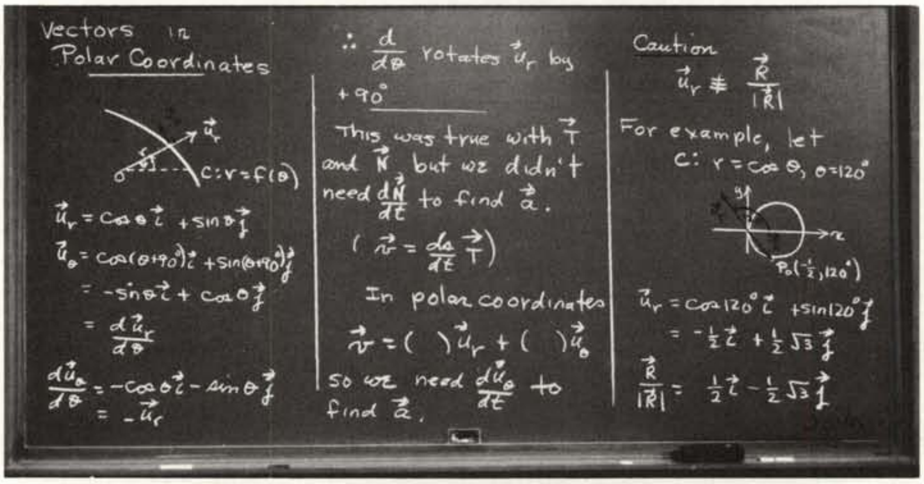
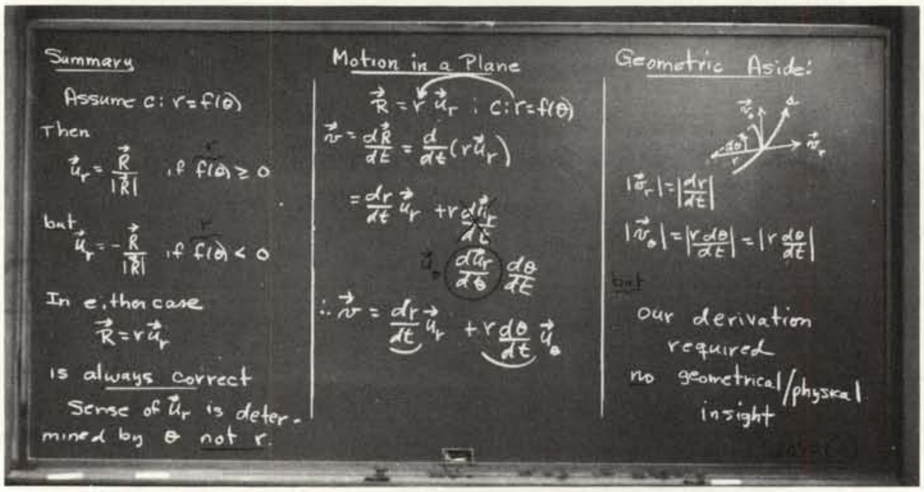
Unit 6: Vectors in Terms of Polar Coordinates

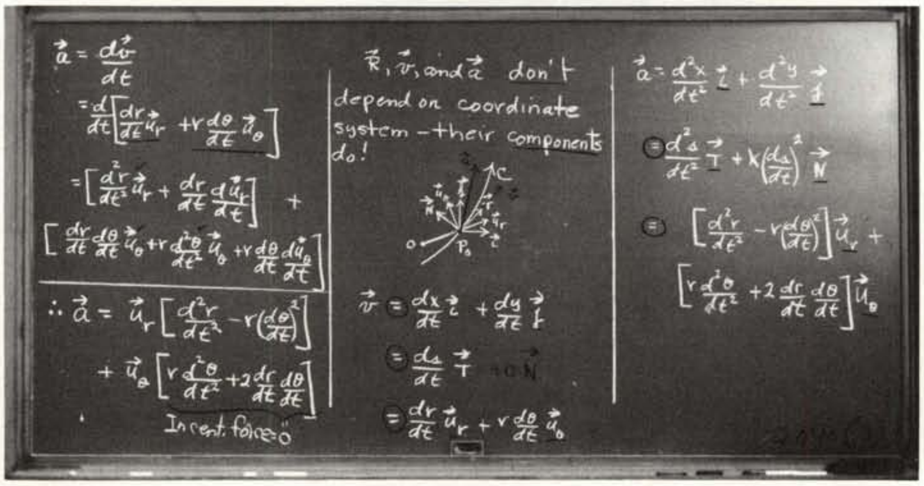
1. Lecture 2.040



a.



b.



c.

2. Read Thomas, Section 14.6.

3. Exercises:

2.6.1(L)

A particle moves in the xy-plane according to some equation of motion, $\vec{R}(t)$. Write $\vec{R}$ as $r\,\vec{u}_r$, and then express the velocity and the acceleration of the particle at any time t in terms of $\vec{u}_r$ and $\vec{u}_\theta$ components.

2.6.2

A particle moves according to the equation of motion

$$r = \sin^2\theta,\ \theta = \frac{t}{2}\ .$$

a. Describe the path of the particle.

b. Find the $\vec{u}_r$ and $\vec{u}_\theta$ components of the acceleration of the particle at any time t.

c. Where is the particle at time $t = \frac{2\pi}{3}$, and what are its components of acceleration at this point?

d. Use the results of (c) to construct the acceleration vector of the particle at the point indicated in (c).

e. What is the magnitude of the acceleration at the point indicated in (c)?

2.6.3

A particle moves according to the equation

$$r = 1 + \cos\theta,\ \theta = e^t.$$

a. Compute its $\vec{u}_r$ and $\vec{u}_\theta$ components of acceleration.

b. Evaluate these components when $t = \ln\frac{\pi}{2}$.

c. Graph the results of (b).

2.6.4(L)

Use the fact that in a central force field (with the force always directed toward the origin) the $\vec{u}_\theta$ component of the acceleration of a particle is zero, to show that if a particle moves in this central force field in the xy-plane, its radius vector sweeps out area at a constant rate. (This is known as Kepler's 2nd Law, i.e. heavenly bodies sweep out equal areas in equal time intervals.) (In this and the following exercises, assume that Newton's Second Law, $\vec{F} = m\vec{a}$, applies.)

2.6.5(L)

A particle moves in the xy-plane in an elliptic path, where the polar equation for the ellipse is

$$r = \frac{c}{1 - e\cos\theta} \quad (0 \leqslant e \leqslant 1).$$

If the motion is always under the influence of the single force $\vec{F}$ which is always directed toward the origin (again, a central force field), show that $|\vec{F}|$ must be proportional to $1/r^2$.

2.6.6(L)

Assume that a particle travels in a central force field. Prove that the radius vector, $\vec{R}$, and the velocity vector, $\vec{v}$, determine the <u>same</u> plane for every time, t.

Quiz

1. Determine $\vec{F}(t)$ if $\vec{F}'(t) = 5t^4\,\vec{i} - 2\sin t\,\vec{j}$ and $\vec{F}(0) = 2\vec{i} + 3\vec{j}$.

2. (a) Compute the curvature, κ, at the point (x, e^x) on the curve $y = e^x$.

   (b) Find the curvature of $y = e^x$ at the point $(\ln\sqrt{3}, \sqrt{3})$.

   (c) At what point on the curve $y = e^x$ is its curvature the greatest?

3. A particle moves in the xy-plane according to the vector equation of motion

   $\vec{R} = \ln(t^2+1)\vec{i} + (t - 2\tan^{-1} t)\vec{j}$ (units are feet and seconds)

   (a) Compute the speed of the particle at any time t.

   (b) Express the acceleration vector, $\vec{a}$, in $\vec{i}$ and $\vec{j}$ components at any time t.

   (c) At what time does the acceleration vector have the greatest magnitude and what is the maximum value?

   (d) What are the tangental and normal components of acceleration (a_T and a_N) at any time t?

   (e) In light of the answer to part (a) of this exercise, explain why a_T had to be zero.

   (f) Determine the curvature of the path traced by the particle at any time t.

4. A particle moves in the xy-plane in such a way that at t=0 it is at the point $P_0(-5,-12)$. At this time its velocity is $3\vec{i}+4\vec{j}$ and its acceleration is $\vec{i}+6\vec{j}$. Moreover when the equation of the path of the particle is given in polar coordinates r is negative when t=0.

4. continued

(a) Describe the vectors $\vec{u}_r$ and $\vec{u}_\theta$ (in terms of $\vec{i}$ and $\vec{j}$) when $t=0$.

(b) Express $\vec{v}$ at $t=0$ in terms of $\vec{u}_r$ and $\vec{u}_\theta$.

(c) Evaluate $\frac{dr}{dt}$ and $\frac{d\theta}{dt}$ at $t=0$.

(d) (Optional) What is the curvature of the path at the point P_0 (i.e., at the point corresponding to $t=0$)?

5. A particle moves in the xy-plane according to the equation of motion $r = \sin 2\theta$, $\theta = 3t^2$. Express $\vec{a}$ in $\vec{u}_r$ and $\vec{u}_\theta$ components and find the $\vec{u}_r$ and $\vec{u}_\theta$ components of $\vec{a}$ at time $t = \frac{\sqrt{3\pi}}{6}$.

6. (a) The polar equation for the curve C is $r = \sin\theta + \cos\theta$. Express C in Cartesian form and then describe the curve C.

(b) Find the area enclosed by the curve $(x^2+y^2)^3 = y^4$ by expressing the curve in polar form.

7. The curve C has as its polar equation

$r = \sin^2\theta$.

Find the slope of the line tangent to C at the point $P_0(\frac{3}{4}, 60°)$.

8. The curve C_1 has as its polar equation $r = 1+\cos\theta$ while the polar equation of C_2 is $r = \sin\frac{\theta}{2}$. Find all points of intersection between C_1 and C_2.