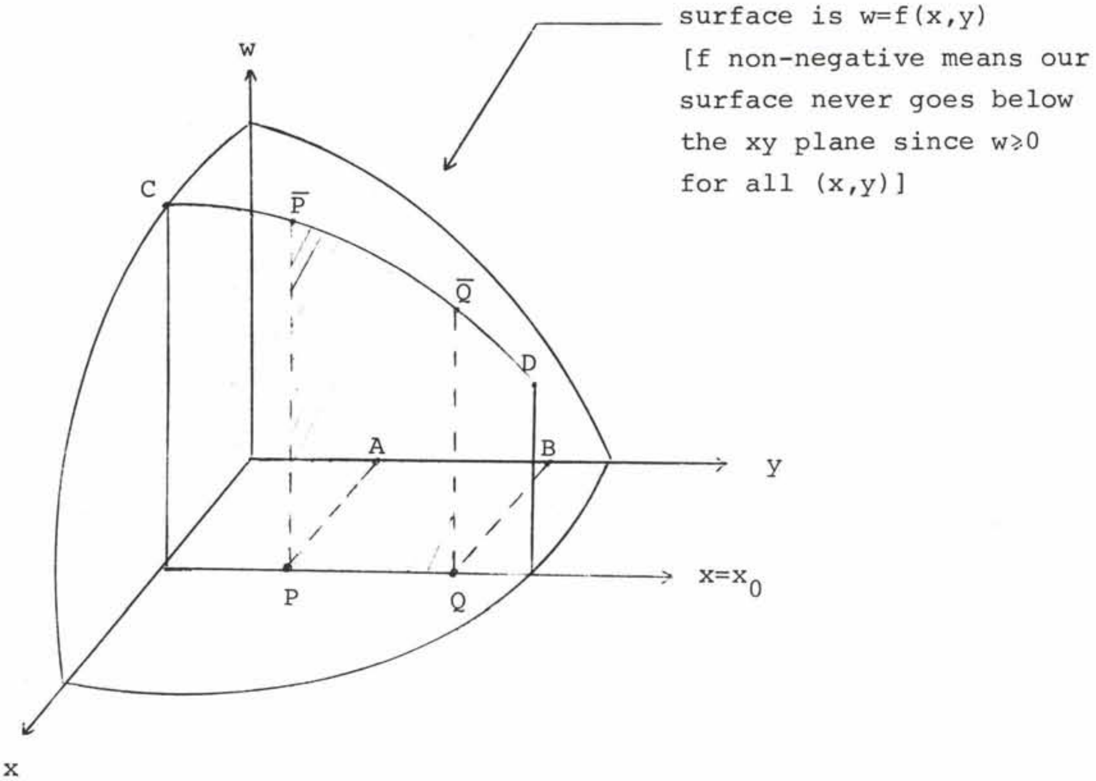
Unit 7: More on Derivatives of Integrals

3.7.1



(Figure 1)

1. We pick $x=x_0$ (subject only to the condition, of course, that our choice is in the domain of f) and form the curve CD which is the intersection of the surface $w=f(x,y)$ and the plane $x=x_0$.

2. $\int_a^b f(x_0,y)\,dy$ implies, among other things, that $a \leqslant y \leqslant b$.

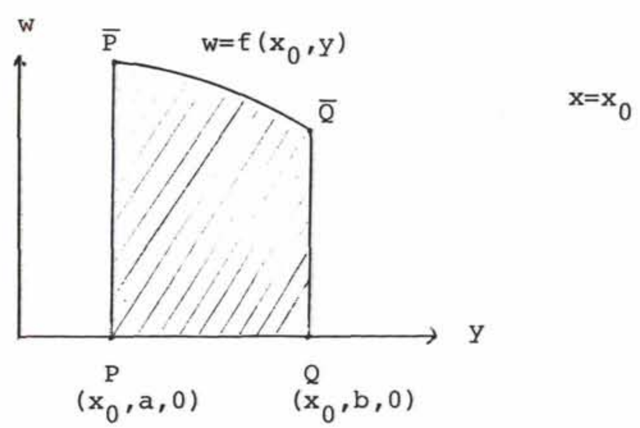
   Hence we locate A(0,a,0) and B(0,b,0) on the y-axis and project them onto the plane $x=x_0$ as the points $P(x_0,a,0)$ and $Q(x_0,b,0)$. We them project P and Q onto the points $\overline{P}$ and $\overline{Q}$ (on the curve CD)

3.7.1 continued

whose coordinates are $\overline{P}(x_0,a,f(x_0,a))$, $\overline{Q}(x_0,b,f(x_0,b))$. (Notice that the line PQ must lie in the domain of f).

3. In terms of f, the equation of the curve $\overline{P}\,\overline{Q}$ is given by $w = f(x_0,y)$, $a \leqslant y \leqslant b$. [Notice that since x_0 is fixed, $f(x_0,y)$ is a function of only y, but what this function looks like depends on x_0. Pictorially this means that the plane $x=x_0$ fixed the curve CD, but if we vary x_0 (that is, we look at the plane $x=x_0$, for different choices x_0) the shape of CD varies].

4. The portion of the plane $x=x_0$ whose vertices are $\overline{P}$, P, $\overline{Q}$, and $\overline{Q}$ has the appearance



(Figure 2)

so by the results of our study of calculus of a single variable the area of the region bounded above by $w=f(x_0,y)$, below by the xy-plane, on the left by y=a and on the right by y=b is

$$\int_a^b f(x_0,y)\,dy$$

provided only that $w=f(x_0,y)$ is piecewise continuous.

3.7.1 continued

In summary, then, with reference to Figure 1,

$$\int_a^b f(x_0,y)\,dy$$

is the area of the shaded region $\overline{P}\ P\ Q\ \overline{Q}$, and once a and b are fixed, the region (hence also its area) is completely determined by x_0. That is,

$$\int_a^b f(x_0,y)\,dy = g(x_0).$$

3.7.2

We have that f(x,y) is continuous on the rectangle $a \leqslant y \leqslant b$, $c \leqslant x \leqslant d$, and for $c<x<d$ we define g(x) by

$$g(x) = \int_a^b f(x,y)\,dy \tag{1}$$

(The geometric meaning of equation (1) was discussed in the previous exercise).

If we now look at $g(x+\Delta x)$, we have

$$g(x+\Delta x) = \int_a^b f(x+\Delta x,y)\,dy \tag{2}$$

(where Δx must be sufficiently small so that $(x+\Delta x,y)$ is in the domain of f for all $a \leqslant y \leqslant b$; and since ultimately we shall let $\Delta x \to 0$, there is no loss of generality in choosing Δx to be small).

3.7.2 continued

From (1) and (2) we see that

$$g(x+\Delta x) - g(x) = \int_a^b f(x+\Delta x,y)\,dy - \int_a^b f(x,y)\,dy \tag{3}$$

and again it is crucial that we realize that both x and $x+\Delta x$ are arbitrary but fixed numbers so that the definite integrals in (3) are functions of the single variable y. (We could have used x_0 and x_0+h rather than x and Δx but we prefer to use the more generally accepted convention of x and Δx).

Then, since the integral of a sum (difference) is the sum (difference) of the integrals, we conclude from (3) that

$$g(x+\Delta x)-g(x) = \int_a^b [f(x+\Delta x,y)-f(x,y)]\,dy \tag{4}$$

Consequently

$$\frac{g(x+\Delta x)-g(x)}{\Delta x} = \frac{1}{\Delta x}\int_a^b [f(x+\Delta x,y)-f(x,y)]\,dy \tag{5}$$

Since Δx is a (non-zero) constant so also is $\frac{1}{\Delta x}$, and a constant factor may be moved inside the integral sign (i.e.

$$c\int_a^b f(y)\,dy = \int_a^b cf(y)\,dy \quad) \quad ,$$

we obtain from (5) that for a fixed x and fixed $\Delta x \neq 0$,

$$\frac{g(x+\Delta x)-g(x)}{\Delta x} = \int_a^b \left[\frac{f(x+\Delta x,y)-f(x,y)}{\Delta x}\right] dy \tag{6}$$

3.7.2 continued

so that

$$\lim_{\Delta x \to 0} \left[\frac{g(x+\Delta x)-g(x)}{\Delta x} \right] = \lim_{\Delta x \to 0} \left\{ \int_a^b \left[\frac{f(x+\Delta x,y)-f(x,y)}{\Delta x} \right] dy \right\} \qquad (7)$$

Now if we are permitted to interchange the order of the limit and the integral on the right side of (7) [hopefully, no one feels, especially after our discussion of uniform convergence in part 1 of our course, that such a step is automatically valid] we would obtain

$$g'(x) = \lim_{\Delta x \to 0} \left[\frac{g(x+\Delta x)-g(x)}{\Delta x} \right] = \int_a^b \left\{ \lim_{\Delta x \to 0} \left[\frac{f(x+\Delta x,y)-f(x,y)}{\Delta x} \right] \right\} dy \qquad (8)$$

We now observe that

$$\lim_{\Delta x \to 0} \left[\frac{f(x+\Delta x,y)-f(x,y)}{\Delta x} \right] \qquad (9)$$

is by definition

$$f_x(x,y) \qquad (10)$$

[provided that f_x exists at the points (x,y), $a \leqslant y \leqslant b$].

Thus if we assume that f_x exists [which will allow us to replace (9) by (10) in (8)] and that f_x is continuous [which will allow us to interchange the order of limit and integration in (7)], we obtain

$$g'(x) = \frac{d}{dx} \int_a^b f(x,y)\,dy = \int_a^b f_x(x,y)\,dy$$

3.7.2 continued

which says (as we might have "guessed") that we differentiate

$$\int_a^b f(x,y)\,dx$$

with respect to x by first taking the (partial) derivative of f(x,y) with respect to x and then performing the integration.

3.7.3

In order to be able to utilize the chain rule, we think of

$$\int_{a(x)}^{b(x)} f(x,y)\,dy$$

as being

$$\int_u^v f(x,y)\,dy$$

(where x, u, and v are considered as being independent) and then letting u=a(x) and v=b(x). In other words, we let

$$F(x,u,v) = \int_u^v f(x,y)\,dy; \; u=a(x), \text{ and } v=b(x)$$

Then

$$g(x) = F(x,a(x),\, b(x)) = \int_{a(x)}^{b(x)} f(x,y)\,dy$$

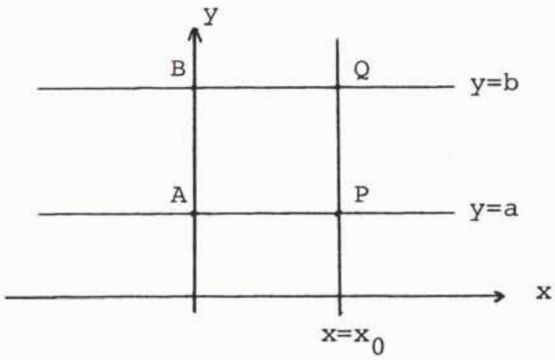
and we wish to compute g'(x).

[This is a generalization of Exercise 3.7.2 in which our limits of integration were constants. Now our limits of integration may be any differentiable function of x. In particular, constants are differentiable functions of x.]

3.7.3 continued

Pictorially, the distinction between this problem and the previous ones is as follows.

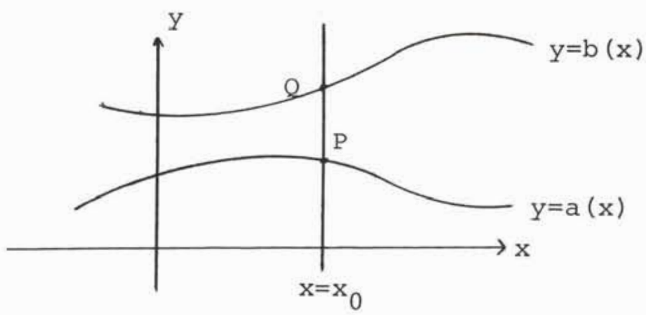
When we looked at $\int_a^b f(x_0,y)dy$, (see Figure 1, Exercise 3.7.1), the view in the xy-plane was



(Figure 1)

That is, g(y) was the segment of $x=x_0$ between the <u>lines</u> y=a and y=b.

If we now think of a and b as being functions of x, the lines y=a and y=b are replaced by the <u>curves</u> y=a(x) and y=b(x). Consequently our view in the xy-plane has the form



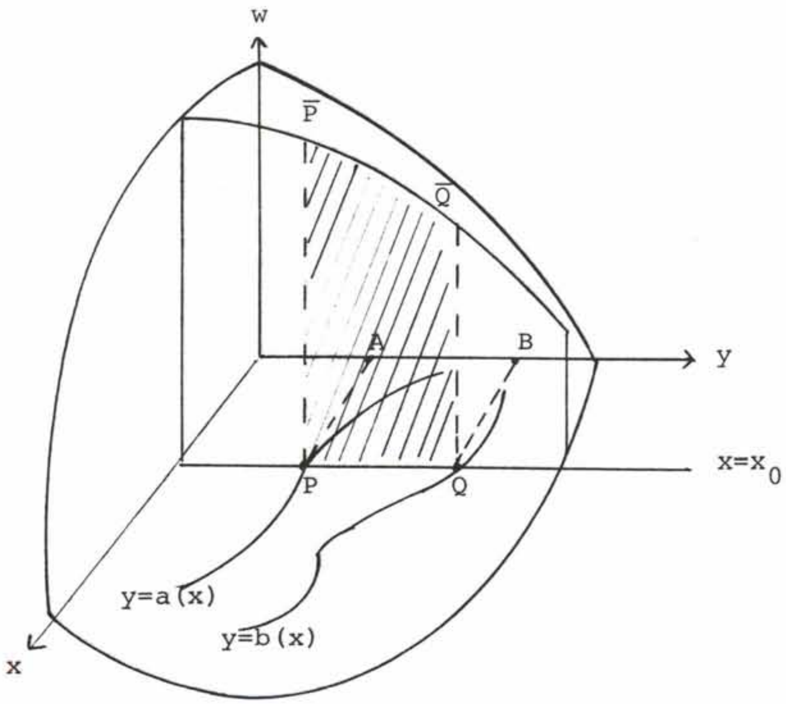
(Figure 2)

3.7.3 continued

That is, P and Q still denote the end points of the segment of $x=x_0$ lying between the curves $y=a(x)$ and $y=b(x)$.

The main difference between Figure 1 and Figure 2 is in the fact that the y-coordinates of P and Q do not depend on x_0 in Figure 1, but in Figure 2, they do.

In three dimensions:



(Figure 3)

[Comparing Figure 3 here with Figure 1 in Exercise 3.7.1, notice that in both cases the shape of $\overline{PQ}$ depends on x_0, but in Figure 3 the y-coordinate of P (and Q) also depends on the choice of x_0].

3.7.3 continued

The geometric interpretation is supplied only to show pictorially that our new problem is more complicated than our original one. Analytically, we are saying that now our limits of integration vary with the choice of parameter, whereas previously these limits were constants (with respect to both x and y).

At any rate, we have

$$g(x) = F(x,a(x),b(x)) = \int_{a(x)}^{b(x)} f(x,y)\,dy \qquad (1)$$

or to emphasize the chain rule format,

$$\left.\begin{array}{l} g(x) = F(x,u,v) \\ u = a(x),\ v = b(x) \end{array}\right\} \qquad (1')$$

Applying the chain rule to (1') we obtain

$$g'(x) = F_x(x,u,v)\,\frac{dx}{dx} + F_u(x,u,v)\,\frac{du}{dx} + F_v(x,u,v)\,\frac{dv}{dx} \qquad (2)$$

Since u=a(x) and v=b(x) where a and b are differentiable functions of x, $\frac{du}{dx} = a'(x)$ while $\frac{dv}{dx} = b'(x)$.

Putting this into (2) yields

$$g'(x) = F_x(x,u,v) + [F_u(x,u,v)]a'(x) + [F_v(x,u,v)]b'(x) \qquad (3)$$

The key step now lies in the fact that

$$F(x,u,v) = \int_u^v f(x,y)\,dy \quad \text{where } u,v, \text{ and } x \text{ are } \underline{\text{independent variables}} \qquad (4)$$

Therefore [from (4)]

3.7.3 continued

$F_x(x,u,v)$ means $\frac{\partial}{\partial x}\left[\int_u^v f(x,y)\,dy\right]$ and since u and v are independent constants as far as varying x is concerned, we may use our previous results concerning constant limits of integration to conclude that

$$F_x(x,u,v) = \int_u^v f_x(x,y)\,dy \qquad (5)$$

We also see from (4) that

$$F_v(x,u,v) = \frac{\partial}{\partial v}\left[\int_u^v f(x,y)\,dy\right]$$

and since u and x are constants with respect to v, we may use the result

$$\frac{\partial}{\partial v}\int_c^v h(y)\,dy = h(v)$$

to conclude

$$F_v(x,u,v) = f(x,v)^* \qquad (6)$$

Similarly

*That is, $f(x,y) = h(y)$ since x is constant with respect to our integration. Then because $h(y) = f(x,y)$, $h(v) = f(x,v)$.

3.7.3 continued

$$F_u(x,u,v) = \frac{\partial}{\partial u}\left[\int_u^v f(x,y)\,dy\right]$$

$$= -f(x,u)^* \tag{7}$$

Putting (5), (6), and (7) into (3) we obtain

$$g'(x) = \int_u^v f_x(x,y)\,dy + [f(x,v)]b'(x) - [f(x,u)]a'(x) \tag{8}$$

Finally, recalling that u=a(x) and v=b(x) we see from (8) that

$$g'(x) = \int_{a(x)}^{b(x)} f_x(x,y)\,dy + b'(x)f(x,b(x)) - a'(x)f(x,a(x))$$

or remembering that

$$g(x) = \int_{a(x)}^{b(x)} f(x,y)\,dy$$

we finally obtain

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,y)\,dy = \int_{a(x)}^{b(x)} f_x(x,y)\,dy + b'(x)f(x,b(x)) - a'(x)f(x,a(x)) \tag{9}$$

*That is, $\frac{d}{du}\int_u^c h(y)\,dy = -h(u)$, so since v and x are being treated as constants, we may let f(x,y) = h(y) and c = v so that $\frac{d}{du}\int_u^v f(x,y)\,dy = \frac{d}{du}\int_u^v h(y)\,dy = -h(u) = -f(x,u)$.

3.7.3 continued

[Notice in the special case for which a(x) and b(x) are constants, $a'(x) = b'(x) = 0$ so that if $a(x) = a$ and $b(x) = b$ equation (9) yields

$$\frac{d}{dx}\int_a^b f(x,y)\,dy = \int_a^b f_x(x,y)\,dy$$

which agrees with our earlier result.]

The important point is that if our limits of integration are non-constant functions of x, the "correction factor" for our constant-limit result is

$$b'(x)f(x,b(x)) - a'(x)f(x,a(x))$$

3.7.4(L)

a. Our aim here is to pick an example wherein

$$\int_a^b f(x,y)\,dx$$

is easy to compute so that we may "check" the result

$$\frac{d}{dx}\int_a^b f(x,y)\,dy = \int_a^b f_x(x,y)\,dy$$

without recourse to any abstract theory.

We have

$$f(x,y) = x^2y + y^3x^5 + 3 \tag{1}$$

Hence

$$\int_0^1 f(x,y)\,dy = \frac{1}{2}x^2y^2 + \frac{1}{4}x^5y^4 + 3y \Big|_{y=0}^{1}$$

$$= \frac{1}{2}x^2 + \frac{1}{4}x^5 + 3^* \tag{2}$$

*Remember, even though the symbol suggests a variable, x is being treated as a constant when we compute $\int_0^1 f(x,y)\,dy$.

3.7.4 continued

so that

$$\frac{d}{dx}\left[\int_0^1 f(x,y)\,dy\right] = \frac{d}{dx}\left(\frac{1}{2}x^2 + \frac{1}{4}x^5 + 3\right)$$

$$= x + \frac{5}{4}x^4 \qquad (3)$$

On the other hand, from (1) we have that

$$f_x(x,y) = 2xy + 5x^4y^3$$

so that

$$\int_0^1 f_x(x,y)\,dy = xy^2 + \frac{5}{4}x^4y^4 \Big|_{y=0}^{1}$$

$$= x + \frac{5}{4}x^4 \qquad (4)$$

whereupon a comparison of (3) and (4) establishes, at least in this example, that

$$\frac{d}{dx}\int_0^1 f(x,y)\,dy = \int_0^1 f_x(x,y)\,dy$$

b. In part (a) we picked a rather trivial example in which it made little if any difference whether we first computed

$$\int_a^b f(x,y)\,dy$$

and differentiated this with respect to x, or whether we computed

$$\int_a^b f_x(x,y)\,dy.$$

3.7.4 continued

In other words there are times when the "old" way will be at least as easy as the "new" way. On the other hand there are times where we work abstractly with functions of two variables and we have no explicit recipe for simplifying $f(x,y)$. [This is much the same as using the formula $(a+b)^2 = a^2 + 2ab + b^2$. For example, to compute the specific number $(3+2)^2$ we (hopefully) would not resort to the recipe, but when we had no specific values for a and b we would, by necessity, have to use the recipe].

Aside from this, there are also times when $f(x,y)$ is given explicitly, but

$$\int_a^b f(x,y)dy$$

is extremely difficult (or even impossible) to compute, while

$$\int_a^b f_x(x,y)\ dy$$

is relatively simple to compute. (There are, of course, many times when neither

$$\int_a^b f(x,y)\ dy \quad \text{nor} \int_a^b f_x(x,y)\ dy$$

are convenient to compute). Thus the aim of part (b) is to emphasize the case in which

$$\int_a^b f_x(x,y)\ dy$$

is more accessible to us than is

$$\int_a^b f(x,y)\ dy.$$

3.7.4 continued

Notice first of all that the formula stated in Exercise 3.7.2 applies since $\sin(xe^y)$ and its partial with respect to x are continuous for all values of x and y (i.e., the rectangle R in this case may be drawn without restriction).

We have

$$g(x) = \int_1^2 \sin(xe^y)\,dy$$

therefore

$$g'(x) = \int_1^2 \frac{\partial}{\partial x}[\sin(xe^y)]\,dy,$$

and

$$\frac{\partial}{\partial x}[\sin(xe^y)] = [\cos(xe^y)]\frac{\partial}{\partial x}(xe^y)$$

Therefore $g'(x) = \int_1^2 e^y \cos(xe^y)\,dy$ (1)

$$= \frac{1}{x}\int_1^2 \cos(xe^y)\,[xe^y dy]^*$$

$$= \frac{1}{x}\int_1^2 \cos(xe^y)\,d(xe^y)^{**} \qquad (2)$$

$$= \frac{1}{x}\sin(xe^y)\Big|_{y=1}^{y=2}$$

$$= \frac{\sin(e^2x) - \sin(ex)}{x}$$

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*Again, keep in mind that x is a constant with respect to the integration where the variable of integration is y.

**Treating x as a constant notice that we could have obtained (2) from (1) by the substitution $u=xe^y$ and proceeded in the usual way for integrating a function of a single variable.

3.7.4 continued

Notice that $\int \sin(ce^y)\,dy$ is not readily evaluated. Thus in this example

$$\int_a^b f_x(x,y)\,dy$$

is easier to compute than is

$$\frac{d}{dx}\left[\int_a^b f(x,y)\,dy\right].$$

3.7.5(L)

This is a new "wrinkle" to our present discussion. The point is that

$$\int_1^2 \frac{\sin cy}{y}\,dy$$

is a function of c. That is, we may write

$$g(c) = \int_1^2 \frac{\sin cy}{y}\,dy \qquad (1)$$

To find the values of c* which produce max-min values for g it is necessary only to compute g'(c) and g"(c) and this is much easier than computing g(c) itself.

*As mentioned in the lecture there is no need to think specifically in terms of the variables x and dy. Thus we may also write $\frac{d}{dc}\int_a^b f(c,y)\,dy = \int_a^b f_c(c,y)\,dy$. If we are inclined to think of c as being a constant notice that once chosen, it is. That is, while $\int_a^b f(c,y)\,dy$ may vary with the choice of c, c is fixed in value inside the integrand once it is chosen. This is precisely why c is called a parameter (i.e., a "variable constant").

3.7.5(L) continued

Now, since the choice of c is independent of y we have that

$$\frac{\partial}{\partial c}\left[\frac{\sin cy}{y}\right] = \frac{1}{y}\ \frac{\partial}{\partial c}\ [\sin cy] = \frac{1}{y}\ (y)\ \cos cy \quad .$$

Since $\int_1^2 \frac{\sin cy}{y}\ dy$ guarantees that $1 \leqslant y \leqslant 2$, $y \neq 0$ and hence $(\frac{1}{y})y = 1$. Therefore

$$\frac{\partial}{\partial c}\left[\frac{\sin cy}{y}\right] = \cos cy \tag{2}$$

Applying the result of Exercise 3.7.2 to equation (1) and using result (2), we obtain

$$g'(c) = \int_1^2 \cos cy\ dy$$

Therefore

$$g'(c) = \frac{1}{c}\ \sin cy\ \Big|_{y=1}^{2} = \frac{\sin 2c - \sin c^*}{c} \tag{3}$$

Therefore

$$\begin{aligned} g'(c) = 0 &\leftrightarrow \sin 2c - \sin c = 0 \\ &\leftrightarrow 2 \sin c \cos c - \sin c = 0 \\ &\leftrightarrow \sin c\ (2 \cos c - 1) = 0 \\ &\leftrightarrow \sin c = 0 \text{ or } \cos c = \frac{1}{2} \end{aligned}$$

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*This expression is invalid if c=0. Notice that if c=0 $g(c) = \int_1^2 \frac{\sin 0y}{y}\ dy = 0$. Since $0 \leqslant c \leqslant \frac{\pi}{2}$ and $1 \leqslant y \leqslant 2$, $\int_1^2 \frac{\sin cy}{y}\ dy$ is non-negative because $0 \leqslant cy \leqslant \pi \rightarrow \sin cy \geqslant 0$. Hence $g(c)=0$ must be a minimum of $g(c)$ for $0 \leqslant c \leqslant \frac{\pi}{2}$. With this in mind, we may use (3) with the understanding that $c \neq 0$.

3.7.5(L) continued

Since the validity of (3) requires that $c \neq 0$ we have that for $0<c\leq\frac{\pi}{2}$, $\sin c\neq 0$. We have, however, in our previous footnote established that $c=0$ yields a minimum, namely 0, for

$$\int_1^2 \frac{\sin cy}{y}\,dy \quad , \; 0\leq c\leq\frac{\pi}{2}\;^{*}$$

Finally for $0<c\leq\frac{\pi}{2}$, $\cos c = \frac{1}{2} \leftrightarrow c = \frac{\pi}{3}$. To see whether this is a max or min we compute $g''(\frac{\pi}{3})$ from equation (3). This yields

$$g''(c) = \frac{c[2\cos 2c - \cos c]-[\sin 2c - \sin c]}{c^2}$$

Therefore

$$g''(\tfrac{\pi}{3}) = \frac{\frac{\pi}{3}\left[2\cos\frac{2\pi}{3} - \cos\frac{\pi}{3}\right] - \left[\sin\frac{2\pi}{3} - \sin\frac{\pi}{3}\right]}{\frac{\pi^2}{9}}$$

$$= \frac{\frac{\pi}{3}\left[-2\left(\frac{1}{2}\right) - \frac{1}{2}\right] - \left[\frac{1}{2}\sqrt{3} - \frac{1}{2}\sqrt{3}\right]}{\frac{\pi^2}{9}}$$

Therefore

$$g''(\tfrac{\pi}{3}) = \frac{\frac{\pi}{3}\left[-\frac{3}{2}\right] - 0}{\frac{\pi^2}{9}} = -\frac{9}{2\pi} < 0$$

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*Notice also that we could have "lucked out" here. Had we ignored c=0 and equated g'(c) to 0 in (3), the value c=0 for a minimum would still have "turned up."

3.7.5(L) continued

Therefore $g''(\frac{\pi}{3}) < 0$; consequently $g(c)$ is (a relative) maximum* when $c = \frac{\pi}{3}$. In summary for $0 \leq c \leq \frac{\pi}{2}$

$$\int_1^2 \frac{\sin cy}{y}\,dy \text{ is least when } c=0$$
$$\text{greatest when } c = \frac{\pi}{3}$$

The minimum value is 0 and the maximum value is $\int_1^2 \frac{\sin \frac{\pi}{3} y}{y}\,dy$

(which may be evaluated by series, etc.).

3.7.6

a.

$$g(y) = \int_0^1 \frac{x^y - x^b}{\ln x}\,dx \quad ^{**} \qquad (1)$$

*Actually equation (3) is easier to use and it would spare us the grief of distinguishing between a relative maximum and an absolute maximum at $x=\frac{\pi}{3}$. Namely since $c>0$ we have from (3) that sign $[g'(c)]$ = sign $[\sin 2c - \sin c]$ = sign $[\sin c\,(2\cos c-1)]$. Since $0<c\leq\frac{\pi}{2}$, $\sin c>0$, hence sign $[\sin c\,(2\cos c-1)]$ = sign $(2\cos c-1)$. Therefore, $g'(c)$ is positive for $0<c<\frac{\pi}{3}$ and negative for $\frac{\pi}{3}<c<\frac{\pi}{2}$. Hence for $0\leq c\leq\frac{\pi}{2}$, $g(\frac{\pi}{3})$ is an absolute maximum.

**When $x=1$, $\ln x=0$ so that our integrand is "∞." Similarly when $x=0$, $\ln x=-\infty$ so that $\ln x$ again gives us trouble. The point is that $\int_0^1 \frac{x^y-x^b}{\ln x}\,dx$ is a (convergent) improper integral and really denotes

$$\lim_{h,k\to 0} \int_h^{1+k} \frac{x^y-x^b}{\ln x}\,dx$$ (where the convergence requires that $y>b>-1$).

3.7.6 continued

Therefore

$$g'(y) = \int_0^1 \frac{\partial}{\partial y}\left[\frac{x^y - x^b}{\ln x}\right] dx$$

$$= \int_0^1 \frac{1}{\ln x} \frac{\partial}{\partial y} [x^y]\ dx$$

$$= \int_0^1 \frac{1}{\ln x} \frac{\partial}{\partial y} [e^{y\ln x}]\ dx$$

$$= \int_0^1 \frac{1}{\ln x} \{\ln x\ e^{y\ln x}\}\ dx$$

$$= \int_0^1 e^{y\ln x}\ dx$$

$$= \int_0^1 x^y\ dx \qquad (2)$$

Since y is being treated as a constant in our integration and since $y \gneq -1$, we see from (2) that

$$g'(y) = \frac{x^{y+1}}{y+1}\Bigg|_{x=0}^{1}$$

so

$$g'(y) = \frac{1}{y+1} \qquad (3)$$

3.7.6 continued

b. $\ln(y+1)$ is a rather simple function whose derivative with respect to y is $\frac{1}{y+1}$. Comparing this with (3) we see that since g(y) and $\ln(y+1)$ have identical derivatives they differ by at most a constant. Thus

$$g(y) = \ln(y+1) + c \tag{4}$$

or, from (1),

$$\int_0^1 \frac{x^y - x^b}{\ln x}\, dx = \ln(y+1) + c \tag{5}$$

To determine c, we let y=b in (5) to obtain

$$\int_0^1 \frac{x^b - x^b}{\ln x}\, dx = \ln(b+1) + c$$

or

$$0 = \ln(b+1) + c \quad \text{therefore}$$

$$c = -\ln(b+1) \tag{6}$$

Putting the value of c, as indicated by (6), into (5) we obtain

$$\int_0^1 \frac{x^y - x^b}{\ln x}\, dx = \ln(y+1) - \ln(b+1)$$

or

$$\int_0^1 \frac{x^y - x^b}{\ln x}\, dx = \ln \frac{y+1}{b+1}\ , \text{ where } y>b>-1 \tag{7}$$

3.7.6 continued

c. In particular with y=3 and b=2 (so that y>b>-1 is clearly satisfied), equation (7) yields

$$\int_0^1 \frac{x^3 - x^2}{\ln x}\, dx = \frac{\ln(3+1)}{\ln(2+1)}$$

$$= \ln \frac{4}{3}$$

[Notice that $\int \frac{x^y - x^b}{\ln x}\, dx$ is difficult to evaluate. Thus

parts (b) and (c) show us how the result in Exercise 3.7.2 can be utilized to evaluate

$$\int_0^1 \frac{x^y - x^b}{\ln x}\, dx;\ y>b>-1 \quad .]$$

3.7.7

a. Using the "direct" approach first we have that

$$g(x) = \int_x^{x^2} xy\, dy = \frac{1}{2} xy^2 \Big|_{y=x}^{x^2}$$

$$= \frac{1}{2} x(x^2)^2 - \frac{1}{2} x(x)^2$$

$$= \frac{1}{2} x^5 - \frac{1}{2} x^3$$

Hence

$$g'(x) = \frac{5}{2} x^4 - \frac{3}{2} x^2 \qquad (1)$$

3.7.7 continued

If we use the recipe, we have

$$g'(x) = \int_{a(x)}^{b(x)} f_x(x,y)\,dy + b'(x)f(x,b(x)) - a'(x)f(x,a(x))$$

where $f(x,y) = xy$, $a(x) = x$, and $b(x) = x^2$.

Therefore

$$g'(x) = \int_x^{x^2} \frac{\partial(xy)}{\partial x}\,dy + 2x[x(x^2)]^* - 1[x(x)]^*$$

$$= \int_x^{x^2} y\,dy + 2x^4 - x^2$$

Therefore

$$g'(x) = \frac{1}{2}y^2 \Big|_{y=x}^{x^2} + 2x^4 - x^2$$

$$= \frac{1}{2}\left[(x^2)^2 - x^2\right] + 2x^4 - x^2$$

$$= \frac{5}{2}x^4 - \frac{3}{2}x^2 \qquad (2)$$

Comparing (1) and (2) we see that $g'(x)$ is the same for both methods.

b. Here we have

---

*Since $f(x,y) = xy$; $f(x,t) = xt$ for <u>any</u> number t. In particular $f(x,b(x)) = f(x,x^2) = x(x^2) = x^3$ and $f(x,a(x)) = f(x,x) = x(x) = x^2$.

3.7.7 continued

$a(x) = -x$ therefore $a'(x) = -1$

$b(x) = x$ therefore $b'(x) = 1$

$f(x,y) = \dfrac{1-e^{-xy}}{y}$ therefore $f(x,b(x)) = f(x,x) = \dfrac{1-e^{-x^2}}{x}$

$f_x(x,y) = \dfrac{1}{y}[-(-y)e^{-xy}]$ $\quad f(x,a(x)) = f(x,-x) = \dfrac{1-e^{-x(-x)}}{-x} = \dfrac{e^{x^2}-1}{x}$

Using the formula

$$g'(x) = \int_{a(x)}^{b(x)} f_x(x,y)\,dy + b'(x)f(x,b(x)) - a'(x)f(x,a(x))$$

$$= \int_{-x}^{x} e^{-xy}dy + \frac{1-e^{-x^2}}{x} - (-1)\left(\frac{e^{x^2}-1}{x}\right)$$

$$= -\frac{1}{x}e^{-xy}\Big|_{y=-x}^{x} + \frac{1-e^{-x^2}+e^{x^2}-1}{x}$$

$$= \left[-\frac{1}{x}e^{-x^2}\right] - \left[-\frac{1}{x}e^{-x(-x)}\right] + \frac{e^{x^2}-e^{-x^2}}{x}$$

Therefore

$$g'(x) = \frac{2\left(e^{x^2}-e^{-x^2}\right)}{x} \qquad (3)$$

Equation (3) can be rewritten to utilize the definition that

$$\sinh x^2 = \frac{e^{x^2}-e^{-x^2}}{2} \quad .$$

3.7.7 continued

That is,

$$g'(x) = \frac{4}{x}\left[\frac{e^{x^2}-e^{-x^2}}{2}\right]$$

$$= \frac{4 \sinh x^2}{x}$$

In summary

$$\frac{d}{dx}\left[\int_{-x}^{x} \frac{1-e^{-xy}}{y}\,dy\right] = \frac{4 \sinh (x^2)}{x}, \quad x \neq 0$$

(Note: $\int_{-x}^{x} \frac{1-e^{-xy}}{y}\,dy$ is an improper integral since $0 \epsilon [-x,x]$ but the integral converges so no harm is done.)

3.7.8(L)

a. Given that y is some function of x such that

$$y(x) = \int_{a}^{x} h(t) \sin(x-t)\,dt \qquad (1)$$

we have

3.7.8(L) continued

$$y'(x) = \int_a^x \frac{\partial[h(t)\sin(x-t)]}{\partial x}\,dt + \frac{dx}{dx}[h(x)\sin\underbrace{(x-x)}_{=0}] - \underbrace{\frac{da}{dx}}_{=0}[h(a)\sin(x-a)]$$

$$= \int_a^x h(t)\cos(x-t)dt + 0 - 0$$

$$= \int_a^x h(t)\cos(x-t)dt \qquad (2)$$

From (2),

$$y''(x) = \frac{d}{dx}\int_a^x h(t)\cos(x-t)dt$$

$$= \int_a^x \frac{\partial}{\partial x}[h(t)\cos(x-t)]dt + \overbrace{\frac{dx}{dx}}^{=1}[h(x)\cos\underbrace{(x-x)}_{=1}] - \overbrace{\frac{da}{dx}}^{=0}[h(a)\cos(x-a)]$$

$$= \int_a^x -h(t)\sin(x-t) + h(x) \qquad (3)$$

From (1) we have that $\int_a^x -h(t)\sin(x-t)dt = -y(x)$, so that equation (3) becomes

3.7.8(L) continued

$$y''(x) = -y(x) + h(x)$$

Therefore

$$y''(x) + y(x) = h(x) \tag{4}$$

Moreover when x=a, y' and y are both 0 since our integral goes between a and a as limits.

Hence y(x) is determined by the equation

$$y''(x) + y(x) = h(x)$$

where $y(a) = y'(a) = 0$.

b. This is a concrete illustration of part (a)

$$y(x) = \int_0^x 2e^t \sin(x-t)\,dt$$

therefore

$$y'(x) = \int_0^x \frac{\partial}{\partial x}[2e^t \sin(x-t)]dt + \frac{dx}{dx}[2e^x \sin\underbrace{(x-x)}_{=0}] - \underbrace{\frac{d0}{dx}}_{=0}[2e^0 \sin(x-0)]$$

$$= \int_0^x 2e^t \cos(x-t)\,dt$$

therefore

3.7.8(L) continued

$$y''(x) = \int_0^x \frac{\partial}{\partial x}[2e^t\cos(x-t)]dt + \frac{dx}{dx}[2e^x \underbrace{\cos(x-x)}_{=1}] - \overbrace{\frac{d0}{dx}}^{0}[2e^0\cos(x-0)]$$

$$= \int_0^x -2e^t\sin(x-t)dt + 2e^x$$

$$= -y(x) + 2e^x$$

therefore

$$\left.\begin{aligned} &y''(x) + y(x) = 2e^x \\ &\text{and } y(0) = y'(0) = 0 \end{aligned}\right\} \qquad (5)$$

3.7.9

a.

$$y(x) = \int_0^x (t-x)[y(t)-e^t]dt \qquad (1)$$

therefore

$$y'(x) = \int_0^x \frac{\partial}{\partial x}[(t-x)\{y(t)-e^t\}]dt + 0 + 0$$

$$= \int_0^x [e^t-y(t)]dt \qquad (2)$$

3.7.9 continued

therefore

$y''(x) = e^x - y(x)$

therefore

$$\left.\begin{aligned} & y''(x) + y(x) = e^x \\ & \text{and } y(0) = y'(x) = 0 \text{ [from (1) and (2)]} \end{aligned}\right\} \quad (3)$$

b.

$$y(x) = \frac{1}{2}(e^x - \sin x - \cos x) \quad (4)$$

$$y'(x) = \frac{1}{2}(e^x - \cos x + \sin x) \quad (5)$$

$$y''(x) = \frac{1}{2}(e^x + \sin x + \cos x) \quad (6)$$

Adding (4) and (6) yields

$y''(x) + y(x) = e^x$

Moreover, letting $x=0$ in (4) and (5) shows that

$y(0) = y'(0) = 0$

so $y(x)$ satisfies the conditions of (3).

In other words, if y is defined by

$$y(x) = \int_0^x (t-x)[y(t)-e^t]dt$$

then

$$y(x) = \frac{1}{2}(e^x - \sin x - \cos x)$$

[How we determined that $y(x) = \frac{1}{2}(e^x - \sin x - \cos x)$ would satisfy the given conditions, without solving the given integral equation, is a subject covered later during our discussion of differential equations in Block 7.]