Unit 6: The Jacobian

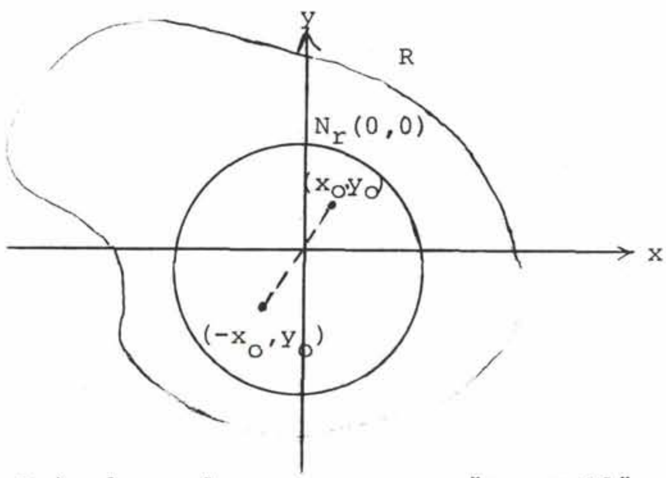
4.6.1

a. Notice that

$$u = x^2 - y^2$$
$$v = 2xy$$

implies that $\underline{f}(x,y) = \underline{f}(-x,-y)$, since $x^2 - y^2 = (-x^2) - (-y)^2$ and $2xy = 2(-x)(-y)$. Thus $\underline{f}$ maps (x,y) and $(-x,-y)$ into the same element, namely, $\underline{f}(x,y) = \underline{f}(-x,-y) = (x^2 - y^2, 2xy)$.

Geometrically (x,y) and $(-x,-y)$ lie on the same line through the origin. Thus, given any (non-zero) neighborhood R of $(0,0)$, it contains a circular neighborhood $N_r(0,0)$ centered at $(0,0)$, having radius r. Pick $(x_0,y_0) \in N_r(0,0)$ subject only to the condition that $(x_0,y_0) \neq (0,0)$. Then $(-x_0,-y_0) \in N_r(0,0) \subset R$ and $\underline{f}(x_0,y_0) = \underline{f}(-x_0,-y_0)$. Therefore $\underline{f}$ is not 1-1 on R. Pictorially



(1) R is drawn large so we can "see well".

(2) $(x_0,y_0) \in R$ does not guarantee that $(-x_0,-y_0) \in R$ since R need not be symmetric with the origin.

(3) The reason for constructing $N_r(0,0)$ was simply to replace R by a region which is symmetric with respect to the origin (we did not have to pick a circular region). The key point is that such a symmetric region can be inscribed in <u>any</u>

4.6.1 continued

non-zero region R.

(4) (x_o, y_o) and $(-x_o, -y_o) \varepsilon R$ [since they are in the subregion $N_r(0,0)$] and $\underline{f}(x_o, y_o) = \underline{f}(-x_o, -y_o) = (x_o^2 - y_o^2, 2x_o y_o)$.

Note:

Using the technique described in Lecture 4.040 we would have that

$$du = 2xdx - 2ydy$$
$$dv = 2ydx + 2xdy \quad .$$

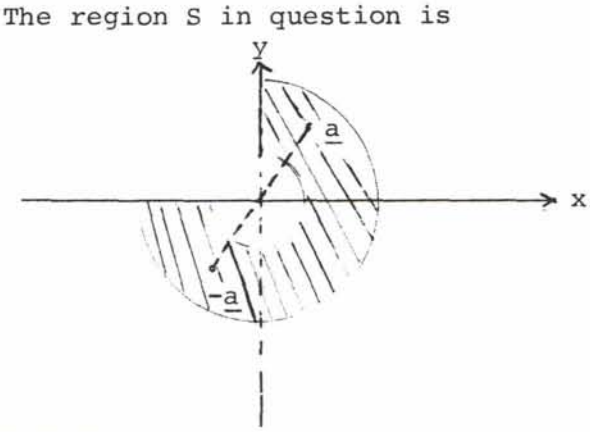
Therefore,

$$dx = \frac{x}{2(x^2 + y^2)} du + \frac{y}{2(x^2 + y^2)} dv$$
$$dy = \frac{-y}{2(x^2 + y^2)} du + \frac{x}{2(x^2 + y^2)} dv \qquad (1)$$

From (1) we would expect trouble when $(x,y) = (0,0)$, since then and only then is $2(x^2 + y^2) = 0$. The inversion theorem verifies that $\underline{f}$ is not invertible in any neighborhood of (0,0) but is invertible in sufficiently small neighborhoods (i.e., is locally invertible) of any other point, since

$$\left|\frac{\partial(u,v)}{\partial(x,y)}\right| = 0 \leftrightarrow (x,y) = (0,0).$$

b. The region S in question is



4.6.1 continued

The region S does not include the point (0,0). Therefore $|\underline{f}'(\underline{a})| \neq 0$ for every $\underline{a} \in S$. Yet $\underline{f}$ is not 1-1 on S. Indeed if $\underline{a}$ is any first quadrant point in S then $-\underline{a}$ is a third quadrant point in S and $\underline{f}(\underline{a}) = \underline{f}(-\underline{a})$ even though $\underline{a} \neq -\underline{a}$.

All the inversion theorem allows us to conclude is that for each $\underline{a} \in S$ there is a sufficiently small neighborhood of $\underline{a}$ on which $\underline{f}$ is 1-1. The trouble with S is that it is too large a region for $\underline{f}$ to be 1-1.

In summary, then

(1) $\underline{f}: E^r \to E^r$ is locally invertible in a sufficiently small neighborhood of $\underline{a}$ provided only that $|\underline{f}'(\underline{a})| \neq 0$.

(2) The fact that $|\underline{f}'(\underline{a})| \neq 0$ guarantees us that $\underline{f}$ is invertible near $\underline{a}$, but the size of the neighborhood must be suitably restricted. That is, knowing that $|\underline{f}'(\underline{a})| \neq 0$ for every $\underline{a} \in S$ is not enough to insure that $\underline{f}$ is 1-1 on S.

4.6.2

a. We have $\underline{f}: E^2 \to E^2$ defined by $\underline{f}(x,y) = (x^3 - y^3, 2xy)$. That is,

$$\left.\begin{aligned} u &= x^3 - y^3 \\ v &= 2xy \end{aligned}\right\} \qquad (1)$$

From (1),

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \begin{bmatrix} 3x^2 & -3y^2 \\ 2y & 2x \end{bmatrix}$$

therefore,

$$\left|\frac{\partial(u,v)}{\partial(x,y)}\right| = 6x^3 + 6y^3 = 6(x^3 + y^3)$$

4.6.2 continued

therefore,

$$\left|\frac{\partial(u,v)}{\partial(x,y)}\right| = 0 \leftrightarrow x^3 + y^3 = 0 \leftrightarrow \underline{y = -x}\ .$$

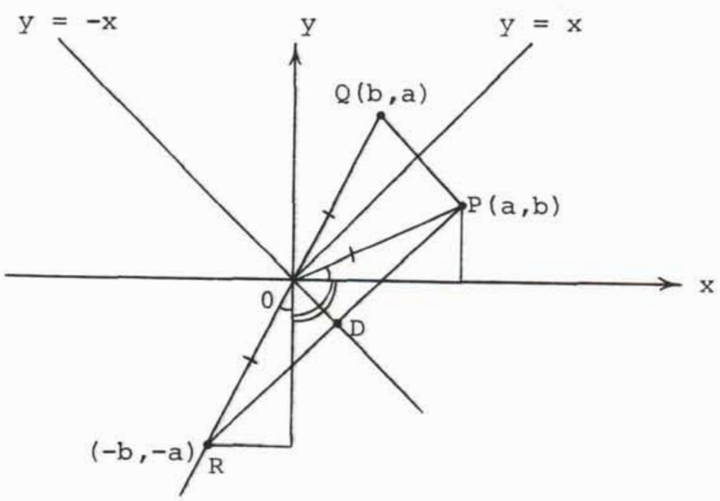
Therefore, $\underline{f}$ is locally invertible at any point $(x,y) \in E^2$ provided only the point is not on the line $y = -x$.

b. $x^3 - y^3 = (-y)^3 - (-x)^3$

$2xy = 2(-y)(-x)$.

Hence $\underline{f}$ maps (x,y) and $(-y,-x)$ into the same element $(x^3-y^3, 2xy)$.

Notice that (x,y) and $(-y,-x)$ are symmetric with respect to the line $y = -x$. [I.e.



(1) P(a,b) and Q(b,a) are symmetric w.r.t. $y = x$.

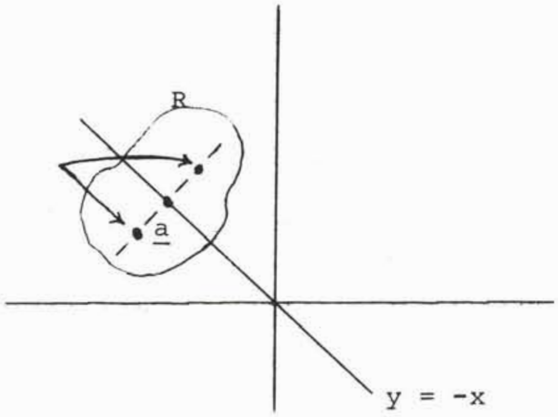
(2) Q(b,a) and R(-b,-a) are symmetric w.r.t. the origin (i.e. r and -r are symmetric with respect to origin)

(3) $\triangle$ORP is isosceles and OA is an angle bisector. Therefore OA is perpendicular bisector of RP.

(4) P(a,b) and R(-b,-a) are symmetric w.r.t $y = -x$]

4.6.2 continued

Hence, for any point $\underline{a}$ on $y = -x$ we have



(1) These two points have the same image under $\underline{f}$ since they are symmetric with respect to $y = -x$.

(2) No matter how small we choose R, once it contains $\underline{a}$ it contains (at least) 2 points symmetric with respect to $y = -x$.

(3) Therefore, $\underline{f}$ cannot be 1-1 in <u>any</u> neighborhood of $\underline{a}$ if $\underline{a}$ is on the line $y = -x$.

4.6.3

Since $x_1,\ldots,x_n$ are assumed to be independent $\partial x_i/\partial x_j = 0$ if $i \neq j$. and of course $\partial x_i/\partial x_j = 1$ if $i = j$.

Accordingly

$$\frac{\partial(x_1,\ldots,x_n)}{\partial(x_1,\ldots,x_n)} = \left[\frac{\partial x_i}{\partial x_j}\right] = \begin{bmatrix} \frac{\partial x_1}{\partial x_1} & \cdots & \frac{\partial x_1}{\partial x_n} \\ \frac{\partial x_n}{\partial x_1} & \cdots & \frac{\partial x_n}{\partial x_n} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 1 \end{bmatrix} = I_n .$$

Thus

$$\frac{\partial(x_1,\ldots x_n)}{\partial(x_1,\ldots x_n)}$$

4.6.3 continued

can be cancelled (as if it were the fraction $\frac{a}{a}$) as a "factor" from any <u>matrix</u> equation since it represents the identity matrix.

In the language of determinants

$$\left|\frac{\partial(x_1,\ldots,x_n)}{\partial(x_1,\ldots,x_n)}\right| = 1$$

and this is why some people like the notation

$$\frac{\partial(y_1,\ldots,y_n)}{\partial(x_1,\ldots,x_n)}$$

to denote the Jacobian <u>determinant</u> rather than the Jacobian matrix. That is, with this notation,

$$\frac{\partial(x_1,\ldots,x_n)}{\partial(x_1,\ldots,x_n)} = 1.$$

4.6.4

a. In Exercise 4.6.2 we saw that

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{bmatrix} 3x^2 & -3y^2 \\ 2y & 2x \end{bmatrix}. \qquad (1)$$

b. Now, from

$$\begin{cases} u = x^3 - y^3 \\ v = 2xy \end{cases}$$

we have

$$\left.\begin{aligned} du &= 3x^2dx - 3y^2dy \\ dv &= 2ydx + 2xdy \end{aligned}\right\}. \qquad (2)$$

4.6.4 continued

Solving (2) for dx and dy in terms of du and dv we obtain

$$\left.\begin{aligned} 6(x^3 + y^3)dx &= 2xdu + 3y^2dv \\ 6(x^3 + y^3)dy &= -2ydu + 3x^2dv \end{aligned}\right\} . \qquad (3)$$

Therefore unless $6(x^3 + y^3) = 0$, we may divide by it in (2) to obtain

$$\left.\begin{aligned} dx &= \frac{x}{3(x^3+ y^3)}\,du + \frac{y^2}{2(x^3+ y^3)}\,dv \\ dy &= \frac{-y}{3(x^3+ y^2)}\,du + \frac{x^2}{2(x^3+ y^3)}\,dv \end{aligned}\right\} . \qquad (4)$$

[Notice again how the mechanical approach yields (4) directly, that is, $6(x^3 + y^3)$ is the Jacobian determinant. Hence we know it cannot be zero at a point where $\underline{f}$ is invertible. In other words we can derive (4) without (consciously) referring to the Jacobian

At any rate, from (4) we deduce that

$$\left.\begin{aligned} x_u &= \frac{x}{3(x^3+ y^3)}, \quad x_v = \frac{y^2}{2(x^3+ y^3)}, \\ y_u &= \frac{-y}{3(x^3+ y^3)}, \text{ and } y_v = \frac{x^2}{2(x^3+ y^3)} \end{aligned}\right\} . \qquad (5)$$

From (5) we conclude that

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix} = \begin{bmatrix} \frac{x}{3(x^3+ y^3)} & \frac{y^2}{2(x^3+ y^3)} \\ \frac{-y}{3(x^3+ y^3)} & \frac{x^2}{2(x^3+ y^3)} \end{bmatrix} . \qquad (6)$$

c. Combining (1) and (6) we have

4.6.4 continued

$$\left[\frac{\partial(u,v)}{\partial(x,y)}\right]\left[\frac{\partial(x,y)}{\partial(u,v)}\right] = \begin{bmatrix} 3x^2 & -3y^2 \\ 2y & 2x \end{bmatrix} \begin{bmatrix} \frac{x}{3(x^3+y^3)} & \frac{y^2}{2(x^3+y^3)} \\ \frac{-y}{3(x^3+y^3)} & \frac{x^2}{2(x^3+y^3)} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{x^3}{x^3+y^3} + \frac{y^3}{x^3+y^3} & \frac{3x^2y^2}{2(x^3+y^3)} - \frac{3x^2y^2}{2(x^3+y^3)} \\ \frac{2xy}{3(x^3+y^3)} - \frac{2xy}{3(x^3+y^3)} & \frac{y^3}{x^3+y^3} + \frac{x^3}{x^3+y^3} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(provided $x^3 + y^3 \neq 0$) (7)

d. Since $\det I_n = 1$ and $\det AB = (\det A)(\det B)$, equation (7) yields

$$\left|\frac{\partial(u,v)}{\partial(x,y)}\right| \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = 1.$$

Therefore

$$\left|\frac{\partial(u,v)}{\partial(x,y)}\right| = \left|\frac{\partial(x,y)}{\partial(u,v)}\right|^{-1}$$

and this makes sense since

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| \neq 0$$

when $x^3 + y^3 \neq 0$.

4.6.5

a. We must examine

$$\left|\frac{\partial(u,v)}{\partial(x,y)}\right|.$$

4.6.5 continued

Since

$$u = \frac{x^2 - y^2}{x^2 + y^2}$$

and

$$v = \frac{xy}{x^2 + y^2} \quad , \ (x,y) \neq (0,0), \text{ we have}$$

$$u_x = \frac{(x^2 + y^2)2x - (x^2 - y^2)2x}{(x^2 + y^2)^2} = \frac{4xy^2}{(x^2 + y^2)^2}$$

$$u_y = \frac{(x^2 + y^2)(-2y) - (x^2 - y^2)2y}{(x^2 + y^2)^2} = \frac{-4x^2y}{(x^2 + y^2)^2}$$

$$v_x = \frac{(x^2 + y^2)y - xy(2x)}{(x^2 + y^2)^2} = \frac{y^3 - x^2y}{(x^2 + y^2)^2}$$

$$v_y = \frac{(x^2 + y^2)x - xy(2y)}{(x^2 + y^2)^2} \quad \frac{x^3 - xy^2}{(x^2 + y^2)^2} \ .$$

Therefore,

$$\left|\frac{\partial(u,v)}{\partial(x,y)}\right| =$$

$$\begin{vmatrix} \frac{4xy^2}{(x^2 + y^2)^2} & \frac{-4x^2y}{(x^2 + y^2)^2} \\ \frac{y^3 - x^2y}{(x^2 + y^2)^2} & \frac{x^3 - xy^2}{(x^2 + y^2)^2} \end{vmatrix} = \frac{4xy^2(x^3 - xy^2) + 4x^2y(y^3 - x^2y)}{(x^2 + y^2)^4} \equiv 0$$

Therefore u and v are functionally dependent.

4.6.5 continued

b. $u^2 = \dfrac{(x^2 - y^2)^2}{(x^2 + y^2)^2}$

$4v^2 = \dfrac{4x^2y^2}{(x^2 + y^2)^2}$.

Therefore

$$u^2 + 4v^2 = \frac{(x^2 - y^2) + 4x^2y^2}{(x^2 + y^2)^2}$$

$$= \frac{(x^4 + 2x^2y^2 + y^4}{(x^2 + y^2)^2}$$

$$= \frac{(x^2 + y^2)^2}{(x^2 + y^2)^2},$$

and since $x^2 + y^2 \neq 0$,

$$u^2 + 4v^2 \equiv 1 \qquad (1)$$

therefore,

$\underline{u^2 + 4v^2 - 1 = 0}$

Let $f(u,v) = u^2 + 4v^2 - 1$.

Therefore $f \neq 0$ but $f[u(x,y),\ v(x,y)] \equiv 0$.

c. $\underline{f}:E^2 \to E^2$ is defined by

$$\underline{f}(x,y) = \left(\frac{x^2 - y^2}{x^2 + y^2},\ \frac{xy}{x^2 + y^2} \right)$$

if $(x,y) \neq (0,0)$ and $\underline{f}(0,0) = (0,0)$; hence the image of $\underline{f}$ in the uv-plane [from equation (1)] is the ellipse $u^2 + 4v^2 = 1$ <u>together with the point (0,0)</u> [which does not belong to the ellipse; i.e., equation (1) was derived under the assumption that $(x,y) \neq (0,0)$]

4.6.6

a. $f(x,y,z,y,v) = 0$
$g(x,y,z,u,v) = 0$ (1)

We want to know whether (1) allows us to view u and v as functions of the independent variables x, y, and z * in some neighborhood of (x_0,y_0,z_0,u_0,v_0). According to our theory this can be done at any point satisfied by (1) provided only that

$$\left|\frac{\partial(f,g)}{\partial(u,v)}\right| \neq 0.$$

b. To show that u and v are functions of x,y, and z it is sufficient to show that u_x, u_y, u_z as well as v_x, v_y and v_z are determined. From a purely mechanical point of view, as we did in Block 3, we may differentiate system (1) implicitly with respect to x,y, and z to obtain

$$\left.\begin{aligned} f_x + f_u u_x + f_v v_x &= 0 \\ f_y + f_u u_y + f_v v_y &= 0 \\ f_z + f_u u_z + f_v v_z &= 0 \end{aligned}\right\} \qquad (2)$$

and

$$\left.\begin{aligned} g_x + g_u u_x + g_v v_x &= 0 \\ g_y + g_u u_y + g_v v_y &= 0 \\ g_z + g_u u_z + g_v v_z &= 0 \end{aligned}\right\}. \qquad (3)$$

*We named the variables x,y,z,u,v rather than x_1, x_2, x_3, x_4, x_5 to make it easier to remember "which variables are which". That is, thing of x,y, and z as denoting whatever three of the variables x_1,x_2,x_3,x_4,x_5 are being chosen at random (i.e., the independent variables).

4.6.6

Appropriately pairing members of (2) with members of (3) we obtain

$$\left.\begin{array}{l} \left.\begin{array}{l} f_x + f_u u_x + f_v v_x = 0 \\ g_x + g_u u_x + g_v v_x = 0 \end{array}\right\} \\ \\ \left.\begin{array}{l} f_y + f_u u_y + f_v v_y = 0 \\ g_y + g_u u_y + g_v v_y = 0 \end{array}\right\} \\ \\ \left.\begin{array}{l} f_z + f_u u_z + f_v v_z = 0 \\ g_z + g_u u_z + g_v v_z = 0 \end{array}\right\} \end{array}\right\} \qquad (4)$$

From the left side of (1) we can explicitly compute f_x, f_y, $f_z, f_u, f_v, g_x, g_y, g_z, g_u$ and g_v.

Therefore at any solution of (1) the unknowns in (4) are u_x, v_x, u_y, v_y, u_z, and v_z. In each case, then, our determinant of coefficients for each pair of equations is

$$\begin{vmatrix} f_u & f_v \\ g_u & g_v \end{vmatrix} = \left|\frac{\partial(f,g)}{\partial(u,v)}\right|$$

and unless this determinant is o we can solve (4) for u_x, v_x, u_y, v_y, u_z, and v_z.

4.6.7

a. Without the knowledge contained in Chapter 7 of the Supplementary Notes, we could have proceeded as follows:

<u>Assuming</u> that the system

$$\left.\begin{array}{l} x + y + z = 0 \\ x^2 + y^2 + z^2 + 2xz - 1 = 0 \end{array}\right\} \qquad (1)$$

determines y and z as functions of x, we can differentiate (1) implicitly with respect to x to obtain

4.6.7 continued

$$1 + \frac{dy}{dx} + \frac{dz}{dx} = 0$$

$$2x + 2y\frac{dy}{dx} + 2z\frac{dz}{dx} + 2x\frac{dz}{dx} + 2z = 0$$

or

$$\left.\begin{aligned} \frac{dy}{dx} + \frac{dz}{dx} &= -1 \\ y\frac{dy}{dx} + (x+z)\frac{dz}{dx} &= -(x+z) \end{aligned}\right\}. \qquad (2)$$

Solving (2) for dy/dx and dz/dx we obtain

$$\left.\begin{aligned} -y\frac{dy}{dx} - y\frac{dz}{dx} &= y \\ y\frac{dy}{dx} + (x+z)\frac{dz}{dx} &= -(x+z) \end{aligned}\right\}.$$

Therefore,

$$(x + z - y)\frac{dz}{dx} = (-x-z+y) \qquad (3)$$

or

$$\underline{\frac{dz}{dx} \equiv -1} \qquad (4)$$

and correspondingly

$$\frac{dy}{dx} \equiv 0 \qquad (5)$$

unless $x + z - y = 0$ [since in this case we cannot solve (3) for $\frac{dz}{dx}$].

That is, <u>once we know</u> that system (1) determines y and z as functions of x, equations (4) and (5) follow inescapably.

The point of this exercise is to show that the general theory supports what we already suspect to be true - that y and z are function of x unless $x + z - y = 0$.

4.6.7 continued

More specifically, the general theory tells us that y and z are functions of x if

$$\left|\frac{\partial(f,g)}{\partial(y,z)}\right| \neq 0$$

where $f(x,y,z) = x + y + z$ and $g(x,y,z) = x^2 + y^2 + z^2 + 2xz - 1$.

Now

$$\left|\frac{\partial(f,g)}{\partial(y,z)}\right| = \begin{vmatrix} 1 & 1 \\ 2y & 2z + 2x \end{vmatrix} = 2z + 2x - 2y.$$

Therefore,

$$\left|\frac{\partial(f,g)}{\partial(y,z)}\right| = 0 \leftrightarrow x + z - y = 0.$$

Therefore, the theory tells us, just as we suspected, that unless $x + z - y = 0$ (i.e., unless we are on the plane $z = y - x$) system (1) determines y and z as differentiable functions of x.

As a direct check notice that $x + z - y = 0$ says $y = x + z$, whereupon (1) becomes

$$\left.\begin{aligned} &x + (x + z) + z = 0 \\ &x^2 + (x^2 + 2xz + z^2) + z^2 + 2xz - 1 = 0 \end{aligned}\right\}. \qquad (6)$$

From (6) it follows that

$$\left.\begin{aligned} &2(x + z) = 0 \\ &2(x + z)^2 = 1 \end{aligned}\right\} \qquad (7)$$

Since $2(x + z)^2 = 1$ implies $x + z = \pm \frac{1}{\sqrt{2}}$, system (7) is inconsistent since it implies $0 = \pm\frac{1}{\sqrt{2}}$.

Hence system (1) is incompatible when $x + z - y = 0$; otherwise system (1) determines y and z as differentiable functions of x

4.6.7 continued

with dy/dx and dz/dx as given by (4) and (5).

b. If we look at

$$\left|\frac{\partial(f,g)}{\partial(x,z)}\right|$$

we obtain

$$\left|\frac{\partial(f,g)}{\partial(x,z)}\right| = \begin{vmatrix} 1 & 1 \\ 2x + 2z & 2x + 2z \end{vmatrix} \equiv 0 .$$

Therefore x and z are never expressible as functions of y. In particular, notice that (1) can be written as

$$\left.\begin{aligned} (x + z) + y &= 0 \\ (x + z)^2 + y^2 - 1 &= 0 \end{aligned}\right\} . \qquad (1')$$

4.6.8

a. Row reducing our given system of equations yields

$$\left.\begin{aligned} x - y + 2z + u &= 0 \\ 2x + 2y - 3z + 2u &= 0 \\ 3x + y - z + u^2 &= 0 \end{aligned}\right\} \sim \left.\begin{aligned} x - y + 2z + u &= 0 \\ 4y - 7z &= 0 \\ 4y - 7z + u^2 - 3u &= 0 \end{aligned}\right\}. \qquad (1)$$

The last two equations on the right side of (1) imply that

$u^2 - 3u = 0$; or, $u = 0$ or $u = 3$.

Therefore the given system of equations is incompatible unless $u = 0$ or $u = 3$.

b. With $u = 0$, (1) becomes

$$\left.\begin{aligned} x - y + 2z &= 0 \\ 4y - 7z &= 0 \end{aligned}\right\} \qquad (2)$$

4.6.8 continued

and from (2) we see that we may pick x or y or z at random and then solve for the remaining two.

For example, if we let $z = z_0$ then $4y - 7z = 0$ implies $y = \frac{7}{4} z_0$; whereupon $x - y + 2z = 0$ then tells us that

$$x - \frac{7}{4} z_0 + 2z_0 = 0$$

or

$$x = -\frac{z_0}{4} .$$

Therefore with $z = z_0$, $u = 0$ implies that $y = \frac{7}{4}z_0$ and $x = -\frac{z_0}{4}$. Hence, one set of points (4-tuples) which satisfy the given system of equations is

$$\left\{\left(-\frac{z_0}{4}, \frac{7z_0}{4}, z_0, 0\right)\right\}. \qquad (3)$$

Therefore, "near" $(x,y,z,0) \in S$, we have

$$\left.\begin{aligned} u &= 0 \\ y &= \frac{7}{4} z \\ x &= -\frac{1}{4} z \end{aligned}\right\} . \qquad (4)$$

Hence y and x can all be expressed as functions of z.

With $u = 3$, system (1) yields

$$\left.\begin{aligned} x - y + 2z &= -3 \\ 4y - 7z &= 0 \end{aligned}\right\} . \qquad (5)$$

So from (5) we obtain for $z = z_0$

$$y = \frac{7}{4} z_0$$

and

4.6.8 continued

$$x - \frac{7}{4} z_o + 2z_o = - 3, \text{ or } x = - \frac{z_o}{4} - 3.$$

Therefore near $(x,y,z,3) \varepsilon S$,

$$\left.\begin{aligned} u &= 3 \\ y &= \frac{7}{4} z \\ x &= - \frac{1}{4} z - 3 \end{aligned}\right\} . \qquad (6)$$

Therefore the given system can always be solved for u, x, and y in terms of z since the given system is equivalent to the union of (4) and (6).

c. If $u \neq 0$ or $u \neq 3$ the given system is incompatible. Hence we cannot view x,y, and z as functions of u since any such function would have the domain {0,3} . That is, u assumes only the two discrete values 0 and 3.

d. We now want to show how these results are obtained in terms of the general theory.

First of all, in order that the given system determines x,y, and u in terms of z we must look at

$$\left|\frac{\partial (f,g,h)}{\partial (x,y,u)}\right|$$

where

$$\begin{aligned} f(x,y,z,u) &= x - y + 2z + u \\ g(x,y,z,y) &= 2x + 2y = 3z + 2u \\ h(x,y,z,u) &= 3x + y - z + u^2 \quad . \end{aligned}$$

Therefore

$$\left|\frac{\partial (f,g,h)}{\partial (x,y,u)}\right| = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 2 & 2 \\ 3 & 1 & 2u \end{vmatrix}$$

4.6.8 continued

$$= \begin{vmatrix} 2 & 2 \\ 1 & 2u \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 3 & 2u \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix}$$

$$= 4u - 2 + 4u - 6 + 2 - 6$$

$$= 8u - 12.$$

Therefore

$$\left| \frac{\partial(f,g,h)}{\partial(x,y,u)} \right| \neq 0 \text{ unless } u = \frac{3}{2}\,. \tag{7}$$

Hence in a neighborhood of any solution (x_0,y_0,z_0,u_0) of the given system, we may express x,y, and u as functions of z, <u>except when $u_0 = 3/2$</u>.

But in part (a) we showed that $u_0 = 3/2$ means (x_0,y_0,z_0,u_0) is not a solution of our system. In fact we showed that (x_0,y_0,z_0,u_0) could not be a solution unless $u_0 = 0$ or $u_0 = 3$.

In any event, since $u_0 \neq 3/2$, equation (7) tells us x,y, and u may be expressed as functions of z near (x_0,y_0,z_0,u_0).

Finally

$$\left| \frac{\partial(f,g,h)}{\partial(x,y,z)} \right| = \begin{vmatrix} 1 & -1 & 2 \\ 2 & 2 & -3 \\ 3 & 1 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & -3 \\ 1 & -1 \end{vmatrix} + \begin{vmatrix} 2 & -3 \\ 3 & -1 \end{vmatrix} + 2 \begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix}$$

$$= 1 + 7 - 8$$

$$\equiv 0.$$

Hence, x,y, and z cannot be expressed in terms of u.

4.6.9

a. Here

$$\left.\begin{aligned} f(x,y,z) &= x + y + z \\ g(x,y,z) &= \tfrac{1}{3}x^3 + x - \tfrac{1}{3}y^3 - z^2y \end{aligned}\right\} \qquad (1)$$

Therefore

$$\left|\frac{\partial(f,g)}{\partial(x,y)}\right| = \begin{vmatrix} 1 & 1 \\ x^2+1 & -y^2-z^2 \end{vmatrix}$$

$$= -y^2 - z^2 - x^2 - 1.$$

Therefore

$$\left|\frac{\partial(f,g)}{\partial(x,y)}\right| = 0 \quad \leftrightarrow -(x^2 + y^2 + z^2 + 1) = 0.$$

Therefore

$$\left|\frac{\partial(f,g)}{\partial(x,y)}\right| \neq 0 \text{ for all } (x,y,z): f(x,y,z) = g(x,y,z) = 0$$

since $x^2 + y^2 + z^2 + 1 \geq 1$.

Therefore in the given system x and y are always determined as functions of z.

b. From (1)

$$\left.\begin{aligned} &\frac{dx}{dz} + \frac{dy}{dz} + 1 = 0 \\ &(x^2+1)\frac{dx}{dz} - y^2\frac{dy}{dz} - z^2\frac{dy}{dz} - 2zy = 0 \end{aligned}\right\}.$$

Therefore,

$$\left.\begin{aligned} \frac{dx}{dz} &+ \frac{dy}{dz} = -1 \\ (x^2+1)\frac{dx}{dz} &- (y^2+z^2)\frac{dy}{dz} = 2zy \end{aligned}\right\}$$

4.6.9 continued

$$\left.\begin{aligned} (y^2 + z^2)\frac{dx}{dz} + (y^2 + z^2)\frac{dy}{dz} &= -(y^2 + z^2) \\ (x^2 + 1)\frac{dx}{dz} - (y^2 + z^2)\frac{dy}{dz} &= 2yz \end{aligned}\right\}.$$

Therefore,

$$(x^2 + y^2 + z^2 + 1)\frac{dx}{dz} = -y^2 - z^2 + 2yz = -(y - z)^2.$$

Therefore

$$\frac{dx}{dz} = \frac{-(y - z)^2}{x^2 + y^2 + z^2 + 1}. \tag{2}$$

[Notice that on the right side of (2) x,y, and z are not independent; rather they satisfy $f(x,y,z) = 0$ <u>and</u> $g(x,y,z) = 0$. In other words, x and y are functions of z for which $f(x,y,z) \equiv g(x,y,z) \equiv 0$. The fact that x and y are differentiable functions of z was established in part (a).]