. Vertical strips are of two kinds: from $y = x$ up to $y = 2x$ or to $y = 4$.
Area is $\int_{0}^{2} \int_{y}^{2y} dx \ dy + \int_{2}^{4} \int_{y}^{2y} dx \ dy = 2 + 2 = 4$.
22 (Hard Problem) The boundary lines are $y = \frac{1}{2} x$ from $(-2,-1)$ to $(0,0)$, and $y = -2x$ from $(0,0)$ to 
$(1,-2)$, and $y = -x - \frac{5}{2}$ or $x = -3y - 5$ from $(-2,-1)$ to $(1,-2)$. (This is the hardest one: note first the 
slope $-\frac{1}{2}$.) Vertical strips go from the third line up to the first or second: area is $\int_{-2}^{0} \int_{-2x}^{3/2} dy \ dx + 
\int_{0}^{1} \int_{-2x}^{3/2} dy \ dx = \frac{5}{3} + \frac{5}{6} = \frac{10}{3}$. Horizontal strips cross from the first or third lines to the second:
area is $\int_{-1}^{1} \int_{-2x}^{y} dy \ dx + \int_{0}^{1} \int_{-2x}^{y} dy \ dx = \frac{1}{4} + \frac{1}{4} = \frac{1}{4}$.
24 The top of the triangle is $(a,b)$. From $x = 0$ to $a$ the vertical strips lead to $\int_{a}^{0} \int_{dx/c}^{a} dy \ dx dx = 
\left[\frac{b - 1}{2a} - \frac{dx}{2ac}\right]_{0}^{a} = \frac{b}{2} - \frac{d}{2c}$. From $x = a$ to $c$ the strips go up to the third side:
$\int_{a}^{c} \int_{b - \frac{d}{c}(c-a) + (c-a)(x-a)}^{dx/c} dy \ dx = \left[\frac{b}{2} - \frac{d}{2c}\right]_{a}^{c} = b(c-a) + \frac{(c-a)(x-a)}{2} - \frac{dx}{c} + \frac{d}{c}$. 
The sum is $\frac{b}{2} + \frac{b(c-a)}{2} + \frac{d(c-a)}{2} - \frac{d}{c} = \frac{bc-ad}{2c}$. This is half of a parallelogram.
26 $\int_{0}^{a} \int_{0}^{y} dy \ dx = \int_{0}^{a} f(s, y) - f(0, y) dy$. 
28 Over the square $\int_{0}^{a} \int_{0}^{b} (x^{2} + y^{2}) dy \ dx = \int_{0}^{a} \int_{0}^{b} (x^{2} + y^{2}) (x - z) dx = |\frac{x^{2} + y^{2}}{2} - \frac{x^{2} + y^{2}}{2}| = \frac{5}{2} - \frac{5}{2} - \frac{1}{2} + \frac{1}{2} = 0$.
(Looking back: zero is not a surprise because of symmetry.) Over the triangle the integral up to 
y = $z$ is $\int_{0}^{z} \int_{0}^{z} (x^{2} + y^{2}) dy \ dx$. Over the triangle across to $y = z$ the integral is $\int_{0}^{z} \int_{0}^{z} (x^{2} + y^{2}) dy \ dx$.
Exchange $y$ and $x$ in the second double integral to get minus the first double integral.
30 $\int_{0}^{a} \int_{0}^{1 - x^{2}} dx \ dy = \int_{0}^{1} 2\sqrt{1-y} \ dy = \frac{1}{2} (1 - y)^{3/2}]_{0}^{1} = \frac{1}{3}$.
With horizontal strips this is $\int_{0}^{1} \int_{0}^{\sqrt{1-y}} dy \ dx =$ 
$\int_{0}^{1} 2\sqrt{1-y} \ dy = \frac{1}{2} (1 - y)^{3/2}]_{0}^{1} = \frac{1}{3}$. 
32 The height is $x = \frac{1}{a} - \frac{b}{b}y$. Integrate over the triangular base ($x = 0$ gives the side $ax + by = 1$) :
volume is $\int_{x=0}^{a} \int_{y=0}^{b/a} 1 - \frac{a}{b} by \ dy \ dx = \int_{x=0}^{a} \int_{y=0}^{b/a} dy \ dx = \int_{x=0}^{a} \left[\frac{1}{b} (y - ax y) - \frac{1}{2} by^{2} \right]_{0}^{b/a} = \frac{1}{a} - \frac{1}{2} b[a^{2} - 2b^{2}] = \frac{1}{2} b[a^{2} - 2b^{2}] = 0 a b c$.
34 From Problem 33 the mass is $\frac{1}{2} \pi$. The moments are $\int_{0}^{3} \int_{0}^{x^{2}} dy \ dx = \int_{0}^{3} 2\frac{x^{2}}{2} dy \ dx = \frac{15}{2}$ and $\int_{0}^{3} \int_{0}^{y^{2}} dz \ dx = \int_{0}^{3} \frac{z^{2}}{2} dy \ dx = 28 \frac{3}{2}$. Then $x = \frac{15}{2} \frac{3}{2} = \frac{45}{2} \frac{3}{2}$ and $y = 28 \frac{3}{2} = 2$. 
36 The area of the quarter-circle is $\frac{\pi}{4}$. The moment is zero around the axis $y = 0$ (by symmetry): $R = 0$.
The other moment, with a factor 2 that accounts for symmetry of left and right, is 
$2 \int_{0}^{\sqrt{2} / 2} \int_{x}^{1 - x^{2}} y \ dy \ dx = 2 \int_{0}^{\sqrt{2} / 2} \left[\frac{x^{2}}{2} - \frac{x^{2}}{2}\right] \ dy \ dx = \frac{2}{2} \frac{4}{2}$ \quad \text{Then $g = \frac{4}{2} = 4/2$.}
38 The integral $\int_{0}^{a} \int_{0}^{x} x^{2} dy \ dx$ has the usual midpoint error $-\frac{(a x)^{2}}{12}$ for the integral of $x^{2}$ (see Section 5.8). 
The $y$ integral $\int_{0}^{x} dy = 1$ is done exactly. So the error is $-\frac{1}{12} x^{3}$ (and the same for $\int_{0}^{a} x^{2} dy \ dx$). The integral of $xy$ is computed exactly. Errors decrease with exponent $p = 2$, the order of accuracy.
40 The exact integral is
\[ \int_0^1 \int_0^1 \frac{dx \, dy}{\sqrt{x^2 + y^2}} = 2 \int_0^{\pi/4} \int_0^r \frac{r \, dr \, d\theta}{r} = 2 \int_0^{\pi/4} \sec \theta \, d\theta = 2 \ln(\sec \theta + \tan \theta)\bigg|_0^{\pi/4} = 2 \ln(\sqrt{2} + 1). \]

42 The exact integral is
\[ \int_0^1 \int_0^1 e^x \sin \pi y \, dx \, dy = \int_0^1 (e - 1) \sin \pi y \, dy = \frac{e-1}{\pi} (-\cos \pi y)\bigg|_0^1 = \frac{2}{\pi} (e - 1). \]

14.2 Change to Better Coordinates (page 534)

We change variables to improve the limits of integration. The disk \( x^2 + y^2 \leq 9 \) becomes the rectangle \( 0 \leq r \leq 3, 0 \leq \theta \leq 2\pi \). The inner limits of \( \int \int dy \, dx \) are \( y = \pm \sqrt{9 - x^2} \). In polar coordinates this area integral becomes \( \int \int r \, dr \, d\theta = 9\pi \).

A polar rectangle has sides \( dr \) and \( r \, d\theta \). Two sides are not straight but the angles are still \( 90^\circ \). The area between the circles \( r = 1 \) and \( r = 3 \) and the rays \( \theta = 0 \) and \( \theta = \pi/4 \) is \( \frac{1}{2} (8^2 - 1^2) = 15 \). The integral \( \int x \, dy \, dx \) changes to \( \int x^2 \cos \theta \, dr \, d\theta \). This is the moment around the \( y \) axis. Then \( \overline{z} \) is the ratio \( M_y/M \). This is the \( z \) coordinate of the centroid, and it is the average value of \( x \).

In a rotation through \( \alpha \), the point that reaches \( (u, v) \) starts at \( x = u \cos \alpha - v \sin \alpha, y = u \sin \alpha + v \cos \alpha \). A rectangle in the \( uv \) plane comes from a rectangle in \( xy \). The areas are equal so the stretching factor is \( J = 1 \). This is the determinant of the matrix
\[ \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}. \]

The moment of inertia \( \int z^2 \, dy \, dx \) changes to \( \int (u \sin \alpha \cos \alpha)^2 \, du \, dv \).

For single integrals \( dx \) changes to \( (dx/du) \, du \). For double integrals \( dx \, dy \) changes to \( J \, du \, dv \) with \( J = \partial (x, y)/\partial (u, v) \). The stretching factor \( J \) is the determinant of the 2 by 2 matrix
\[ \begin{bmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{bmatrix}. \]

The functions \( x(u, v) \) and \( y(u, v) \) connect an \( xy \) region \( R \) to a \( uv \) region \( S \), and \( \int \int_R \, dx \, dy = \int \int_S J \, du \, dv \) = area of \( R \).

For polar coordinates \( x = u \cos \theta \) and \( y = u \sin \theta \) (or \( r \sin \theta \)). For \( z = u, y = u + 4v \) the 2 by 2 determinant is \( J = 4 \). A square in the \( uv \) plane comes from a parallelogram in \( xy \). In the opposite direction the change has \( u = x \) and \( v = \frac{1}{4} (y - x) \) and a new \( J = \frac{1}{4} \). This \( J \) is constant because this change of variables is linear.

1. \( \int_{3\pi/4}^{\pi/4} \int_0^1 r \, dr \, d\theta = \frac{\pi}{4} \)
2. \( S \) is quarter-circle with \( u \geq 0 \) and \( v \geq 0; \int_0^{\pi/4} \int_0^{1 - r^2} du \, dv \)
3. \( R \) is symmetric across the \( y \) axis; \( \int_0^{\pi/4} \int_0^{\sqrt{1 - r^2}} u \, du \, dv = \frac{1}{3} \)
4. \( \text{divided by area gives } (\theta, \phi) = (4/3, 4/3\pi) \)
5. \( \int_0^{\pi/4} \int_0^{\sqrt{1 - r^2}} \theta \, d\theta = \frac{1}{2} \)
6. \( \text{when region moves} \)
7. \( \int_0^{\pi/4} \int_0^{\pi/2} \theta \, d\theta = \frac{\pi}{2} \)
8. \( J = \left| \begin{array}{cc} \partial x/\partial r^* & \partial x/\partial \theta^* \\ \partial y/\partial r^* & \partial y/\partial \theta^* \end{array} \right| = \frac{\cos \theta^* - r^* \sin \theta^*}{\sin \theta^* \cos \theta^*} = r^*; \int_{\pi/4}^{\pi/2} \int_0^{r^*} r^* \, dr^* \, d\theta^* \)
9. \( I_y = \int_R z^2 \, dx \, dy = \int_{\pi/4}^{\pi/4} \int_0^{1 - r^2} r^2 \, d\theta \, dr = \frac{\pi}{16} \)
10. \( I_x = \frac{\pi}{16} + \frac{1}{4}; I_0 = \frac{\pi}{8} \)
11. \( I_y = \int_R z^2 \, dx \, dy = \int_{\pi/4}^{\pi/4} \int_0^{1 - r^2} r^2 \, d\theta \, dr = \frac{\pi}{16} - \frac{\pi}{8} \)
12. \( I_x = \frac{\pi}{16} + \frac{1}{4}; I_0 = \frac{\pi}{8} \)
13. \( (0,0), (1,2), (1,3), (0,1); \text{area of parallelogram is 1} \)
14. \( I_y = \int_R z^2 \, dx \, dy = \int_{\pi/4}^{\pi/4} \int_0^{1 - r^2} r^2 \, d\theta \, dr = \frac{\pi}{16} - \frac{\pi}{8} \)
15. \( I_x = \frac{\pi}{16} + \frac{1}{4}; I_0 = \frac{\pi}{8} \)
16. \( z = u, y = u + 3v + uv; \text{then } (u, v) = (1, 0), (1, 1), (0, 1) \) give corners \( (x, y) = (1, 0), (1, 5), (0, 3) \)
17. \( \text{Corners } (0,0), (2,1), (3,3), (1,2); \text{sides } y = \frac{1}{2} x, y = 2x - 3, y = \frac{1}{2} x + \frac{3}{2} \) \( y = 2x \)
18. \( \text{Corners } (1,1), (e^2, e), (e^2, -e^2), (e, e^2); \text{sides } x = y^2, y = x^2/e^2, z = y^2/e^4, y = x^2 \)
19. \( \text{Corners } (0,0), (1,0), (1,2), (0,1); \text{sides } y = 0, z = 1, y = 1 + x^2, z = 0 \)

147
23 $J = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, area $\int_0^1 \int_0^1 3dudv = 3; J = \begin{bmatrix} 2e^{2u+2v} \\ e^{2u+2v} \end{bmatrix} \begin{bmatrix} 2e^{3u+3v} \\ 2e^{3u+3v} \end{bmatrix} = 3e^{3u+3v}, J_0^1 \int_0^1 3e^{3u+3v}dudv =$

$\int_0^1 (e^{3u} - e^{3v})du = \frac{1}{2}(e^6 - 2e^3 + 1)$

25 Corners $(x,y) = (0,0), (1,0), (1,1), (0,0); (\frac{1}{2},1)$ gives $x = \frac{1}{2}, y = f(\frac{1}{2}); J = \begin{bmatrix} 1 \\ f(u) \\ f(u) \end{bmatrix} = f(u)$

26 $B^2 = 2 \int_0^{\pi/4} \int_0^{\sin \phi} e^{-r^2}rdrd\theta = \int_0^{\pi/4} (e^{-\sin \phi^2} - 1)d\phi$

29 $r = \int r^2dr/d\theta \int r \cos \theta d\theta = \frac{\pi}{8}\sin^3 \theta \theta^2/\pi^2 = \frac{32\pi}{9}$

31 $j_0^{2\pi} \int_0^{2\pi} r^2 r dr d\theta = \frac{\pi}{4}$

33 $25 \text{comers (2,} Y) = (0,0), (1,0), (1,1), (0,0); (t,1) \text{gives } 2\pi \text{of } +1; J = \int r^2dr dB.$

27 $\int f^2 \int_0^{\sin \phi} e^{-r^2}rdrd\theta = \int f^2 (e^{-\sin \phi^2} - 1)d\phi$

29 $r = \int r^2dr/d\theta \int r \cos \theta d\theta = \frac{\pi}{8}\sin^3 \theta \theta^2/\pi^2 = \frac{32\pi}{9}$

31 $j_0^{2\pi} \int_0^{2\pi} r^2 r dr d\theta = \frac{\pi}{4}$

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29 $r = \int r^2dr/d\theta \int r \cos \theta d\theta = \frac{\pi}{8}\sin^3 \theta \theta^2/\pi^2 = \frac{32\pi}{9}$

31 $j_0^{2\pi} \int_0^{2\pi} r^2 r dr d\theta = \frac{\pi}{4}$

33 $\int f^2 \int_0^{\sin \phi} e^{-r^2}rdrd\theta = \int f^2 (e^{-\sin \phi^2} - 1)d\phi$
24 Problem 18 has \( J = \begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} = 1 \). So the area of \( R \) is \( 1 \times \) area of unit square = 1. Problem 20 has

\[
\int_0^1 \int_0^{\sqrt{1-x^2}} dv \ dx = \frac{1}{4}.
\]

Check in \( z, y \) coordinates: area of \( R \) = \( \int_0^z \int_0^{\sqrt{1-y^2}} dv \ dy = \frac{1}{3} \).

28 \[ \int_0^\infty x^2 e^{-x^2/2} \, dx = (u)(v) \]

By Example 5. Divide by \( 6 \) to find \( f = 1. \)

So \( R \) is an infinite strip above the interval \([0,1]\) on the \( z \) axis. Its boundary \( z = 1 \) is \( r \cos \theta = 1 \) or \( r = \sec \theta. \)

The limits are 0 to \( \theta \leq \frac{\pi}{2} \). The integral is \( \int_0^{\pi/2} \int_0^\infty r \ dr \ d\theta = \frac{\pi}{2} \) \( \theta \) is infinite.

Equation (3) with \( y \) instead of \( z \) has \( \int \int y^2 \, dA = \int_0^1 \int_0^{\sqrt{1-x^2}} u \sin \alpha + v \cos \alpha \, du \, dv = \sin^2 \alpha \int_0^1 u^2 \, du \, dv + \sin \alpha \cos \alpha \int_0^1 v^2 \, dv \, du + \cos^2 \alpha \int_0^1 v^2 \, du \, dv = \frac{\sin^2 \alpha}{3} + \frac{\sin \alpha \cos \alpha}{2} + \frac{\cos^2 \alpha}{3}.

(a) False (forgot the stretching factor \( J \)) (b) False (\( z \) can be larger than \( x^2 \)) (c) False (forgot to divide by the area) (d) True (odd function integrated over symmetric interval) (e) False (the straight-sided region is a trapezoid; angle from 0 to \( \theta \) and radius from \( r_1 \) to \( r_2 \) yields area \( \frac{1}{2}(r_2^2 - r_1^2) \sin \theta \cos \theta \)).

30 \( R \) is an infinite strip above the interval \([0,1]\) on the \( x \) axis. Its boundary \( x = 1 \) is \( r \cos \theta = 1 \) or \( r = \sec \theta. \)

The limits are 0 to \( r \leq \sec \theta \) and \( 0 \leq \theta \leq \frac{\pi}{2}. \) The integral is \( \int_0^{\pi/2} \int_0^\infty r \ dr \ d\theta = \frac{\pi}{2} \) \( \theta \) is infinite.

32 \[ \int \int y^2 \, dA = \int_0^1 \int_0^{\sqrt{1-x^2}} (u \sin \alpha + v \cos \alpha)^2 \, du \, dv = \sin^2 \alpha \int_0^1 u^2 \, du \, dv + \sin \alpha \cos \alpha \int_0^1 v^2 \, dv \, du + \cos^2 \alpha \int_0^1 v^2 \, du \, dv = \frac{\sin^2 \alpha}{3} + \frac{\sin \alpha \cos \alpha}{2} + \frac{\cos^2 \alpha}{3}.

34 Six important solid shapes are a box, prism, cone, cylinder, tetrahedron, and sphere. The integral \( \int \int \int dx \, dy \, dz \) adds the volume \( \int dA \) of small boxes. For computation it becomes three single integrals. The inner integral \( \int dz \) is the length of a line through the solid. The variables \( y \) and \( z \) are held constant. The double integral \( \int dy \, dz \) is the area of a slice, with \( z \) held constant. Then the \( z \) integral adds up the volumes of slices.

If the solid region \( V \) is bounded by the planes \( z = 0, y = 0, z = 0, \) and \( x + 2y + 3z = 1, \) the limits on the inner \( z \) integral are 0 and \( 1 - 2y - 3z. \) The limits on \( y \) are 0 and \( \frac{1}{2}(1 - 3z). \) The limits on \( z \) are 0 and \( \frac{1}{3}. \) In the new variables \( u = x, v = 2y, w = 3z, \) the equation of the outer boundary is \( u + v + w = 1. \) The volume of the tetrahedron in \( uvw \) space is \( \frac{1}{6}. \) From \( dx = du \) and \( dy = dv/2 \) and \( dz = dw/3, \) the volume of an \( xyz \) box is \( dx \, dy \, dz = \frac{1}{6} \, du \, dv \, dw. \) So the volume of \( V \) is \( \frac{1}{56}. \)

To find the average height \( \bar{z} \) in \( V \) we compute \( \int \int \int z \, dV \) / \( \int \int \int dV. \) To find the total mass if the density is \( \rho = e^z \) we compute the integral \( \int \int \int e^z \, dx \, dy \, dz. \) To find the average density we compute \( \int \int \int e^z \, dV / \int \int \int dV. \) In the order \( \int \int \int dx \, dz \, dy \) the limits on the inner integral can depend on \( x \) and \( y. \) The limits on the middle integral can depend on \( y. \) The outer limits for the ellipsoid \( x^2 + 2y^2 + 3z^2 \leq 8 \) are \( -2 \leq y \leq 2. \)

1 \[ \int_0^1 \int_0^1 \int_0^1 dx \, dy \, dz = \frac{1}{6} \]

3 \[ 0 \leq y \leq z \leq 1 \] and all other orders \( xzy, yzx, zyx, xyz; \) all six contain \((0,0,0); \) to contain \((1,0,1)\)
14.3 Triple Integrals (page 540)

5 \int_0^1 \int_0^1 \int_0^1 f(x, y, z) \, dx \, dy \, dz = 8 \quad 7 \int_0^1 \int_0^1 \int_0^1 f(x, y, z) \, dx \, dy \, dz = 4 \quad 9 \int_0^1 \int_0^1 \int_0^1 f(x, y, z) \, dx \, dy \, dz = \frac{3}{4}

11 \int_0^1 \int_0^{2\pi} \int_0^2 f(r, \theta, z) \, r \, dr \, d\theta \, dz = \frac{3}{2} \quad 13 \int_0^{1/2} \int_0^{2\pi} \int_0^{2\pi} f(\rho, \theta, \phi) \, \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi = \frac{7}{12}

15 \int_0^1 \int_0^1 \int_0^{1-z^2} f(x, y, z) \, dx \, dy \, dz = \frac{\pi}{3} \quad 17 \int_0^1 \int_0^1 \int_0^{1-y^2} f(x, y, z) \, dx \, dy \, dz = 6\pi \quad 19 \int_0^1 \int_0^1 \int_0^{1-z} f(x, y, z) \, dx \, dy \, dz = \pi

21 Corner of cube at \( (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \); sides \( \frac{2}{3} \); area \( \frac{8}{3} \)

23 Horizontal slices are circles of area \( \pi r^2 = \pi (4-z)^2 \); volume = \( \int_0^4 \pi (4-z) \, dz = 8\pi \); centroid has \( x = 0, y = z, z = y, z = 1 \).

25 \int \frac{y}{x} \, f \, dx \, dy \, dz = \int_0^1 \int_0^2 f(x, y, z) \, dx \, dy \, dz \quad 27 \int_0^1 \int_0^1 \int_0^1 (y^2 + z^2) \, dx \, dy \, dz = \frac{1}{3} \quad 29 \int_0^3 \int_0^2 \int_0^1 f(x, y, z) \, dx \, dy \, dz = \frac{1}{3}

31 Trapesoidal rule is second-order; correct for \( 1, x, y, z, xy, xz, yz, xyz \)

2 The area of \( 0 \leq x \leq y \leq z \leq 1 \) is \( \int_0^1 \int_0^1 f(x, y, z) \, dx \, dy \). The four faces are \( z = 0, y = x, x = 0, z = 0 \) and \( z = 1 \).

4 \int_0^1 \int_0^1 f(x, y, z) \, dx \, dy \, dz = \int_0^1 \int_0^{x+1} y^2 \, dx \, dy \, dz = \int_0^1 \frac{x+1}{3} \, dx \, dy \, dz = \frac{1}{6}

10 \int_1^2 \int_0^1 \int_0^1 f(x, y, z) \, dx \, dy \, dz = \int_1^2 \int_0^1 (y+1) \, dy \, dz \, dx = \int_1^2 \frac{(x+1)^2}{2} \, dx \, dy \, dz = \frac{1}{6} \quad \text{(tetrahedron)}

12 The plane faces are \( z = 0, y = 0, x = 0, \) and \( 2x + y + z = 4 \). The volume is \( \int_0^2 \int_0^{4-z} \int_0^{4-2z} f(x, y, z) \, dx \, dy \, dz = \int_0^2 \int_0^{4-z} \frac{1}{2} \, dx \, dy \, dz = \frac{1}{12} \)

18 The plane \( z = c \) cuts the circle base in half, leaving \( z \geq 0 \). Volume = \( \int_0^1 \int_{\sqrt{1-z^2}}^1 \int_0^{1-z-2} f(x, y, z) \, dx \, dy \, dz = \int_0^1 \int_{\sqrt{1-z^2}}^1 2x \, dx \, dy \, dz = \frac{1}{3} \)

22 Change variables to \( X = \frac{x}{a}, Y = \frac{y}{b}, Z = \frac{z}{c} \); then \( dXdYdZ = \frac{dxdydz}{abc} \). Volume = \( \int\int\int abc \, dXdYdZ = \frac{1}{a}\frac{1}{b}\frac{1}{c} \). (Recall volume \( \frac{1}{6} \) and centroid \( \frac{1}{4}, \frac{1}{4}, \frac{1}{4} \) of standard tetrahedron: this is Example 2.)

24 (a) Change variables to \( X = \frac{x}{a}, Y = \frac{y}{b}, Z = \frac{z}{c} \). Then the solid is \( X^2 + Y^2 + Z^2 = 1 \), a unit sphere of volume \( \frac{4\pi}{3} \). Therefore the original volume is \( \frac{4\pi}{3}(\frac{3}{4}) = \frac{12\pi}{9} \). (b) The hypervolume in 4 dimensions is \( \frac{1}{24} \) following the pattern of 1 for interval, \( \frac{1}{3} \) for triangle, \( \frac{1}{6} \) for tetrahedron.

26 Average of \( f = \int\int\int f(x, y, z) \, dV \) over \( f \) is \( \int\int\int \, dV \) divided by the volume.

28 Volume of unit cube = \( \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N (\Delta x)^2 = 1 \).

30 In one variable, the midpoint rule is correct for the functions 1 and x. In three variables it is correct for \( 1, x, y, z, xy, xz, yz, xyz \).

32 Simpson's Rule has coefficients \( \frac{1}{6}, \frac{4}{6}, \frac{1}{6} \) over a unit interval. In three dimensions the 8 corners of the cube will have coefficients \( \left( \frac{1}{6} \right)^3 = \frac{8}{216} \). The center will have \( \frac{4}{27} = \frac{8}{216} \). The centers of the 12 edges will have \( \frac{1}{3} = \frac{4}{27} \). The centers of the 6 faces have \( \frac{1}{3} = \frac{4}{27} \). (Check: 8(1) + 64 + 12(4) + 6(16) = 216.) When \( N^3 \) cubic cells are stacked together, with \( N \) small cubes each way, there are only \( 2N + 1 \) meshpoints
along each direction. This makes \((2N + 1)^3\) points or about 8 per cube. (Visualize the 8 new points of the cube as having \(x, y, z\) equal to zero or \(\frac{1}{2}\).)

### 14.4 Cylindrical and Spherical Coordinates

The three cylindrical coordinates are \(r\theta z\). The point at \(x = y = z = 1\) has \(r = \sqrt{2}, \theta = \pi/4, z = 1\). The volume integral is \(\iiint r \, dr \, d\theta \, dz\). The solid region \(1 \leq r \leq 2, 0 \leq \theta \leq 2\pi, 0 \leq z \leq 4\) is a hollow cylinder (a pipe). Its volume is \(12\pi\). From the \(r\) and \(\theta\) integrals the area of a ring (or washer) equals \(3\pi\). From the \(z\) and \(\theta\) integrals the area of a shell equals \(2\pi r\). In \(r\theta z\) coordinates the shapes of cylinders are convenient, while boxes are not.

The three spherical coordinates are \(\rho \phi \theta\). The point at \(x = y = z = 1\) has \(\rho = \sqrt{3}, \phi = \cos^{-1}(1/\sqrt{3}), \theta = \pi/4\). The angle \(\phi\) is measured from the \(x\) axis. \(\theta\) is measured from the \(x\) axis. \(\rho\) is the distance to the origin, where \(r\) was the distance to the \(r\) axis.

If \(\rho \phi \theta\) are known then \(x = \rho \sin \phi \cos \theta, y = \rho \sin \phi \sin \theta, z = \rho \cos \phi\). The stretching factor \(J\) is a 3 by 3 determinant and volume is \(\iiint \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta\).

The solid region \(1 \leq \rho \leq 2, 0 \leq \phi \leq \pi, 0 \leq \theta \leq 2\pi\) is a hollow sphere. Its volume is \(4\pi(2^3 - 1^3)/3\). From the \(\phi\) and \(\theta\) integrals the area of a spherical shell at radius \(\rho\) equals \(4\pi \rho^2\). Newton discovered that the outside gravitational attraction of a sphere is the same as for an equal mass located at the center.

\[
\begin{align*}
1 & (r, \theta, z) = (D, 0, 0); (\rho, \phi, \theta) = (D, \frac{\pi}{2}, 0) & 3 & (r, \theta, z) = (0, \text{any angle}, D); (\rho, \phi, \theta) = (D, 0, \text{any angle}) \\
5 & (x, y, z) = (2, -2, 2\sqrt{2}); (r, \theta, z) = (2\sqrt{2}, -\frac{\pi}{4}, 2\sqrt{2}) & 7 & (x, y, z) = (0, 0, 1); (r, \theta, z) = (0, \text{any angle}, -1) \\
9 & \phi = \tan^{-1}\left(\frac{z}{x}\right) & 11 & 45^\circ \text{ cone in unit sphere: } \frac{2\pi}{3}(1 - \frac{1}{\sqrt{2}}) & 13 & \text{cone without top: } \frac{7\pi}{3} \\
15 & \frac{\pi}{8} \text{ hemisphere: } \frac{8\pi}{3} & 17 & \frac{\pi}{8} & 19 & \text{Hemisphere of radius } \pi: \frac{\pi}{2}x^2 & 21 & \pi(R^2 - z^2); 4\pi r\sqrt{R^2 - r^2} \\
23 & \frac{2\pi}{3} \tan \alpha \text{ (see 8.1.39)} & 27 & \frac{d\phi}{dB} = e^{-D \cos \phi} \frac{\text{near side}}{\text{hypotenuse}} = \cos \alpha \\
31 & \text{Wedges are not exactly similar; the error is higher order} \Rightarrow \text{proof is correct} & 35 & \text{Proportional to } 1 + \frac{1}{h}(\sqrt{a^2 + (D-h)^2} - \sqrt{a^2 + D^2}) \\
37 & J = \begin{vmatrix} a & 0 & 0 \\ b & \cos \phi & -r \sin \phi \\ c & r \sin \phi & 0 \end{vmatrix} = abc; \text{ straight edges at right angles} \\
39 & \begin{pmatrix} \frac{8\pi a^2}{3} \end{pmatrix} \begin{pmatrix} \frac{2\pi}{3} \end{pmatrix} & 41 & \rho^2; \rho^3; \text{ force } = 0 \text{ inside hollow sphere} \\
\end{align*}
\]

\[
\begin{align*}
2 & (r, \theta, z) = (D, \frac{3\pi}{4}, 0); (\rho, \phi, \theta) = (D, \frac{\pi}{2}, \frac{3\pi}{2}) & 4 & (r, \theta, z) = (5, \cos^{-1}\frac{3}{5}, 5); (\rho, \phi, \theta) = (\sqrt{2}, \frac{\pi}{6}, \cos^{-1}\frac{3}{5}) \\
6 & (x, y, z) = (\frac{3}{2}, \frac{\sqrt{3}}{2} - 1); (r, \theta, z) = (\sqrt{3}, \frac{\pi}{6}, 1) & 8 & x = r \text{ on the positive } x \text{ axis } (x \geq 0, y = 0(= \theta), z = 0) \\
10 & x = \cos t, y = \frac{\sqrt{2}}{2} \sin t, z = \frac{\sqrt{2}}{2} \sin t. \text{ The unit sphere intersects the plane } y = z. \\
12 & \text{The surface } z = 1 + r^2 = 1 + x^2 + y^2 \text{ is a paraboloid (parabola rotated around the } x \text{ axis). The region is above the half-disk } 0 \leq & 14 & \text{This is the volume of a half-cylinder (because of } 0 \leq \theta \leq \pi): \text{ height } \pi, \text{ radius } \pi, \text{ volume } \frac{1}{2}\pi^4. \\
16 & \text{The upper surface } \rho = 2 \text{ is the top of a sphere. The lower surface } \rho = \sec \phi \text{ is the plane } z = \rho \cos \phi = 1. \text{ (The angle } \phi = \frac{\pi}{2} \text{ is the meeting of sphere and plane, where sec } \phi = 2.) \text{ The volume is } 2\pi \int_{\rho=0}^{\rho=2} \frac{(8-\sec^2\phi) \sin \phi \, d\phi}{3} = 2\pi(\frac{8-\sec^2\phi}{3})^{\pi/3} = 2\pi(\frac{8-\sec^2\phi}{3})^{\pi/3} = 2\pi(\frac{8-\sec^2\phi}{3})^{\pi/3} = (\frac{5\pi}{2})^3.
\end{align*}
\]
18 The region 1 ≤ ρ ≤ 3 is a hollow sphere (spherical shell). The limits 0 ≤ Φ ≤ π/4 keep the part that lies above a 45° cone. The volume is $\frac{52\pi}{3}(1 - \sqrt{2})$.

20 From the unit ball ρ ≤ 1 keep the part above the cone φ = 1 radian and inside the wedge 0 ≤ θ ≤ 1 radian.
Volume = $\frac{1}{4} \int_0^1 \sin \phi d\phi = \frac{1}{4}(1 - \cos 1)$.

22 The curve ρ = 1 − cos φ is a cardioid in the xz plane (like r = 1 − cos θ in the xy plane). So we have a cardioid of revolution. Its volume is $\frac{8\pi}{3}$ as in Problem 9.3.35.

24 Mass = $\int_0^{2\pi} \int_0^\rho \rho \sin \phi (\rho + 1) d\rho d\phi d\theta = \frac{4\pi}{3} R^3 + 2\pi R^2$.

26 Newton's achievement The cosine law (see hint) gives $\cos \alpha = \frac{D^2 + \rho^2 - \rho^2}{2\rho D}$. Then integrate $\frac{\cos \alpha}{\sqrt{\rho}}$:

\[
\int \int (\frac{D^2 - \rho^2}{2D} + \frac{1}{2\rho D}) dV.
\]

The second integral is $\frac{1}{2D} \int \int \frac{dV}{\sqrt{\rho}} = \frac{4\pi R^3}{2D^2}$. The first integral over \(\phi\) uses the same u = $D^2 - 2\rho D \cos \phi + \rho^2 = q^2$ as in the text: $\int \int \frac{\sin \phi \phi}{\sqrt{\rho}} d\phi = \int \frac{du/2\rho D}{u^{1/2}} = [\frac{-1}{\rho^D u^{1/2}}]_{\phi=0}^{\pi} = \frac{1}{\rho D} (\frac{1}{D - \rho} - \frac{1}{D + \rho}) = \frac{2}{D(D^2 - \rho^2)}$. The \(\theta\) integral gives $2\pi$ and then the \(\rho\) integral is $\int_0^R 2\pi \frac{D^2 - \rho^2}{2D} \rho^2 d\rho = \frac{4\pi R^4}{3}$. The two integrals give $\frac{4\pi R^4}{3}$ as Newton hoped and expected.

28 The small movement produces a right triangle with hypotenuse \(\Delta D\) and almost the same angle \(\alpha\). So the new small side \(\Delta q\) is $\Delta D \cos \alpha$.

30 $\int \int q dA = 4\pi \rho^2 D + \frac{4\pi}{3} \rho^3 D^2$. Divide by $4\pi \rho^2$ to find $q = D + \frac{\rho^2}{3D}$ for the shell. Then the integral over \(\rho\) gives $\int \int q dV = \frac{4\pi}{3} R^3 D + \frac{4\pi}{15} R^5 D^2$. Divide by the volume $4\pi R^3$ to find $q = D + \frac{R^2}{3D}$ for the solid ball.

32 Yes. First concentrate the Earth to a point at its center -- this is OK for each point in the Sun. Then concentrate the Sun at its center -- this does not change the force on the (concentrated) Earth.

34 $J = ae + bfg + cdh - ceg - afh - bdi$.

36 Column 1: $\sqrt{\sin^2 \phi (\cos^2 \theta + \sin^2 \theta)} = 1$; Column 2: $\sqrt{\rho^2 \cos^2 \phi (\cos^2 \theta + \sin^2 \theta)} = 1$; Column 3: $\frac{\rho^2 \sin^2 \phi (\sin^2 \theta + \cos^2 \theta)}{\rho} = \rho \sin \phi$. These are the edge lengths of the box. The dot products of these columns are zero; so $J = \text{volume of box} = (1)(r)(\rho \sin \phi)$ as before.

38 Column 1: $\sqrt{\cos^2 \theta + \sin^2 \theta} = 1$; Column 2: $\sqrt{\rho^2 \sin^2 \theta + \rho^2 \cos^2 \theta} = r$; Column 3: $\sqrt{\rho^2 + \rho^2 + 1} = 1$.

Again the dot products of the columns are zero and $J = \text{volume of box} = (1)(r)(1) = r$.

40 $I = \frac{8\pi R^5}{15}; J = \frac{2}{5}$; the mass is closer to the axis.

42 The ball comes to a stop at Australia and returns to its starting point. It continues to oscillate in harmonic motion $y = R \cos(\sqrt{c/m} t)$.