PROFESSOR: Hi. I thought that today what I'd like to do is to in a sense clean up a few little things that were implied by our last lecture, things that bothered me a little bit and which I thought were worthy of discussing in more detail today. You may recall that last time we discussed the inverse of differentiation. And towards the end of the lecture, we pointed out that traditionally this operation was called the 'indefinite integral'. Now, you see, two points bothered me here. One is why the word integral? In other words, why not just 'inverse derivative'? And we're going to talk about that in more detail later in the course.

The other part that bothered me was the inference of what do you mean by the 'indefinite' integral, say, as opposed to the 'definite' integral? In essence then, I think that when you say indefinite integral, one probably assumes that there was something called the definite integral that exists. And in order not to prejudice anything and not to use the word definite integral in a context that I don't really want it in, I have entitled today's lecture The "Definite" Indefinite Integral. And let's see what this really means now.

Recall that last time we used the notation that 'D inverse' 'f of x' meant-- and that was what? This is notation we used, and we said in the book but this was denoted by integral ''f of x' dx', and that this was called the indefinite integral. And what it meant was it was a name for all function 'G' whose derivative with respect to 'x' was 'f'. And by the mean value theorem, we also showed what? That another name for that set was simply what? If we knew one function whose derivative was 'f', call that capital 'F'. Then the family capital 'F of x' plus 'c' was another name for this set of functions.

And by way of a review, let's do a typical problem that we could tackle last time. Let's see where the word 'indefinite' comes from. You see, in a sense, the indefiniteness comes from the arbitrary constant 'c'. You see, let's suppose that we know that 'f prime of x' is 'x squared', and we now want to find out what 'f of x' looks like. You see the inverse of differentiation again. We have the derivative.

Well, we know that one function whose derivative is 'x squared' is '1/3 x cubed'. Therefore, any

other function whose derivative is 'x squared' must differ from '1/3 x cubed' by a constant. Hence, we know that 'f of x' is '1/3 x cubed' plus a constant. And it's this 'c' that makes this thing called the 'indefinite' integral. You see, at this particular stage, 'c' could be any constant whatsoever, and this would still be a correct answer. We have only got a correct answer up to a family of functions which differ only by a constant.

Now, if I specify one piece of information, which is called an 'initial condition', shall we say. In other words, let's suppose I specify that when 'x' is equal to 'x0', 'f of x' is equal to 'y sub 0', the subscript 0 here just being the abbreviation for a beginning point. In other words, I now specify one additional piece of information. Well, if I plug this back into the equation before, I get that 'f of x0', which is 'y0', equals '1/3 'x sub 0' cubed plus c', from which I can determine that 'c' is equal to ''y0' minus '1/3 x0 cubed''.

Now, notice that 'x0' and 'y0' are given numbers. Consequently, 'c' here is a particular number. And 'c' being a constant, it means that whatever 'c' is for one given value of 'x', it must be that for all. So now what I can do is replace the indefinite 'c' by the definite specific value ''y0' minus '1/3 x0 cubed'' and arrive at the fact that 'f of x' is that unique function, the only function that there is: '1/3 x cubed' plus 'y0' minus '1/3 x0 cubed'. In other words, no more arbitrary constant in here.

What is this function 'f of x'? Well, it's the function whose derivative is 'x squared' and satisfies the condition that when 'x' is equal to 'x0', 'y' is equal to 'y0'. And if you want to see that in terms of our usual analogy of resorting to curve plotting, observe that what we can say here is what? The slope is 'x squared'. Therefore, the curve has the form 'y' equals '1/3 x cubed plus c'. We now specify that the curve passes through the point (x0, y0). That helps us determine that 'c' is 'y0' minus '1/3 x0 cubed'. We put that value 'c' back into our original equation, and we wind up with 'y' equals '1/3 x cubed' plus 'y0' minus '1/3 x0 cubed'.

And what is that particular curve? It's the curve that's characterized by the fact that its slope at any point (x , y) is given by 'x squared' and it passes through the point (x0, y0). In other words, notice then that if we know the slope of a curve and we know a point on the curve, in other words, we know the slope everywhere and we know a point that the curve passes through, it shouldn't seem too surprising that we can uniquely locate where the curve is at any time. And that's exactly what this says. We have a unique expression for this particular curve.

Let's see if we can't generalize this result. And instead of talking about 'f of x' equals 'x

squared', let's consider the following. Suppose we just generalize it. We have that we're given that 'dy dx' is 'f of x' and the domain of 'f' is the closed interval from 'a' to 'b'. And now what we'd like to do is to find out what function 'y' is specifically.

Well, assume that 'g prime' is any function whose derivative is 'f'. And by the way, this is a pretty big assumption. In other words, it's one thing to say let's assume that we have a function whose derivative is 'f'. But for a given 'f', it may not be quite that simple to find what 'G' must be.

And we'll talk about that in more detail as our course unfolds. But it's not always that simple to find a function which has a given derivative. We mentioned that in the last lecture, too. At any rate, all we're assuming now though is that by hook or by crook, somehow or other, we know a function 'G' whose derivative is 'f'. Then therefore, what we conclude is that since 'G' has the same derivative as 'y', that 'y' must be 'G of x' plus a constant.

Now, here's where the word 'initial condition' comes in. We're starting this problem at 'x' equals 'a'. See, 'x' equals 'a'. What we assume is the curve has to be someplace when 'x' equals 'a'. Let's call that y-coordinate 'y' of 'a'. In other words, 'y' of 'a' is equal to 'G of a' plus 'c', from which we determine that 'c' is equal to 'y of a' minus 'G of a'.

You see, the only indefinite thing in here is the 'y of a'. Because, you see, all we're saying is we know the x-coordinate when 'x' is 'a', and the curve can then pass through any point whose x-coordinate is 'a'. And all we're saying is let's call the y-coordinate of that point 'y of a'. At any rate, knowing what 'c' is, we now have that 'y of x' is 'G of x' plus 'y of a' minus 'G of a'.

Now, in particular, from this, you see, notice from this, we can now find out what 'y of b' is. Namely, we just replace 'x' by 'b', and we get that 'y of b' is 'G of b' plus 'y of a' minus 'G of a'. And again, the only thing that's indefinite over here is 'y of a'. Now, to get rid of the only indefinite part, it seems rather clever that maybe all we should do is subtract 'y of a' from 'y to b'. If we do that, we get that 'y of b' minus 'y of a' is 'G of b' minus 'G of a'.

Notice that name for 'y of b' minus 'y of a' is just the change in 'y' as 'x' moved from 'a' to 'b'. In other words, if we know the derivative of 'y' with respect to 'x' and we want to find out how much 'y' changed by as we moved from 'x' equals 'a' to 'x' equals 'b', all we have to do is to compute 'G of b' minus 'G of a' where 'G' is 'a' function whose derivative is 'f'. And by the way, that's all this notation means over here. This is read this evaluated at the upper value minus the value of this evaluated at the lower value.

In fact, now I suppose the only problem that's dangling, as we said, you know, let 'G' be the function whose derivative is 'f'. There are many functions whose derivative is 'f' once we know one function whose derivative is 'f'. Which function should we use over here? And what I intend to show next is that it really doesn't make any difference. That's how definite this really is.

Namely, let's suppose that 'H' is any other function whose derivative is 'f'. Well, if we use this particular expression, my claim is that the change in 'y' would be a 'H of b' minus 'H of a', the same way as we did with 'G'. And the reason for this is that if 'H' and 'G' have the same derivative, then they differ by a constant. That means that 'H of x' is 'G of x' plus some constant, which I denote by 'c1' to indicate that this is no longer an arbitrary constant. What I'm saying is I've picked a particular 'H'. I have the 'G' of the previous problem. The difference is some specific constant which I call 'c1'.

Well, at any rate, I can now compute 'H to b', which is just 'G of b' plus 'c1', 'H of a', which is 'G of a' plus 'c1'. And if I now subtract, I get 'H of b' minus 'H of a' is equal to 'G of b' minus 'G of a' because the constant 'c1' drops out when I form the subtraction. In other words, notice that 'G of b' minus 'G of a' is a well-determined number independent of what function we choose whose derivative is 'f'. And to see what this thing means pictorially, just observe the following.

Our geometric interpretation is this. Let's suppose that we have the curve 'y' equals 'G of x' on the closed interval from 'a' to 'b'. And what's the property of 'G'? It's derivative is 'f'. And now we look at 'H of x'. Now, what is 'H of x'? 'H of x' is 'G of x' plus a constant. Graphically, what happens to a curve when you add on a constant? You see, adding on a constant just raises the curve. In other words, it displaces the curve parallel to itself vertically, in other words, with respect to the y-axis, that the constant just lifts the curve.

In other words then, what we're saying is that when you're measuring the change in 'y', observe that it really makes no difference whether you're talking about this or whether you're talking about this. Because since these two curves are parallel, you see, the displacements over these intervals are the same. In other words, all we're really saying, I guess, is that these two regions here are congruent.

In fact, it might be wise in a sense to pick a new axis either way here, shift this thing over, and label this the 'delta y' axis. And what you're really saying is the arbitrary constant doesn't really make any difference. Because what you've raised the curve by at this end, you raised it at this

end, and consequently, the arbitrary constant drops out in determining the change in 'y'.

Well, again, this may still seem abstract. Let's give a physical interpretation to what we're talking about right now. Let's suppose now that we have a particle moving along the x-axis. A particle is moving along the x-axis. We know that its speed at any time 't' is given by 't squared', and we know that the particles moves for one second. In other words, the domain of 't' here is the closed interval from 0 to 1.

Now, we know that 'v' is 'dx dt'. Therefore, since '1/3 t cubed' has its derivative equal to 't squared', 'x' must be '1/3 t cubed' plus a constant. And here's where the word 'initial condition' comes in again. We're starting this problem when 't' equals 0. When 't' equals 0, the particle has to be someplace. Let's say it's at 'x' equals 'x sub 0'. That's 'x of 0', you see. And if we do this, notice that when 't' is 0, 'x' is 'x sub 0'. That says that 'c' is 'x sub 0'. Replacing 'c' by 'x sub 0', we find that the particle moves according to the rule 'x' equals '1/3 t cubed' plus 'x sub 0'.

Now, we can find out where the particle is when 't' equals 1. When 't' equals 1, 'x' is '1/3 plus x0''. We know that when the particle started at 't' equals 0, 'x' of 0 was 'x sub 0'. And therefore, the change in 'x', in other words, 'x' evaluated when 't' equals '1 minus x' evaluated when 't' equals 0 is simply what? '1/3 plus 'x0' minus 'x0'', or 1/3. In other words, in this particular problem, if we choose as our unit, say, feet and seconds, if a particle moves for 1 second according to the rule that speed in feet per second is the square of the time, in 1 second it has been displaced by a distance of 1/3 of a foot.

And if I want to draw that for you, let me show you what's happening over here. What we're saying is here is a particle moving along the x-axis. when we start this problem at time 't' equals 0, the particle is at some point 'x' equals 'x0'. We observe that the particle is moving because of the velocity 't squared', and that means it's always moving from left to right. And what we find is that when the time is 1, the particle is now at 'x0 + 1/3'. In other words, in this 1 second, the particle has moved 1/3 of a foot.

By the way, to make a little interesting aside while we're here, and we'll come back to this in more detail also into future lectures, notice that since 'v' equals 't squared', when 't' is 0, the velocity is 0. And when 't' is 1, since 't squared' is then 1, the velocity is 1. Notice that the particle starts with 0 velocity, ends up with a velocity of 1, but the average velocity is not a half a foot per second. In other words, notice that the particle traveled a total distance of 1/3 of a foot in 1 second. In other words, the average speed was just 1/3 of a foot per second. And you

see, notice that in this particular problem, since 'v' is equal to 't squared', the acceleration is not a constant. The acceleration is '2t'. It's a variable.

Notice, for example, that the time is approximately 7/10 of a second before the particle would move a half a foot. See, in other words, notice that the time has to be 7/10 of a second before the velocity is 1/2. Namely, when you square 0.7, you get 0.49. And notice again the nonlinearity of this, which, as I say, we'll come back to later, but I thought it was worth mentioning in passing over here.

At any rate, let's generalize this velocity problem because I think it's much easier to think in terms of distance, rate, and time than in terms of slopes of curves. But let's just take a look here. The generalization is this. Let's oppose that we have a particle moving along the x-axis. Its speed at any time is given by 'v' equals 'f of t', and the time interval is from 't' equals 'a' to 't' equals 'b'. We assume that capital 'G' is any function whose derivative is 'f', and we then can conclude that therefore 'x' must be 'G of t' plus a constant.

We also say OK, initially that's what? Not necessarily 0. Initially is when we start the problem. When 't' equals 'a', 'x of a' is 'G of a' plus 'c'. 'x of t', therefore, is 'G of t'. Notice that 'c' is 'x of a' minus 'G of a', so 'x of t' is 'G of t' plus 'x of a' minus 'G of a'. From this, just replacing 't' by 'b', I find that 'x of b' is 'G of b' plus 'x of a' minus 'G of a'. Again, this is the only indefinite part here.

Notice, by the way, that this is mimicking what I did earlier for 'dy dx', but I wanted to go through this one more time in terms of a more tangible physical example. At any rate, if I now subtract 'x of a' from this result, I wind up with that 'x of b' minus 'x of a' is 'G of b' minus 'G of a'. In other words, the change in 'x' on this time interval from 'a' to 'b' is determined just by evaluating 'G of b' minus 'G of a' where 'G' is any function whose derivative changes. See, a rather easy way of evaluating this particular thing.

Now, again I should point out as an interesting physical aside that this 'delta x' is a displacement, not a total distance, and I'll explain what that means in terms of an example. You see, let's observe that in the problem that I did over here, in this particular problem over here, notice that the velocity was always positive. In given problems, the velocity can oscillate between positive and negative, meaning in terms of moving along the x-axis, the particle can sometimes move from left to right. Other times, it can from right to left.

But at any rate, what I want to point out is that the change in 'x' is a displacement, a net

change. Let me give you an example. Let's suppose again that a particle is moving along the x-axis. Now its speed 'v' is given by 't - 1' where 't' is between 0 and 2.

Now, the function I'm looking for, 'x', must differ from a constant by a constant from any function whose derivative is 't - 1'. In particular, one function whose derivative is 't - 1' is '1/2 t squared' minus 't'. So whatever 'x' is, it must have what form? It's '1/2 t squared' minus 't' plus a constant.

Now, I want to compute 'x' evaluated when 't' is 2, 'x' evaluated when 't' is 0, and take this difference because that will be the change in 'x', you see, as 't' goes from 0 to 2. Well, at any rate, you see 'x of 2', just putting 2 in here, gives me 1/2 of 4, which is 2, minus 2 plus 'c', which is 'c'. 'x of 0' is 'c', and therefore, 'x of 2' minus 'x of 0' is 0. Or written more symbolically, 'delta x' as 't' goes from 0 to 2 is 0. In other words, there is no change in 'x'.

Now, obviously, we don't mean the particle didn't move in this case. For no motion, the velocity should be 0, but the velocity is 't - 1'. Let's see what really happened in this particular problem.

To do this, I have drawn some separate graphs over here. The first one shows 'v' in terms of 't'. And what this shows is what? That when 't' is between 0 and 1, 'v' is negative. Now, there's nothing imaginary, again recall, about a negative velocity. Remember, that since the particle is moving along the x-axis and the positive sense of the x-axis is from left to right, a negative velocity means that the particle is just moving from right to left. So what this says is the particle was moving from right to left for the first second and then from left to right for the second second.

Also, if we now want to plot the distance versus the time, you see, what we see is that 'x' is equal to '1/2 t squared' minus 't'. That's this particular shape. And the plus 'x sub 0' is our initial condition. In other words, this tells us lookit, when we started this problem at 't' equals 0, the particle was positioned at 'x' equals 'x sub 0'. And notice what this thing seems to say. This seems to say that what? You started here. Then for the first second, the displacement was decreasing. That means you were moving in the negative direction. And then you reached a certain peak here, and when you got down to this particular point, you started to come back. And at the end of the second second, your displacement was exactly what it was when you first started here.

And by the way, let me again mention, this is not the graph of how the particle moved. The

particle is moving horizontally along the x-axis. This graph is just how the displacement looks as a function of time.

In fact, by the way, to see how we can eliminate the 'x sub 0' here, I have drawn one more. You'll notice that in our recipe, we said let's compute '1/2 t squared' minus 't' as 't' goes from 0 to 2. That was the special case where I replaced the x-axis by the 'delta x' axis. In other words, the curve 'x' equals '1/2 t squared' minus 't' would be this curve here. In other words, when 't' is 0, 'x' is 0. This says what? The change in 'x' when you start the problem is 0 because the particle hasn't moved yet. And this says what? The particle moves, reaches a minimum value when 't' equals 1. And when 't' equals 1, 'x' equals minus 1/2, and then comes back to 0 when 't' equals 2.

In other words, physically what happened here is the following. We started this problem at 't' equals 0. At 't' equals 0, the particle had a certain displacement, a certain initial position which we'll call 'x sub 0'. And what happened in this problem is that for the first second, the particle moved from right to left. And when 't' was equal to 1, the particle was at the position 'x sub 0' minus 1/2, which is just another way of saying it was 1/2 of a unit, or in this case, 1/2 a foot to the left of its starting point. Then during the next second, the velocity is positive. The particle doubles back, returns to the point 'x0' when 't' equals 2, so that the displacement, meaning the net change in position during the time interval of this problem, the net distance traveled is 0. It started and finished at the same point. On the other hand, the total distance traveled is 1/2 a foot in this direction, 1/2 a foot in this direction, so therefore, a total of 1 foot.

Well, you see, I thought this was a worthwhile discussion to have on distance, rate, and time. But I also thought another thing was rather important. Did you notice that except for my mentioning at the beginning of our lesson that the inverse derivative is called the indefinite integral, that the so-called integral sign has never appeared in anything that I've done. In other words, to emphasize what I was talking about in the last lecture, observe that I could get by wonderfully without ever having heard of an integral sign, without having heard of the phrase indefinite integral.

At any rate, though, let me ask the following query, so to speak, to sort of lead into a summation point of view. Once we invent the notation integral ''f of x' dx' to denote the set of all functions 'G' whose derivative is 'f', why not invent a new symbol, namely, what? We'll still use this integral sign, but now put that lower and upper bound on this thing. We'll call this the definite integral ''f of x' dx', or the definite indefinite integral-- I put these limits on, 'a' to 'b'-- to

denote 'G of b' minus 'G of a' where 'G prime' is any function whose derivative is 'f'.

In other words, to put this in still another perspective, remember the 'G' was a member of the family described by interval ''f of x' dx'. To indicate that I'm going to compute 'G of b', why not put a 'b' on top of this integral sign? And then to indicate that I'm going to subtract from that 'G of a', why not denote 'G of a' by the same integral sign with an a underneath it? And then symbolically, I can think of 'G of b' minus 'G of a' as being this expression here, which I will then abbreviate as-- you see, in other words, I'll take this thing and say OK, this is now an abbreviation for what we have here. In other words, that when you see this thing, which we'll call the definite indefinite integral, ''f of x' dx' from 'a' to 'b', this is just an abbreviation for what? 'G of x', 'G of b' minus 'G of a' where 'G' is a function whose derivative is 'f'.

And with this in mind now, I can summarize our lecture as follows. Suppose I know that 'dy dx' is 'f of x', where 'x' is defined on the closed interval from 'a' to 'b'. Then the change in 'y' on this interval, in other words, the change in 'y' as 'x' goes from 'a' to 'b' is given by the symbol integral 'a' to 'b' ''f of x' dx' where this symbol is defined to be 'G of x' evaluated between 'x' equals 'a' and 'x' equals 'b', meaning 'G of b' minus 'G of a', where 'G' is any function whose derivative is 'f'.

Notice again then, in closing, that this particular symbol here, as far as I'm concerned, is artificial. I don't really need it. And whereas it might have been unfortunate to invent the phrase indefinite integral, it might be even more unfortunate to have two words sounding so much alike that are completely different.

In other words, notice that when I write the integral sign from 'a' to 'b' ''f of x' dx', what this thing denotes is a number. It's a change in 'y'. Without the two numbers here, 'a' and 'b', this denotes a set of functions, namely, all functions whose derivative with respect to 'x' is 'f of x'.

Well, anyway, I hope that this straightens out, if nothing more, the difference between the words definite and indefinite. And we shall return to the subject called-- or turn to the subject called integral calculus and bring this up in more detail from a different context in a little while. In the meantime, we'll have a little sojourn into the circular functions and revisit trigonometry for a few days while we allow this idea here to set more firmly in your mind. So until next time, goodbye.

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