Unit 5: Polar Coordinates II

2.5.1(L)

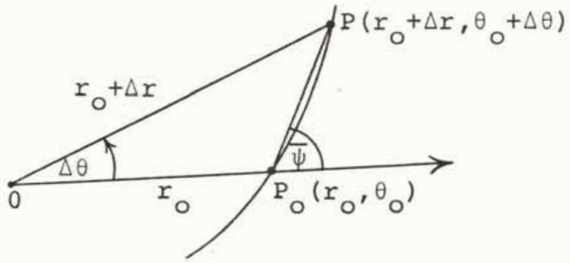
a. The concept of a tangent line can be expressed entirely in terms of a curve without any reference to a coordinate system. We simply look at the limiting position (if such a position exists) of a chord drawn between two points on the curve as the two points are allowed to approach one another to an arbitrary degree of "closeness."

What is true is that we often exploit a particular coordinate system in the sense that we pick, as a parameter for measuring the direction of a tangent line, something that is relatively easy to compute in terms of the variables associated with the particular coordinate system. In Cartesian coordinates, it was particularly convenient to measure direction in terms of the angle the tangent line made with the positive x-axis. Indeed, when the curve is expressed in the Cartesian form $y = f(x)$, $f'(x)$ expresses the direction of the tangent of this angle very nicely.

Notice, however, that talking about the "rise" of a curve versus its "run" seems to presuppose that we are talking in terms of Cartesian coordinates. Suppose, instead, the curve were expressed in polar coordinates. In such a case, there is no longer any great geometric significance to the vertical direction. Rather, it seems that angles should be measured relative to the line that joins our origin (i.e., pole) to the point on the curve. That is, the radius vector $\vec{R}$ is a natural reference line when we deal in polar coordinates.

In terms of this new parameter, let us re-examine the concept of a tangent line. We have a curve C and a point (r_o,θ_o) on C. We denote this point by P_o, and we now look at a "near-by" point, P, on the curve. Diagramatically,

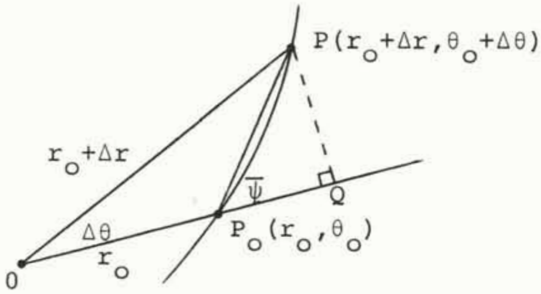
2.5.1(L) continued



Notice here the complete absence of any reference to Cartesian coordinates

Figure 1

As P approaches P_o, the line P_oP approaches the position of the tangent line, and in this respect, the angle $\bar{\psi}$ in the limit becomes the angle ψ between the tangent line and the radius vector. If we now develop the geometry indicated in Figure 1, we obtain,



(i) In the right triangle OPQ,

$\overline{PQ} = \overline{OP} \sin \Delta\theta = (r_o+\Delta r)\sin \Delta\theta$

$\overline{OQ} = \overline{OP} \cos \Delta\theta = (r_o+\Delta r)\cos \Delta\theta$

(ii) $\overline{P_oQ} = \overline{OQ} - \overline{OP_o} = (r_o+\Delta r) \cos \Delta\theta - r_o$

(iii) In the right triangle P_oQP,

$$\tan \bar{\psi} = \frac{\overline{PQ}}{\overline{P_oQ}} = \frac{(r_o+\Delta r)\sin \Delta\theta}{(r_o+\Delta r)\cos \Delta\theta - r_o} \tag{1}$$

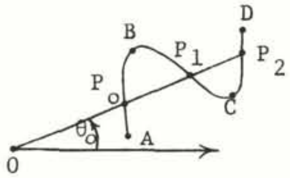
2.5.1(L) continued

Notice that equation (1) tells us explicitly how to compute ψ for given values of r_o, $\Delta\theta$, and Δr. The problem is that we will eventually want to compute $\lim_{P\to P_o} \tan\overline{\psi}$, which is equivalent to $\lim_{\Delta\theta\to o} \tan\overline{\psi}$.

Keeping this in mind and recalling that we have already developed $\lim_{\Delta\theta\to o} \frac{\sin\Delta\theta}{\Delta\theta} = 1$ and $\lim_{\Delta\theta\to o} \left(\frac{\cos\Delta\theta - 1}{\Delta\theta}\right) = 0$ (and neither of these results required any special coordinate system), we now divide both numerator and denominator of (1) by $\Delta\theta$, where, of course, $\Delta\theta \neq 0$.* This yields

$$\tan\overline{\psi} = \frac{\left[(r_o+\Delta r)\ \frac{\sin\Delta\theta}{\Delta\theta}\right]}{\left[r_o\left(\frac{\cos\Delta\theta - 1}{\Delta\theta}\right) + \frac{\Delta r}{\Delta\theta}\cos\Delta\theta\right]} \tag{2}$$

*Notice that had we begun the study of calculus with polar coordinates rather than with Cartesian coordinates, the key philosophical topics would have remained intact. For example, the assumption that $P_o \neq P$ guarantees that $\Delta\theta \neq 0$, only if we restrict our study to single-valued functions. For example, if the polar graph of $r = f(\theta)$ looks like



we are in trouble since $P_o \neq P_1 \neq P_2$ yet $\theta = \theta_o$ (i.e. $\Delta\theta = 0$) for all three points.

Again, as in our earlier study, we can view multi-valued curves as the union of single-valued curves. Of course, the geometric procedure for doing this is different from the Cartesian case, since now we determine "branch points" for a smooth curve by where the radius vector is tangent to the curve. In our above diagram, the pieces of our curve between A and B, B and C, and C and D, are all single valued functions of θ.

2.5.1(L) continued

where the denominator of (2) comes from

$$\frac{(r_o+\Delta r)\cos \Delta\theta - r_o}{\Delta\theta} = \frac{r_o \cos \Delta\theta + \Delta r \cos \Delta\theta - r_o}{\Delta\theta} = \frac{r_o(\cos \Delta\theta - 1) + \Delta r \cos \Delta\theta}{\Delta\theta}$$

$$= \frac{r_o(\cos \Delta\theta - 1)}{\Delta\theta} + \frac{\Delta r \cos \Delta\theta}{\Delta\theta}$$

Applying our limit theorems to (2), we obtain

$$\lim_{\Delta\theta\to o} \tan \overline{\psi} = \frac{\lim\limits_{\Delta\theta\to o} (r_o+\Delta r) \lim\limits_{\Delta\theta\to o} \frac{\sin \Delta\theta}{\Delta\theta}}{r_o \lim\limits_{\Delta\theta\to o} (\frac{\cos \Delta\theta - 1}{\Delta\theta}) + \lim\limits_{\Delta\theta\to o} \frac{\Delta r}{\Delta\theta} \lim\limits_{\Delta\theta\to o} \cos \Delta\theta} \tag{3}$$

Since we are assuming that r is a differentiable function of θ, it follows that $\lim\limits_{\Delta\theta\to o} \Delta r = 0$ and $\lim\limits_{\Delta\theta\to o} \frac{\Delta r}{\Delta\theta} = \frac{dr}{d\theta}\Big]_{\theta=\theta_o}$. Moreover, $\lim\limits_{\Delta\theta\to o} \cos \Delta\theta = \cos 0 = 1$, whereupon (3) takes the form

$$\lim_{\Delta\theta\to o} \tan \overline{\psi} = \frac{r_o(1)}{r_o(o) + (\frac{dr}{d\theta})_{\theta=\theta_o}(1)}$$

or

$$\lim_{\Delta\theta\to o} \tan \overline{\psi} = \frac{r_o}{(\frac{dr}{d\theta})_{\theta=\theta_o}} . \tag{4}$$

Equation (4) yields a rather convenient formula for finding the angle between the tangent line and the radius vector when the curve is given in the polar form $r = f(\theta)$ where f is differentiable. Namely, we need only divide r by $\frac{dr}{d\theta}$.

Finally, it is conventional to denote the limiting position of $\overline{\psi}$ by ψ. That is, ψ is the angle between the tangent line and the radius vector. With this in mind, (4) becomes

$$\tan \psi = \frac{r}{(\frac{dr}{d\theta})} \tag{5}$$

2.5.1(L) continued

where, in (5), it is understood that ψ is evaluated at the point (r,θ) on the curve.

b. While (a) was an important exercise to show that the study of curves in polar coordinates is not dependent on any results of Cartesian coordinates, the fact remains that in many real life situations, any other coordinate system we happen to be dealing with was obtained from some change of variables applied to Cartesian coordinates. For this reason, we often have much information available to us in the form of Cartesian equations, and, in addition to this, we often feel more at home in terms of Cartesian coordinates. Still another thing is that in many cases where the equation of a curve is given initially in terms of polar coordinates, we still want to know where the curve has a greatest height relative to our horizontal reference line. In other words, there are cases in which we might like to use polar information to obtain Cartesian-type results, and vice versa.

In this exercise, we show how the formula for $\tan\psi$ could have been derived from our previous knowledge of Cartesian coordinates. We know that

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}\right\} \qquad (6)$$

Assuming, as in (a), that r is a differentiable function of θ, we may apply the product rule to (6) to obtain

$$\left.\begin{aligned} \frac{dx}{d\theta} &= -r\sin\theta + \cos\theta\,\frac{dr}{d\theta} \\ \frac{dy}{d\theta} &= r\cos\theta + \sin\theta\,\frac{dr}{d\theta} \end{aligned}\right\} \qquad (7)$$

Using the chain rule on (7), we obtain

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{r\cos\theta + \sin\theta\,\frac{dr}{d\theta}}{-r\sin\theta + \cos\theta\,\frac{dr}{d\theta}} \qquad (8)$$

2.5.1(L) continued

It is worth reflecting on equation (8) for a while before we continue. $\frac{dy}{dx}$ yields the slope of the line tangent to the curve or, more simply, what we've defined as the slope of the curve. Since equation (8) is an identity for expressing $\frac{dy}{dx}$ in terms of r and θ, it follows that the right side of (8) is a formula for slope (in the original sense) in terms of polar coordinates as well. That is, we must not feel that the traditional definition of slope does not apply in polar coordinates, but rather, as should be obvious from (8), the formula is rather complicated in polar coordinates. This would serve as another motivation for inventing the angle ψ had we proceeded from the point of view of part (b). That is, recognizing that $\frac{dy}{dx}$ looked extremely complex in polar form, we might want to find a related angle whose tangent was more readily computed in polar coordinates. Of course, had we proceeded from this point of view, it is not clear that we would have discovered that ψ was the angle that simplified things for us (as compared with our approach in part (a) where it was almost self-evident that we should pick ψ as our reference angle).

Quite apart from anything else, this discussion again focuses our attention on how the basic concepts are qualitatively independent of any coordinate system, while quantitatively, the ease of compution will depend on the choice of coordinate system.

At any rate, returning to our immediate problem, we observe that $\frac{dy}{dx}$ is tan ϕ, where ϕ and ψ are related as below.

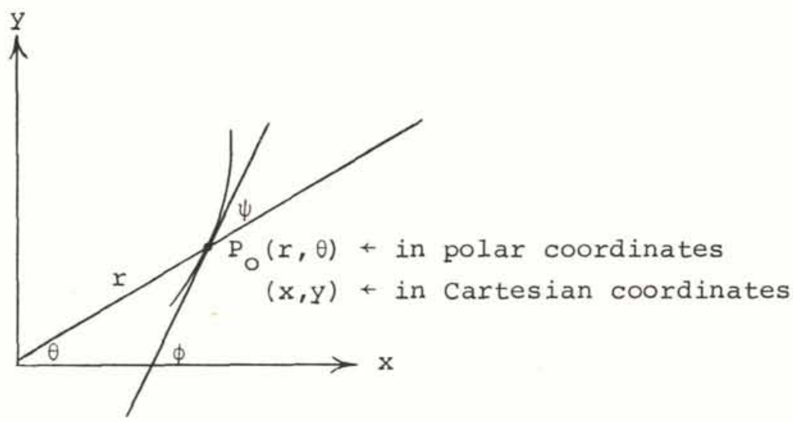


Figure 3

2.5.1(L) continued

From Figure 3, we see that

$\psi = \phi - \theta$

whereupon

$$\tan \psi = \tan(\phi - \theta). \qquad (9)$$

Using the identity for tan(A - B), (9) becomes

$$\tan \psi = \frac{\tan \phi - \tan \theta}{1 + \tan \phi \tan \theta} \qquad (10)$$

By definition, $\frac{dy}{dx} = \tan \phi$. Hence (10) becomes

$$\tan \psi = \frac{\frac{dy}{dx} - \tan \theta}{1 + \frac{dy}{dx} \tan \theta}. \qquad (11)$$

(Notice that our ultimate aim is to express tan ψ as a function of r and θ. For this reason, there is no need to make a substitution for tan θ in either (10) or (11) since this term is already expressed in the proper variables.)

We now put the result of equation (8) into equation (11), and we obtain

$$\tan \psi = \frac{\dfrac{r \cos \theta + \sin \theta \frac{dr}{d\theta}}{-r \sin \theta + \cos \theta \frac{dr}{d\theta}} - \tan \theta}{1 + \left[\dfrac{r \cos \theta + \sin \theta \frac{dr}{d\theta}}{-r \sin \theta + \cos \theta \frac{dr}{d\theta}}\right] \tan \theta} \qquad (12)$$

Writing $\tan \theta = \frac{\sin \theta}{\cos \theta}$, equation (12) becomes

2.5.1(L) continued

$$\tan\psi = \frac{\dfrac{r\cos\theta + \sin\theta\,\dfrac{dr}{d\theta}}{-r\sin\theta + \cos\theta\,\dfrac{dr}{d\theta}} - \dfrac{\sin\theta}{\cos\theta}}{1 + \left[\dfrac{r\cos\theta + \sin\theta\,\dfrac{dr}{d\theta}}{-r\sin\theta + \cos\theta\,\dfrac{dr}{d\theta}}\right]\dfrac{\sin\theta}{\cos\theta}}$$

$$= \frac{\left\{\dfrac{r\cos^2\theta + \sin\theta\cos\theta\,\dfrac{dr}{d\theta} + r\sin^2\theta - \sin\theta\cos\theta\,\dfrac{dr}{d\theta}}{\cos\theta\,[-r\sin\theta + \cos\theta\,\frac{dr}{d\theta}]}\right\}}{\dfrac{-r\sin\theta\cos\theta + \cos^2\theta\,\dfrac{dr}{d\theta} + r\sin\theta\cos\theta + \sin^2\theta\,\dfrac{dr}{d\theta}}{\cos\theta\,(-r\sin\theta + \cos\theta\,\frac{dr}{d\theta})}}$$

$$= \frac{\dfrac{r\,(\sin^2\theta + \cos^2\theta)}{\cos\theta\,[-r\sin\theta + \cos\theta\,\frac{dr}{d\theta}]}}{\dfrac{\frac{dr}{d\theta}\,(\sin^2\theta + \cos^2\theta)}{\cos\theta\,[-r\sin\theta + \cos\theta\,\frac{dr}{d\theta}]}}$$

$$= \frac{r}{\left(\frac{dr}{d\theta}\right)}\,. \tag{13}$$

Notice that while (b) was independent of (a), the amount of "dog work" involved in going from (12) to (13) etc. shows that we can obtain the correct answer mechanically, although we may be sacrificing any feeling as to what is really going on in the procedure. Part (a), on the other hand, while providing us with no connection between polar coordinates and Cartesian coordinates, allows us to develop $\tan\psi$ in a meaningful way.

One final note for those of us who have become "touchy" about dividing by 0. The validity of the last step in going from (12) to (13) requires that

$$\cos\theta\,[-r\sin\theta + \cos\theta\,\frac{dr}{d\theta}] \neq 0.$$

The only way this could equal zero would be

2.5.1(L) continued

(i) $\cos\theta = 0$

or

(ii) $-r\sin\theta + \cos\theta\,\frac{dr}{d\theta} = 0.$

If (i) applies, then $\theta = \frac{\pi}{2}$ and $\psi = \phi - \frac{\pi}{2}$. If (ii) applies, then $r\sin\theta = \cos\theta\,\frac{dr}{d\theta}$, or

$$\frac{r}{\left(\frac{dr}{d\theta}\right)} = \frac{\cos\theta}{\sin\theta}\,.$$

i.e.,

$$\tan\psi = \cot\theta = \tan\left(\frac{\pi}{2} - \theta\right)$$

Therefore,

$$\psi = \frac{\pi}{2} - \theta.$$

This corresponds to $\phi\ (= \psi + \theta) = \frac{\pi}{2}$, whence the tangent line is parallel to the y-axis.

c. The main aim of this part of the exercise is to give you a better feeling for how we use ψ to trace a curve in polar coordinates in a manner analogous to the way we use ϕ to trace a curve in Cartesian coordinates. In this example, we shall make no attempt to trace the curve but rather we shall locate one specific tangent line at one specific point on the curve. It is hoped that the reader will be able to extend the idea to locating other tangent lines to the curve, and we shall provide some experience in this respect in the later exercises.

For now, we have that our curve has as its polar equation, $r = \sin^2\theta + 1$, from which we deduce that $\frac{dr}{d\theta} = 2\sin\theta\cos\theta = \sin 2\theta$. Hence, our formula for $\tan\psi$ yields

$$\tan\psi = \frac{r}{\left(\frac{dr}{d\theta}\right)} = \frac{\sin^2\theta + 1}{\sin 2\theta}\,. \qquad (14)$$

2.5.1(L) continued

To compute $\tan\psi$ at $(\frac{5}{4},\frac{\pi}{6})$, we need only use (14) with $\theta = \frac{\pi}{6}$. This yields

$$\tan\psi = \frac{\left(\frac{1}{2}\right)^2 + 1}{\sin\frac{\pi}{3}} = \frac{\frac{5}{4}}{\frac{1}{2}\sqrt{3}} = \frac{5}{2\sqrt{3}} = \frac{5\sqrt{3}}{6}. \tag{15}$$

Thus, $\tan\psi \approx 1.44$.

The construction proceeds as follows. We know that $P_0(\frac{5}{4},\frac{\pi}{6})$ is a point on C. Pictorially,

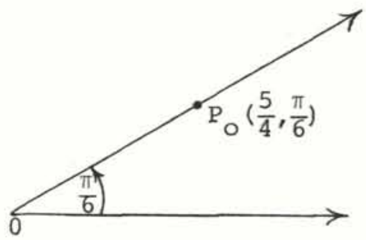


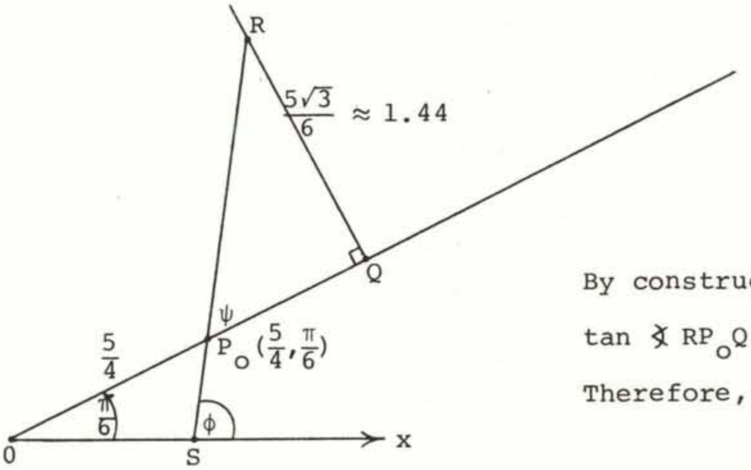
Figure 4

We next recall that ψ measures the angle (measured with $\vec{OP}_0$ as our reference line) between the tangent to C at P_0 and $\vec{OP}_0$. In other words, with P_0 as vertex, we want to construct the angle ψ so that $\tan\psi \approx +1.44$,* and one side of ψ lies along $\vec{OP}_0$.

One way of doing this is to proceed along $\vec{OP}_0$ 1 unit from P_0, calling this point Q. At Q, we construct a line at right angles to $\vec{OP}_0$ in the direction "outside" the angle, and locate R on this line so that $\overline{QR} = 1.44$ (to be precise, $\frac{5\sqrt{3}}{6}$). This guarantees that $\sphericalangle RP_0Q$ is ψ. Again, pictorially,

*We emphasize the sign of $\tan\psi$ to indicate that for positive values of $\tan\psi$, ψ is measured in the positive (counter clockwise) direction relative to $\vec{OP}_0$. If $\tan\psi$ is negative, we construct ψ with the opposite sense.

2.5.1(L) continued



By construction,

$\tan \measuredangle RP_oQ = \frac{5\sqrt{3}}{6} \div 1 = \frac{5\sqrt{3}}{6}$.

Therefore, $\measuredangle RP_oQ = \psi$.

Figure 5

At any rate, the straight line determined by P_o and R is the line tangent to C at P_o.

To relate this discussion to the discussion in part (b), we have extended the line P_oR to intersect the x-axis at S. The line that we have found makes an angle of ϕ with the positive x-axis where $\tan \phi = \frac{dy}{dx}$.

d. Here, we are simply trying to demonstrate the equivalence, conceptually - not computationally - of the methods of part (a) and part (b) in finding tangent lines to polar curves. In (c), we maintained the philosophy of part (a). In (d), we shall maintain the philosophy of (b).

From equation (8), we have

$$\frac{dy}{dx} = \frac{r \cos \theta + \sin \theta \frac{dr}{d\theta}}{-r \sin \theta + \cos \theta \frac{dr}{d\theta}}.$$

In our present example, $r = \sin^2\theta + 1$ and $\frac{dr}{d\theta} = \sin 2\theta$. [This part is the same, of course, as in (c)]. Putting this into (8), yields

$$\frac{dy}{dx} = \frac{(\sin^2\theta + 1) \cos \theta + \sin \theta \sin 2\theta}{-(\sin^2\theta + 1) \sin \theta + \cos \theta \sin 2\theta}. \tag{16}$$

2.5.1(L) continued

We wish to compute $\frac{dy}{dx}$ at P_o, which corresponds to equation (16) with $\theta = \frac{\pi}{6}$. We obtain:

$$\left.\frac{dy}{dx}\right]_{P_o} = \frac{(\frac{1}{4} + 1)\frac{1}{2}\sqrt{3} + \frac{1}{2}(\frac{1}{2}\sqrt{3})}{-(\frac{1}{4} + 1)\frac{1}{2} + (\frac{1}{2}\sqrt{3})(\frac{1}{2}\sqrt{3})} = \frac{\frac{5}{8}\sqrt{3} + \frac{1}{4}\sqrt{3}}{-\frac{5}{8} + \frac{3}{4}}$$

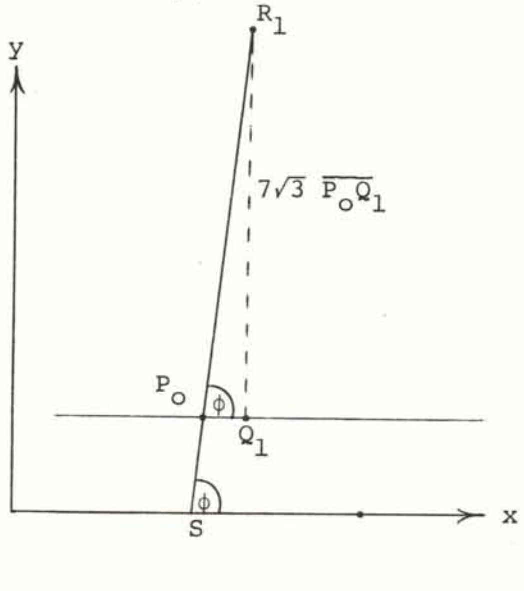
$$= \frac{\frac{7}{8}\sqrt{3}}{\frac{1}{8}}$$

$$= 7\sqrt{3}$$

$$\approx 12.1.$$

What we do now is locate P_o just as before. We then mark off a horizontal segment $\vec{P_oQ_1}$. At Q_1, we move in the direction of the positive y-axis to R_1 such that $|\vec{Q_1R_1}| = 7\sqrt{3}|\vec{P_oQ_1}|$. The line joining P_o to R_1 is the required tangent line.

Pictorially,



φ, as well as the point S, are the same in Figures 5 and 6.

Figure 6

2.5.1(L) continued

Our claim is (and we hope this is quite clear) that the lines SP_o in Figures 5 and 6 coincide if these two figures are superimposed. Doing this we find:

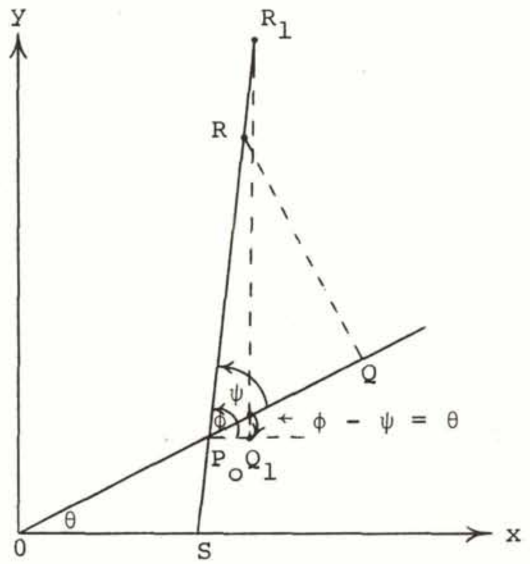


Figure 7

Hopefully, Figure 7 indicates that the methods of parts (a) and (b) are two different computational techniques for finding the same piece of information. It is important that you try to learn to be flexible and be able to use either the Cartesian or the Polar forms, depending upon which lends itself best to the computation you are trying to make.

2.5.2(L)

The main aim of this exercise is two-fold: (1) We want to re-emphasize that $\frac{dy}{dx}$ still exists conceptually, even when we are dealing with polar coordinates, and (2) we would like to reinforce the technique of Exercise 2.4.4 (b), by showing how we can now add slope to our graphing techniques.

In Exercise 2.4.4 (b), we plotted the graph of $r = \sin 2\theta$. One way of getting a further check of our sketch is to check the slope at some indicated point.

2.5.2(L) continued

This is what we are going to do in part (a) of this exercise. We choose as our check point $P_0(\frac{1}{2}\sqrt{3},\frac{\pi}{6})$ on $r = \sin 2\theta$.

a. From the previous exercise, we have that

$$\frac{dy}{dx} = \frac{r\cos\theta + \sin\theta\frac{dr}{d\theta}}{-r\sin\theta + \cos\theta\frac{dr}{d\theta}}. \qquad (1)$$

In this problem, $r = \sin 2\theta$ and $\frac{dr}{d\theta} = 2\cos 2\theta$. Putting this into (1) yields

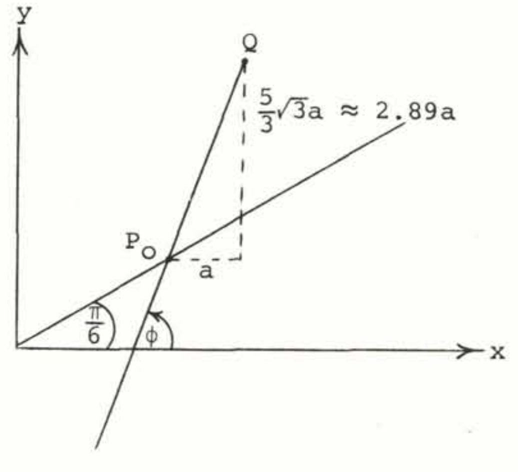
$$\frac{dy}{dx} = \frac{\sin 2\theta\cos\theta + 2\sin\theta\cos 2\theta}{-\sin 2\theta\sin\theta + 2\cos\theta\cos 2\theta}. \qquad (2)$$

At the point $P_0(\frac{1}{2}\sqrt{3},\frac{\pi}{6})$, we find $\frac{dy}{dx}$ by letting $\theta = \frac{\pi}{6}$ in (2). Therefore,

$$\left.\frac{dy}{dx}\right|_{P_0} = \frac{(\frac{1}{2}\sqrt{3})(\frac{1}{2}\sqrt{3}) + 2(\frac{1}{2})(\frac{1}{2})}{(-\frac{1}{2}\sqrt{3})(\frac{1}{2}) + 2(\frac{1}{2}\sqrt{3})(\frac{1}{2})} = \frac{\frac{3}{4}+\frac{1}{2}}{\frac{1}{4}\sqrt{3}} = \frac{5}{\sqrt{3}} = \frac{5}{3}\sqrt{3}$$

$$\approx 2.89.$$

Graphically, this means



Here, "a" refers to an arbitrarily chosen length. The key is that

$$\tan\phi = \frac{\frac{5}{3}\sqrt{3}a}{a} = \frac{5}{3}\sqrt{3}.$$

Therefore, P_0Q is tangent to $r = \sin 2\theta$ at $P_0(\frac{1}{2}\sqrt{3},\frac{\pi}{6})$.

2.5.2(L) continued

Thus, we can now "shape" $r = \sin 2\theta$ exactly at P_o. We have illustrated this in Figure 3 of Exercise 2.4.4.

b. We also sense in our sketch of Figure 1 in Exercise 2.4.4 that the maximum height attained by $r = \sin 2\theta$ in the first quadrant occurs at a point for which θ lies between 45° and 60°. We may now sharpen this result by using equation (2) to see where $\frac{dy}{dx} = 0$. For a fraction to equal 0, its numerator must equal 0. Applying this to (2), we find that $\frac{dy}{dx}$ is 0 provided

$$\sin 2\theta \cos \theta + 2 \sin \theta \cos 2\theta = 0. \qquad (3)$$

Rewriting (3) with $\sin 2\theta = 2 \sin \theta \cos \theta$ and $\cos 2\theta = \cos^2\theta - \sin^2\theta$, we obtain

$$2 \sin \theta \cos^2\theta + 2 \sin \theta (\cos^2\theta - \sin^2\theta) = 0$$

or

$$2 \sin \theta (2 \cos^2\theta - \sin^2\theta) = 0. \qquad (4)$$

Obviously, the case $\sin \theta = 0$ in (4) is not of importance in our present quest. Rather, the information we desire comes from

$$2 \cos^2\theta - \sin^2\theta = 0. \qquad (5)$$

There are a number of ways to solve (5) (for example, replace $\sin^2\theta$ by $1 - \cos^2\theta$ to get an equation involving only $\cos^2\theta$), but perhaps the quickest is to divide both sides of (5) by $\cos^2\theta$. (This is permissible except when $\cos \theta = 0$. Since $\cos \theta = 0$ at $\theta = \frac{\pi}{2}$, this value of θ is not near the point we are seeking here.) We then obtain

$$2 = \frac{\sin^2\theta}{\cos^2\theta}$$

or

$$\tan \theta = \pm\sqrt{2}. \qquad (6)$$

2.5.2(L) continued

Since we are in the first quadrant, $\tan\theta$ is positive; hence, (6) indicates that

$$\tan\theta = \sqrt{2}$$

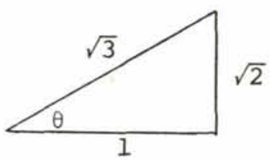
or

$$\theta = \tan^{-1}\sqrt{2}. \qquad (7)$$

To obtain a better numerical feeling for θ, we approximate $\sqrt{2}$ by 1.414 and use tan tables to conclude

$$\theta \approx 54.7°.$$

Moreover, using the reference triangle



$$r = \sin 2\theta = 2\sin\theta\cos\theta = 2\,\frac{\sqrt{2}}{\sqrt{3}}\,\frac{1}{\sqrt{3}} = \frac{2}{3}\sqrt{2} \approx 0.94.$$

Thus, the point we seek is $(\frac{2}{3}\sqrt{2}, \tan^{-1}\sqrt{2})$, or, approximately (0.94, 54.7°). Notice that this point is higher than (1, 45°) even though its r-value is less.

c. For further drill, we now compute the point in the first quadrant at which $r = \sin 2\theta$ has a vertical tangent (i.e. the point at which the curve rises the most rapidly).

In this case, $\frac{dy}{dx} = \infty$ (= tan 90°) and in terms of equation (2), this means our denominator is 0. Thus,

$$-\sin 2\theta\sin\theta + 2\cos\theta\cos 2\theta = 0. \qquad (8)$$

This means

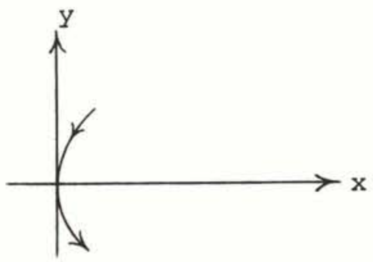
2.5.2(L) continued

$-2 \sin^2\theta \cos\theta + 2 \cos\theta (\cos^2\theta - \sin^2\theta) = 0$

or

$$2 \cos\theta (\cos^2\theta - 2 \sin^2\theta) = 0. \qquad (9)$$

The solution of (9) obtained from $\cos\theta = 0$, so that $\theta = \frac{\pi}{2}$, indicates that the curve is tangent to the y-axis at (0,0) as it enters the fourth quadrant from the first. i.e.



The other factor of (9) yields

$2 \sin^2\theta = \cos^2\theta$

or

$\pm\sqrt{2} \sin\theta = \cos\theta$

Therefore,

$\frac{\sin\theta}{\cos\theta} = \pm\frac{1}{\sqrt{2}}$

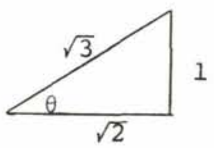
Therefore

$\tan\theta = \frac{1}{\sqrt{2}} = \frac{1}{2}\sqrt{2} \approx 0.707$

(since, again, θ is in the first quadrant).

When $\tan\theta = \frac{1}{\sqrt{2}}$, we have

2.5.2(L) continued



$$r = \sin 2\theta = 2 \sin\theta \cos\theta = 2 \frac{1}{\sqrt{3}} \frac{\sqrt{2}}{\sqrt{3}} = \frac{2}{3}\sqrt{2}$$

Thus, the other point we seek is

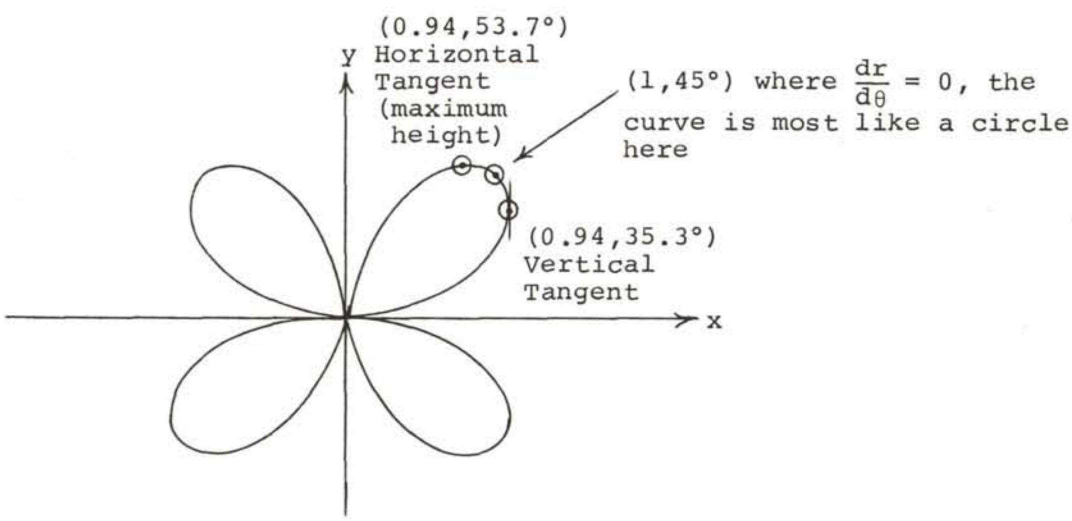
$$\left(\frac{2}{3}\sqrt{2},\ \tan^{-1} \frac{1}{\sqrt{2}}\right)$$

or letting $\frac{1}{\sqrt{2}} = 0.707$, we have approximately

(0.94,35.3°)

[which is symmetric to (0.94,53.7°) with respect to $\theta = \frac{\pi}{4}$].

In any event, it should now be clear how $\frac{dy}{dx}$ (slope) can be used to help us sketch polar curves.



2.5.3

Here we want to emphasize that polar coordinates do not require reference to Cartesian coordinates and also to have you see that some calculations are easier to make in polar form than by converting to Cartesian form.

We have already seen that

$$\tan \psi = \frac{r}{\frac{dr}{d\theta}}$$

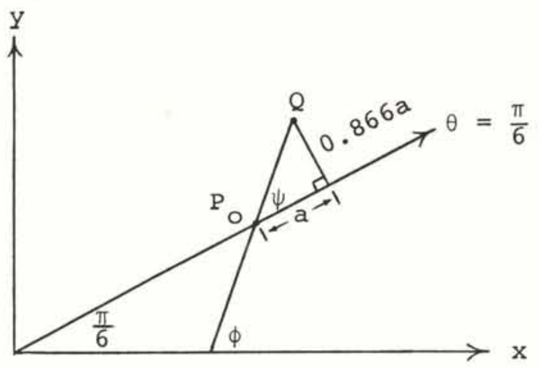
$$= \frac{\sin 2\theta}{2 \cos 2\theta}$$

$$= \frac{1}{2} \tan 2\theta. \qquad (1)$$

Equation (1) now allows us to construct the tangent line to $r = \sin 2\theta$ at any point (r,θ) without ever referring to the angle ϕ. Again, by way of review, if we again pick $(\frac{1}{2}\sqrt{3},\frac{\pi}{6})$, we may obtain from (1) with $\theta = \frac{\pi}{6}$

$$\tan \psi = \frac{1}{2} \tan \frac{\pi}{3} = \frac{\sqrt{3}}{2} = 0.866.$$

Thus,



Again, "a" refers to an arbitrarily chosen length, so that

$$\tan \psi = \frac{0.866a}{a}$$

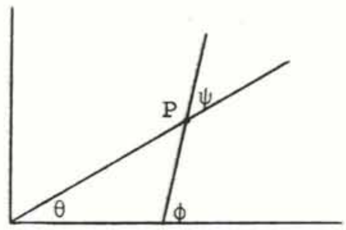
$$= 0.866$$

Figure 1

2.5.3 continued

Notice how our constructions are made with respect to $\vec{R}$ and P_o and make no reference to Cartesian coordinates.

Now, keep in mind that tan ϕ exists independently of any concept of $\frac{dy}{dx}$. In our present context



so that

$$\phi = (\theta + \psi)$$

whence

$$\tan \phi = \tan (\theta + \psi) = \frac{\tan \theta + \tan \psi}{1 - \tan \theta \tan \psi}.$$

From (1), this becomes

$$\tan \phi = \frac{\tan \theta + \frac{1}{2} \tan 2\theta}{1 - \frac{1}{2} \tan \theta \tan 2\theta}. \qquad (2)$$

In other words, aside from the fact that ψ is a natural angle in polar coordinates, the point is that if we even need tan ϕ, it is easily obtained from tan ψ, as shown in our derivation of equation (2).

For example, with $\theta = \frac{\pi}{6}$, (2) yields

$$\tan \phi = \frac{\tan \frac{\pi}{6} + \frac{1}{2} \tan \frac{\pi}{3}}{1 - \frac{1}{2} \tan \frac{\pi}{6} \tan \frac{\pi}{3}} = \frac{\frac{1}{\sqrt{3}} + \frac{1}{2}\sqrt{3}}{1 - \frac{1}{2}\,\frac{1}{\sqrt{3}}\sqrt{3}} = \frac{2 + 3}{2\sqrt{3}\ (\frac{1}{2})}$$

$$= \frac{5}{\sqrt{3}} = \frac{5}{3}\sqrt{3}$$

2.5.3 continued

as in the previous exercise.

2.5.4

a. From $r = \sin^2\theta$, we have that $\frac{dr}{d\theta} = 2 \sin\theta \cos\theta = \sin 2\theta$. Therefore,

$$\tan\psi = \frac{r}{\frac{dr}{d\theta}} = \frac{\sin^2\theta}{\sin 2\theta}. \qquad (1)$$

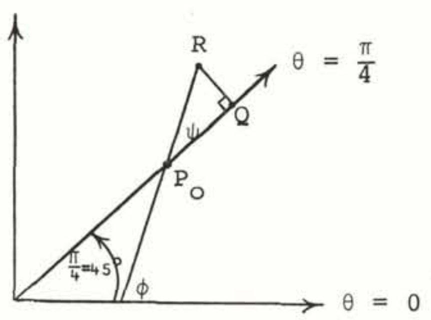
When $\theta = \frac{\pi}{4}$, equation (1) becomes

$$\tan\psi = \frac{\sin^2\frac{\pi}{4}}{\sin\frac{\pi}{2}} = \frac{1}{2} \qquad (2)$$

Therefore,

$$\psi = \tan^{-1}\frac{1}{2} \approx 26.5°.$$

b. From (a), $\tan\psi = \frac{1}{2}$. Hence, at $P_o(\frac{1}{2},\frac{\pi}{4})$



① $\overline{QR} = \frac{1}{2}\,\overline{P_oQ}$. Therefore, $\tan\psi = \frac{1}{2}$, as required.

② Therefore, P_oR is tangent to $r = \sin^2\theta$ at $P_o(\frac{1}{2},\frac{\pi}{4})$.

Figure 1

c. From Figure 1, $\phi = 45° + \psi \approx 45° + 26.5° \approx 71.5°$. Therefore,

$$\tan\phi \approx 3.$$

2.5.4 continued

More analytically,

$$\phi = \theta + \psi \rightarrow \tan \phi = \tan (\theta + \psi) = \frac{\tan \theta + \tan \psi}{1 - \tan \theta \tan \psi}.$$

Picking $\theta = \frac{\pi}{4}$ and using (2) yields

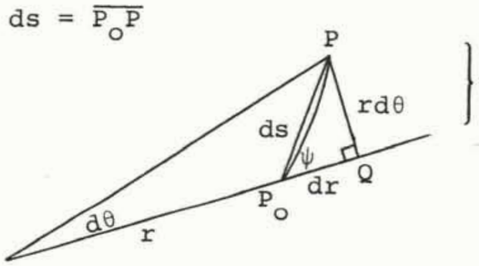
$$\tan \phi = \frac{\tan \frac{\pi}{4} + \frac{1}{2}}{1 - (\tan \frac{\pi}{4}) \frac{1}{2}} = \frac{1 + \frac{1}{2}}{1 - \frac{1}{2}} = 3$$

(exactly, not approximately!).

2.5.5(L)

Our aim here is to show how arc length is computed in polar coordinates. An "intuitive" proof is suggested by our diagram in Exercise 2.5.1(L), namely,

$ds = \overline{P_oP}$



where for small $d\theta$, $rd\theta$ approximates $\overline{PQ}$

Figure 1

As drawn, Figure 1 yields

$$\tan \psi = \frac{rd\theta}{dr} = r\left(\frac{d\theta}{dr}\right) = \frac{r}{\left(\frac{dr}{d\theta}\right)}$$

which checks with the "rigorous" answer obtained in Exercise 2.5.1.

If we accept Figure 1, triangle P_oQP yields

$$ds^2 = (rd\theta)^2 + dr^2 \qquad (1)$$

2.5.5(L) continued

and if our curve is in the form $r = f(\theta)$, so that $\frac{dr}{d\theta}$ is convenient to compute, (1) takes the form

$$\left(\frac{ds}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2 \tag{2}$$

where (2) was obtained from (1) by dividing through by $(d\theta)^2$.

From (2),

$$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}. \tag{3}$$

The trouble with (3) is that it was obtained by taking liberties with "small" amounts. In Part 1 of our course, we saw that the $\frac{0}{0}$ form often entered in subtle forms and as a result, operations which seemed "obvious" were, in fact, false. So (3) was derived under rather nebulous conditions.

On the other hand, to derive (3) rigorously is a difficult chore (just as it was when we tackled it under the heading of Cartesian coordinates in Part 1).

Thus, the aim of part (a) of this exercise is to show how we can convert information in Cartesian coordinates into corresponding information in polar coordinates, thus utilizing familiar knowledge to obtain the unfamiliar. It is important to notice, however, that, while this approach is conceptually simple, ultimately, the serious student would like to verify that certain properties of curves can be expressed in any coordinate system without reference to any other system.

a. We already know from Cartesian coordinates that

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}. \tag{5}$$

In our problem, r is given as a function of θ; hence, x and y may also be viewed as functions of θ. That is, we may differentiate both $x = r\cos\theta$ and $y = r\sin\theta$ implicitly with respect to θ, and as in equation (8) of Exercise 2.5.1, we would obtain

2.5.5(L) continued

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{r \cos\theta + \sin\theta \frac{dr}{d\theta}}{-r \sin\theta + \cos\theta \frac{dr}{d\theta}} . \qquad (6)$$

Squaring (6) yields

$$\left(\frac{dy}{dx}\right)^2 = \frac{r^2 \cos^2\theta + 2r \sin\theta \cos\theta \frac{dr}{d\theta} + \sin^2\theta \left(\frac{dr}{d\theta}\right)^2}{r^2 \sin^2\theta - 2r \sin\theta \cos\theta \frac{dr}{d\theta} + \cos^2\theta \left(\frac{dr}{d\theta}\right)^2} ,$$

whereupon

$$1 + \left(\frac{dy}{dx}\right)^2 = \frac{\left(r^2 \sin^2\theta - 2r \sin\theta \cos\theta \frac{dr}{d\theta} + \cos^2\theta \left(\frac{dr}{d\theta}\right)^2 + r^2 \cos^2\theta + 2r \sin\theta \cos\theta \frac{dr}{d\theta} + \sin^2\theta \left(\frac{dr}{d\theta}\right)^2\right)}{\left(r^2 \sin^2\theta - 2r \sin\theta \cos\theta \frac{dr}{d\theta}\right) + \cos^2\theta \left(\frac{dr}{d\theta}\right)^2}$$

$$= \frac{r^2 + \left(\frac{dr}{d\theta}\right)^2}{\left(-r \sin\theta + \cos\theta \frac{dr}{d\theta}\right)^2} \qquad (7)$$

Putting (7) into (5) yields

$$\frac{ds}{dx} = \frac{\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}}{\left(-r \sin\theta + \cos\theta \frac{dr}{d\theta}\right)} . \qquad (8)$$

By the chain rule, $\frac{ds}{d\theta} = \frac{ds}{dx}\frac{dx}{d\theta}$ and, since we have already seen that $\frac{dx}{d\theta} = -r \sin\theta + \cos\theta \frac{dr}{d\theta}$, we see from (8) that

2.5.5(L) continued

$$\frac{ds}{d\theta} = \frac{ds}{dx}\frac{dx}{d\theta} = \left(\frac{\sqrt{r^2 + (\frac{dr}{d\theta})^2}}{-r \sin\theta + \cos\theta \frac{dr}{d\theta}}\right)\left(-r\sin\theta + \cos\theta\frac{dr}{d\theta}\right)^*$$

Therefore,

$$\frac{ds}{d\theta} = \sqrt{r^2 + (\frac{dr}{d\theta})^2}^{\,**} \tag{9}$$

and this agrees with (3).

b. If $r = \sec\theta$, then $\frac{dr}{d\theta} = \sec\theta\tan\theta$. Therefore, applying (9) to the curve $r = \sec\theta$, $0 \leqslant \theta \leqslant \frac{\pi}{4}$, yields

$$\frac{ds}{d\theta} = \sqrt{\sec^2\theta + \sec^2\theta\tan^2\theta}$$

$$= \sec\theta^{***}\sqrt{1 + \tan^2\theta}$$

and since $1 + \tan^2\theta = \sec^2\theta$, we finally obtain

$$\frac{ds}{d\theta} = \sec^2\theta$$

Therefore,

---

*In obtaining (7), we let $-r\sin\theta + \cos\theta\frac{dr}{d\theta}$ denote the square root of $r^2\sin^2\theta - 2r\sin\theta\cos\theta\frac{dr}{d\theta} + \cos^2\theta(\frac{dr}{d\theta})^2$. We could just as legally have chosen $r\sin\theta - \cos\theta\frac{dr}{d\theta}$. The effect of this would be to change the sign in (9). In this event, we would have used absolute values anyway since we think of s increasing as θ sweeps through the plane.

**Earlier, we mentioned that allowing r to be negative caused some nasty consequences in polar forms of curves. One very nice result in (9) is that r always appears as r^2, whereby we may neglect worrying about whether we are dealing with r or -r since $r^2 = (-r^2)$.

***Technically, we should write $|\sec\theta|$ since $\frac{ds}{d\theta}$ is non-negative. In this problem, no harm is done since $0 \leqslant \theta \leqslant \frac{\pi}{4}$ implies $\sec\theta > 0$.

2.5.5(L) continued

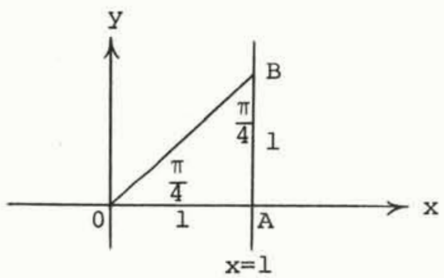
$$s = \int_0^{\frac{\pi}{4}} \sec^2\theta \, d\theta$$

$$= \tan\theta \Big|_0^{\frac{\pi}{4}}$$

$$= \tan\frac{\pi}{4} - \tan 0 = 1 - 0$$

$$= 1. \qquad (10)$$

The more astute student may have noticed that we rigged this problem in the sense that the polar equation $r = \sec\theta$ represents the line $x = 1$ in Cartesian coordinates. We did this so that we would have a nice example to show that equation (9) really does yield arc length in the usual sense. Namely,



That is, (10) represents the length of AB and the above diagram shows that (10) is trivially true.

2.5.6

While this is not meant as a learning exercise, there is a chance that you fell into a trap that confuses you. (If you didn't, take the following note lightly.)

2.5.6 continued

Since $r = 1 + \cos\theta$, $\frac{dr}{d\theta} = -\sin\theta$. Hence

$$\begin{aligned} r^2 + \left(\frac{dr}{d\theta}\right)^2 &= (1 + \cos\theta)^2 + (-\sin\theta)^2 \\ &= 1 + 2\cos\theta + \cos^2\theta + \sin^2\theta \\ &= 1 + 2\cos\theta + 1 \\ &= 2 + 2\cos\theta. \end{aligned} \tag{1}$$

Since our curve is traced out exactly once as θ varies continuously from 0 to 2π, we may use (1) and write

$$s = \int_0^{2\pi} \sqrt{2 + 2\cos\theta}\, d\theta = \sqrt{2}\int_0^{2\pi} \sqrt{1 + \cos\theta}\, d\theta. \tag{2}$$

To handle (2), we recall that $\cos\frac{\theta}{2} = \pm\sqrt{\frac{1 + \cos\theta}{2}}$. Notice that here the ambiguous sign is necessary since, for example, if $\pi \leqslant \theta \leqslant 2\pi$, then $\frac{\pi}{2} \leqslant \frac{\theta}{2} \leqslant \pi$, in which case $\cos\frac{\theta}{2}$ is <u>negative</u>. Thus, if you, as we so often do, merely wrote $\cos\frac{\theta}{2} = \sqrt{\frac{1 + \cos\theta}{2}}$, or $\sqrt{2}\cos\frac{\theta}{2} = \sqrt{1 + \cos\theta}$, equation (2) would have become

$$s = 2\int_0^{2\pi} \cos\frac{\theta}{2}\, d\theta = 4\sin\frac{\theta}{2}\Big|_0^{2\pi} = 0 \tag{3}$$

which is obviously incorrect.

What we should have done, remembering that arc length was non-negative, was to write (3) as

$$s = 2\int_0^{2\pi} \left|\cos\frac{\theta}{2}\right| d\theta. \tag{4}$$

Notice that (4) is compatible with (2) in the sense that since $\sqrt{1 + \cos\theta}$ means the positive square root, the integrand in (2) is never negative.

2.5.6 continued

It is possible that you avoided this pitfall by "lucking out." That is, if you had used symmetry and computed s by

$$s = 2\int_0^{\pi} \sqrt{2 + 2\cos\theta}\, d\theta \tag{5}$$

$$= 4\int_0^{\pi} \cos\frac{\theta}{2}\, d\theta \tag{6}$$

$$= 8 \sin\frac{\theta}{2}\Big|_0^{\pi}$$

$$= 8$$

you would have obtained the correct answer. The reason is that for $0 \leqslant \theta \leqslant \pi$, $\cos\frac{\theta}{2} = |\cos\frac{\theta}{2}|$ since $0 \leqslant \frac{\theta}{2} \leqslant \frac{\pi}{2}$.

As a final note, observe that (4) and (6) are equivalent since $|\cos\frac{\theta}{2}| = \cos\frac{\theta}{2}$ if $0 \leqslant \theta \leqslant \pi$, while $|\cos\frac{\theta}{2}| = -\cos\frac{\theta}{2}$ if $\pi \leqslant \theta \leqslant 2\pi$. Hence, (4) would become

$$s = 2\left[\int_0^{\pi} |\cos\frac{\theta}{2}|\, d\theta + \int_{\pi}^{2\pi} |\cos\frac{\theta}{2}|\, d\theta\right]$$

$$= 2\left[\int_0^{\pi} \cos\frac{\theta}{2}\, d\theta - \int_{\pi}^{2\pi} \cos\frac{\theta}{2}\, d\theta\right] \tag{7}$$

Therefore,

$$s = 2\left[\int_0^{\pi} \cos\frac{\theta}{2}\, d\theta + \int_{2\pi}^{\pi} \cos\frac{\theta}{2}\, d\theta\right],$$

and since $\int_{2\pi}^{\pi} \cos\frac{\theta}{2}\, d\theta = \int_0^{\pi} \cos\frac{\theta}{2}\, d\theta$ (they are lengths of congruent curves), it follows that

2.5.6 continued

$$s = 4 \int_0^{\pi} \cos \frac{\theta}{2} \, d\theta$$

as asserted in (6). Notice also that had (6) not been derived, the correct answer would have followed from (7).

The main caution is to beware where the integrand is algebraically negative when adding positive quantities.

2.5.7

a. Obviously, we know that since we have a circle of radius $\frac{1}{2}$, the answer is clearly $\frac{\pi}{4}$. Pictorially,

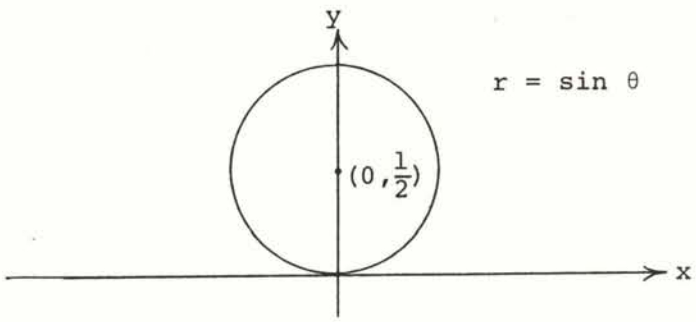


Figure 1

The warning is to remember how θ is measured! To obtain the curve in Figure 1, θ need only vary from 0 to π. Indeed, with this observation, we obtain, as expected,

$$A = \frac{1}{2} \int_0^{\pi} r^2 d\theta = \frac{1}{2} \int_0^{\pi} \sin^2\theta \, d\theta$$

$$= \frac{1}{4} \int_0^{\pi} (1 - \cos 2\theta) d\theta$$

$$= \frac{1}{4} \left(\theta - \frac{\sin 2\theta}{2}\right) \Big|_0^{\pi}$$

$$= \frac{\pi}{4} . \qquad (1)$$

2.5.7 continued

However, had we mechanically written

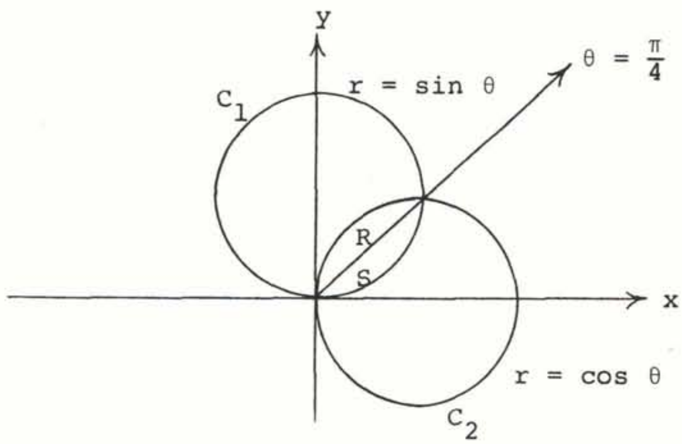
$$A = \frac{1}{2}\int_0^{2\pi} r^2 d\theta$$

we would have obtained

$$A = \frac{1}{4}(\theta - \sin\theta)\Big|_0^{2\pi} = \frac{\pi}{2} \qquad (2)$$

which is twice the desired answer. (In terms of our optional supplementary notes, the back-map of $r = \sin\theta$, $0 \leqslant \theta \leqslant 2\pi$, covers the circle twice.) If we desire to use (2) as the correct answer, we must imagine that each time the curve is traced out, it is covered with a thin shield and the next copy is made on the shield. In our present example, as θ varies from 0 to 2π, two circles, which happen to be congruent, are traced out.

b. Just as in the Cartesian case, we must know where the curves intersect. In this case,



We note that S is that portion of C_1 between $\theta = 0$ and $\theta = \frac{\pi}{4}$. Thus,

$$A_S = \frac{1}{2}\int_0^{\frac{\pi}{4}} r^2 d\theta = \frac{1}{2}\int_0^{\frac{\pi}{4}} \sin^2\theta\, d\theta \quad (\text{since for } C_1,\ r = \sin\theta).$$

2.5.7 continued

On the other hand, R is that portion of C_2 between $\theta = \frac{\pi}{4}$ and $\theta = \frac{\pi}{2}$. Therefore,

$$A_R = \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^2\theta \, d\theta.$$

Therefore,

$$A = \frac{1}{2}\int_0^{\frac{\pi}{4}} \sin^2\theta \, d\theta + \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^2\theta \, d\theta$$

or

$$A = \frac{1}{4}\int_0^{\frac{\pi}{4}} (1 - \cos 2\theta)d\theta + \frac{1}{4}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (1 + \cos 2\theta)d\theta$$

$$= \frac{1}{4}\left(\theta - \frac{1}{2}\sin 2\theta\right)\Big|_0^{\frac{\pi}{4}} + \frac{1}{4}\left(\theta + \frac{1}{2}\sin 2\theta\right)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

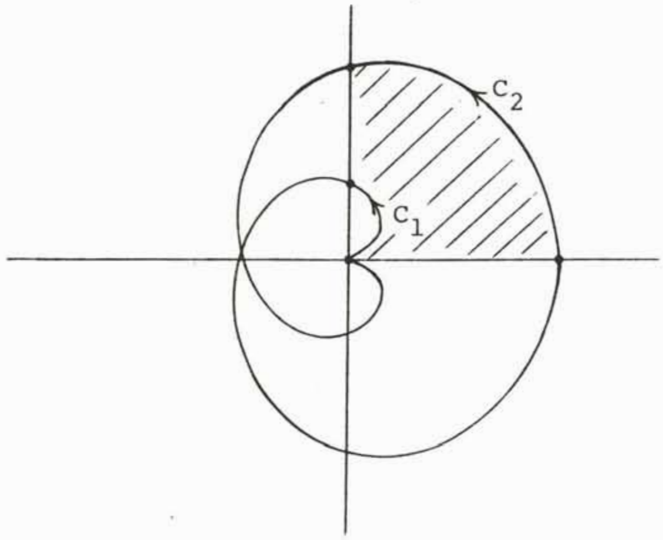
$$= \frac{1}{4}\left(\frac{\pi}{4} - \frac{1}{2}\right) + \frac{1}{4}\left[\left(\frac{\pi}{2} + 0\right) - \left(\frac{\pi}{4} + \frac{1}{2}\right)\right]$$

$$= \frac{\pi}{16} - \frac{1}{8} + \frac{\pi}{8} - \frac{\pi}{16} - \frac{1}{8}$$

$$= \frac{\pi}{8} - \frac{1}{4} \quad .$$

c. From Exercise 2.4.6 (b), we have that the graph of $r = \sin\frac{\theta}{4}$ is given by

2.5.7 continued



and we desire the area of the shaded region.

Now we must be careful of where we are. We notice that C_1 was traced out as θ went from 0 to $\frac{\pi}{2}$. Thus, the area enclosed by C_1 in the first quadrant is given by

$$A_1 = \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin^2 \frac{\theta}{4}\, d\theta$$

(i.e. for C_1 as well as C_2, $r = \sin \frac{\theta}{4}$).

On the other hand, C_2 was traced out as θ went from 2π to $\frac{5\pi}{2}$. Hence, the total area enclosed by C_2 in the first quadrant is

$$A_2 = \frac{1}{2}\int_{2\pi}^{\frac{5\pi}{2}} \sin^2 \frac{\theta}{4}\, d\theta$$

and since the region we want is inside C_2 but outside C_1, our answer is given by

2.5.7 continued

$$A_2 - A_1 = \frac{1}{2}\int_{2\pi}^{\frac{5\pi}{2}} \sin^2 \frac{\theta}{4}\, d\theta - \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin^2 \frac{\theta}{4}\, d\theta$$

$$= \frac{1}{4}\int_{2\pi}^{\frac{5\pi}{2}} (1 - \cos \frac{\theta}{2}) d\theta - \frac{1}{4}\int_0^{\frac{\pi}{2}} (1 - \cos \frac{\theta}{2}) d\theta$$

$$= \frac{1}{4} (\theta - 2 \sin \frac{\theta}{2}) \Big|_{2\pi}^{\frac{5\pi}{2}} - \frac{1}{4} (\theta - 2 \sin \frac{\theta}{2}) \Big|_0^{\frac{\pi}{2}}$$

$$= \frac{1}{4} \left[(\frac{5\pi}{2} - 2 \sin \frac{5\pi}{4}) - (2\pi - 0)\right] - \frac{1}{4} \left[(\frac{\pi}{2} - 2 \sin \frac{\pi}{4}) - 0\right]$$

$$= \frac{1}{4} \left[\frac{\pi}{2} - 2 \left(-\frac{1}{2}\sqrt{2}\right)\right] - \frac{1}{4} \left[\frac{\pi}{2} - 2 \left(\frac{1}{2}\sqrt{2}\right)\right]$$

$$= \frac{1}{4} \left[(\frac{\pi}{2} + \sqrt{2}) - (\frac{\pi}{2} - \sqrt{2})\right]$$

$$= \frac{1}{2}\sqrt{2}.$$