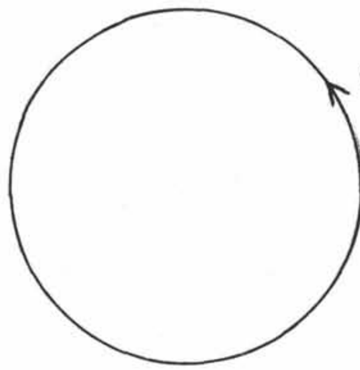
Unit 8: Green's Theorem

5.8.1(L)

a. We have



so that parametrically C is given by

$$\left.\begin{aligned} x &= \cos t \\ y &= \sin t \end{aligned}\right\} \quad 0 \leq t \leq 2\pi. \qquad (1)$$

From (1) $\frac{dx}{dt} = -\sin t$, $\frac{dy}{dt} = \cos t$, so

$$\oint_C -x^2y\,dx + y^2x\,dy$$

$$= \int_0^{2\pi} \left(- x^2y \frac{dx}{dt} + y^2x \frac{dy}{dt}\right) dt$$

$$= \int_0^{2\pi} [- \cos^2 t \sin t(- \sin t) + \sin^2 t \cos t(\cos t)]\,dt$$

$$= \int_0^{2\pi} 2\sin^2 \cos^2 t\,dt$$

5.8.1(L) continued

$$= \int_0^{2\pi} \frac{1}{2} \sin^2 2t dt$$

$$= \frac{1}{2} \int_0^{2\pi} \frac{1 - \cos 4t}{2} dt$$

$$= \frac{1}{4} [t - \frac{1}{4} \sin 4t] \Big|_{t=0}^{2\pi}$$

$$= \frac{\pi}{2} .$$

b. We have $M = -x^2y$ and $N = y^2x$.

Hence,

$$\frac{\partial M}{\partial y} = -x^2$$

$$\frac{\partial N}{\partial x} = y^2.$$

Since Green's Theorem

$$\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA_R ,$$

so

$$\oint_C - x^2ydx + y^2xdy = \iint_R (y^2 + x^2) dA_R \qquad (1)$$

where $R = \{(x,y) = x^2 + y^2 \leq 1\}$.

5.8.1(L) continued

The right side of (1) suggests polar coordinates, and we obtain

$$\oint_C - x^2ydx + y^2xdy = \int_0^{2\pi}\int_0^1 (r^2)rdrd\theta$$

$$= \int_0^{2\pi} d\theta \int_0^1 r^3dr$$

$$= 2\pi \left(\frac{1}{4}\right)$$

$$= \frac{\pi}{2} .$$

While the actual computations involved in this exercise are straight-forward, there are a few remarks that make this exercise worthy of being a learning exercise. First of all, let us observe that part (a) was sufficiently simple so that the knowledge of Green's Theorem was hardly necessary to solve the problem (although we readily admit that there will be times when Green's Theorem will actually be a great computational aid). Rather we used part (b) merely as a check, so to speak, of the validity of Green's Theorem.

Secondly notice that without Green's Theorem we would be able to determine that

$$\oint_C Mdx + Ndy$$

(under the usual suitable conditions) would be zero provided that Mdx + Ndy was exact. This in turn means that $M_y = N_x$. Notice, however, that we had no way of measuring how "close" Mdx + Ndy was to being exact (whatever that might mean) if $M_y \neq N_x$. Green's Theorem, however, now tells us what

$$\oint_C Mdx + Ndy$$

5.8.1(L) continued

looks in terms of $N_y - M_x$. In other words, Green's Theorem does give us a better quantitative idea of how the line integral

$$\oint_C Mdx + Ndy$$

is affected by how "nearly equal" M_y and N_x are.

Finally, notice that Green's Theorem tells us, once and for all, that the value of

$$\oint_C Mdx + Ndy$$

cannot depend on the parametric form of C (a fact we assumed in the previous unit and tried to make seem more plausible through the exercises). Namely the region R enclosed by the (oriented)* curve C is independent of the parametric equation used to represent C. Consequently the fact that

$$\oint_C Mdx + Ndy = \iint_R (N_x - M_y)\,dA_R$$

guarantees that the line integral is the same for all parametrized forms of C since

$\iint_R (N_x - M_y)\,dA_R$ does not depend on C.

---

*Again, if we change the sense of C we change the sign of

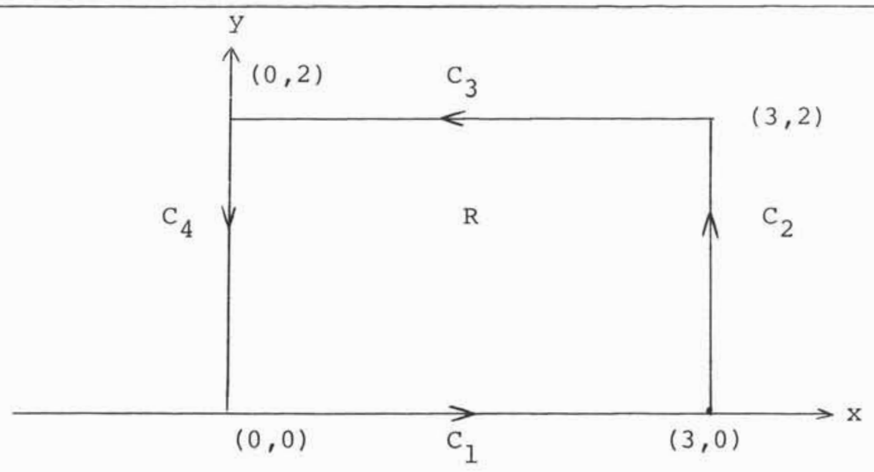
$\int_C Mdx + Ndy$

(just as in the usual definite integral where

$\int_a^b f(x)dx = -\int_b^a f(x)dx$).

What is important is that our form of Green's Theorem hinges on the given orientation, were the orientation reversed the integrand on the right side of Green's Theorem would be $(M_y - N_x)$.

5.8.2



$C = C_1 \cup C_2 \cup C_3 \cup C_4$.

Hence

$$\int_C 2ydx - 2xdy = \sum_{i=1}^{4} \int_{C_i} 2ydx - 3xdy$$

$$= \int_{C_1} 2ydx - 3xdy + \ldots + \int_{C_4} 2ydx - 3xdy. \qquad (1)$$

Now in parametric form we have

C_1: $y = 0$, x varies from 0 to 3; therefore, $\frac{dy}{dx} = 0$
C_2: $x = 3$, y varies from 0 to 2; therefore, $\frac{dx}{dy} = 0$ (2)
C_3: $y = 2$, x varies from 3 to 0; therefore, $\frac{dy}{dx} = 0$
C_4: $x = 0$, y varies from 2 to 0; therefore, $\frac{dx}{dy} = 0$

Putting the results of (2) into (1) we have

$$\int_C 2ydx - 3xdy = \int_0^3 (2y - 3x\,\frac{dy}{dx})\,dx \quad (\text{along } C_1)$$

$$+ \int_0^2 (2y\,\frac{dx}{dy} - 3x)\,dy \quad (\text{along } C_2)$$

5.8.2 continued

$$+ \int_3^0 \left(2y - 3x \frac{dy}{dx}\right) dx \qquad \text{(along } C_3\text{)}$$

$$+ \int_2^0 \left(2y \frac{dx}{dy} - 3x\right) dy \qquad \text{(along } C_4\text{)}$$

$$= \int_0^3 [2(0) - 3x(0)]dx$$

$$+ \int_0^2 [2y(0) - 3(3)]dy$$

$$+ \int_3^0 [2(0) - 3(0)]dy$$

$$+ \int_2^0 [2(2) - 3(0)]dx$$

$$= 0 - \int_0^2 9dx - \int_0^3 4dx + 0$$

$$= -9x \Big|_0^2 - 4x \Big|_0^3$$

$$= -18 - 12 = -30.$$

Now if we use Green's Theorem we have $M = 2y$, $N = -3x$; hence $N_x = -3$ and $M_y = 2$.

Accordingly

$$\iint_R (N_x - M_y)\, dA_R = \iint_R -5dA_R.$$

5.8.2 continued

In our present case $R = \{(x,y): 0 \leq x \leq 3,\ 0 \leq y \leq 2\}$.
Hence,

$$\oint_C 2ydx - 3xdy = \iint_R -5\ dA_R$$

$$= \int_0^2 \int_0^3 -5\ dxdy$$

$$= -5 \int_0^2 dy \int_0^3 dx$$

$$= -5(2)(3)$$

$$= -30.$$

5.8.3(L)

a. Our purpose here is to emphasize that the ordinary definite integral is a special case of a line integral. Namely, given

$$\int_a^b f(x)dx \quad (a < b)$$

we may write this as

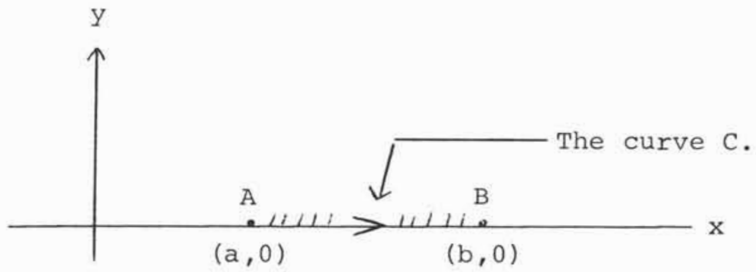
$$\int_C f(x)dx \tag{1}$$

where C is the curve consisting of the segment of the x-axis from (a,0) to (b,0). That is, C is given by

$$y = 0,\ a \leq x \leq b\ . \tag{2}$$

5.8.3(L) continued

Pictorially,



If we now look at (1) and let $M(x,y) = f(x)$ and $N(x,y) \equiv 0$ we have that

$$\int_a^b f(x)dx = \int_C Mdx + Ndy$$

$$= \int_a^b \left(M + N\frac{dy}{dx}\right) dx$$

$$= \int_a^b (f + 0)\ dx$$

$$= \int_a^b f(x)\ dx$$

and this circular chain of steps verifies that $\int_a^b f(x)dx$ may be viewed as a line integral.

b. While equation (1) is a valid interpretation of $\int_a^b f(x)dx$ it should be noted that $\int_C f(x)dx$ is a line integral even when C is not restricted to being a portion of the x-axis.

5.8.3(L) continued

In particular

$$\int_C f(x)\,dx = \int_C f(x)\,dx + 0dy$$

which clearly has the form $\int M dx + N dy$ with $M = f$ and $N = 0$.

For example, if we wished to compute the work done by the particle P moving from (0,0) to (3,9) along the curve $y = x^2$ under the influence of the force $\vec{F}(x,y) = f(x)\vec{i}$, we would have

$$w = \int_C \vec{F} \cdot d\vec{s}$$

$$= \int_C f(x)\vec{i} \cdot [dx\vec{i} + dy\vec{j}]$$

$$= \int_C f(x)\,dx^*.$$

At any rate, if we now let $M(x,y) = f(x)$ and $N(x,y) \equiv 0$, and apply Green's Theorem, we obtain

$$\oint_C f(x)\,dx = \oint_C f(x)\,dx + 0dy = \iint_R \left[\frac{\partial (0)}{\partial x} - \frac{\partial f(x)}{\partial y}\right] dA_R$$

*Notice that our remarks still make sense if $f = f(x,y)$. That is, $\int_C f(x,y)\,dx$ is also a well-defined line integral we are concentrating on $\int_C f(x)\,dx$ since the result we are investigating in this exercise holds for

$$\oint_C f(x)\,dx$$

but, in general, not for

$$\oint_C f(x,y)\,dx.$$

5.8.3(L) continued

$$= \iint_R 0 \, dA_R$$

$$= 0.$$

(Notice that our integrand could not have been identically zero had f depended on y as well as x since in that case $\frac{\partial f}{\partial y} \neq 0$)

A similar process, of course, allows us also to conclude that

$$\oint_C g(y)dy = 0.$$

c. Suppose C is given parametrically by

$$\left.\begin{array}{l} x = f(t) \\ y = g(t) \end{array}\right\} \quad a \leq t \leq b \qquad (3)$$

then

$$\int_C [M_1(x,y) + M_2(x,y)]dx + [N_1(x,y) + N_2(x,y)]dy$$

$$= \int_a^b \{[M_1(f(t),g(t) + M_2(f(t), g(t)]\frac{dx}{dt}$$

$$+ [N_1(f(t),g(t)) + N_2(f(t),g(t)]\frac{dy}{dt}\} dt$$

$$= \int_a^b [(M_1 + M_2)f'(t) + (N_1 + N_2)g'(t)]dt. \qquad (4)$$

The key point is that (4) is a definite integral involving a single real variable, and for this type of integral we already know such theorems as "the integral of a (finite) sum equals the sum of the integrals", etc., so that (4) becomes

5.8.3(L) continued

$$\int_a^b (M_1 + M_2)f'(t)dt + \int_a^b (N_1 + N_2)g'(t)dt$$

$$= \int_a^b M_1 f'(t)dt + \int_a^b M_2 f'(t)dt + \int_a^b N_1 g'(t)dt + \int_a^b N_2 g'(t)dt$$

$$= \int_a^b M_1(f(t), g(t))f'(t)dt + \int_a^b M_2(f(t),g(t))f'(t)dt$$

$$+ \int_a^b N_1(f(t),g(t))g'(t)dt$$

$$+ \int_a^b N_2(f(t),g(t))g'(t)dt. \qquad (5)$$

If we now recall the parametric form of C from equation (3) we see that (5) is equivalent to

$$\int_C M_1(x,y)dx + \int_C M_2(x,y)dx + \int_C N_1(x,y)dy + \int_C N_2(x,y)dy. \qquad (6)$$

[Notice here the tendency to "memorize" formulas in the natural left-to-right order* may make it easier for you to see that (6) implies (5) rather than that (5) implies (6)]

At any rate part (c) should convince us that such familiar theorems as "the integral of a sum is equal to the sum of the integrals" apply to line integrals as well as to the usual definite integral.

*As a more elementary example, the algebra student, given $(a + b)^2$, can usually write at once $a^2 + 2ab + b^2$, but given $a^2 + 2ab + b^2$ he usually takes a bit longer to recognize that this is $(a + b)^2$. That is, he tends to remember $(a + b)^2 = a^2 + 2ab + b^2$ in the left-to-right order.

5.8.3(L) continued

b. Here again we must get away from the left-to-right format and realize that Green's Theorem could have been written in the order

$$\iint_R (N_x - M_y)\,dA_R = \oint_C Mdx + Ndy. \tag{7}$$

Thus, given $\int_R (x^2 + y^2 dA_R$, the left side of (7) suggests that

$$\left.\begin{aligned} N_x &= x^2 \\ M_y &= -y^2 \end{aligned}\right\} \tag{8}$$

[although (8) is not unique; for example, we could have assumed that $N_x = 2x^2 + y^2$ and $M_y = x^2$ but let us not worry about this at the present time]

From equation (8) we obtain by direct integration that

$$\left.\begin{aligned} N(x,y) &= \tfrac{1}{3}x^3 + g(y)^* \\ M(x,y) &= -\tfrac{1}{3}y^3 + h(x)^* \end{aligned}\right\} \tag{9}$$

Putting the results of (8) and (9) into Green's Theorem in (7) we obtain

$$\iint_R (x^2 + y^2)\,dA_R = \oint_C [-\tfrac{1}{3}y^3 + h(x)]dx + [\tfrac{1}{3}x^3 + g(y)dy. \tag{10}$$

By part (c) this becomes

$$\iint_R (x^2 + y^2)\,dA_R = \oint_C -\tfrac{1}{3}y^3 dx + \oint_C h(x)dx$$

$$+ \oint_C \tfrac{1}{3}x^3 dy + \oint_C 3(y)dy. \tag{11}$$

*Recall that for partial derivatives $\frac{\partial F(x)}{\partial y} = 0$, etc. if x and y are independent variables.

5.8.3(L) continued

Now from part (b) we know that

$$\oint_C h(x)dx = 0 = \oint_C g(y)dy$$

so that (11) becomes

$$\iint_R (x^2 + y^2)dA_R = \oint_C -\frac{1}{3} y^3 dx + \oint_C \frac{1}{3} x^3 dy$$

$$= \frac{1}{3} \oint_C - y^3 dx + x^3 dy.$$

Note:

Had we used $N_x = 2x^2 + y^2$ and $M_y = x^2$, we would have obtained

$N = \frac{2}{3} x^3 + y^2 x + g_1(y)$

$M = x^2 y + h_1(x)$

whereupon (7) would have yielded

$$\iint_R (x^2 + y^2)dA_R = \oint_C [x^2 y + h_1(x)]dx + [\frac{2}{3} x^3 + y^2 x + g_1(y)]dy$$

$$= \oint_C x^2 y dx + (\frac{2}{3} x^3 + y^2 x) dy$$

so while

$\iint_R (x^2 + y^2)dA_R$ is a well-defined number, there are many

different integrands for which

$\iint_R (x^2 + y^2)dA_R$ is equal to $\oint_C Mdx + Ndy$.

5.8.4(L)

a. Suppose we let $M(x,y) = -y$ and $N(x,y) = x$, and then apply Green's Theorem. We obtain

$$\oint_c -ydx + xdy = \iint_R (N_x - M_y)^* dA_R$$

$$= 2 \iint_R dA_R$$

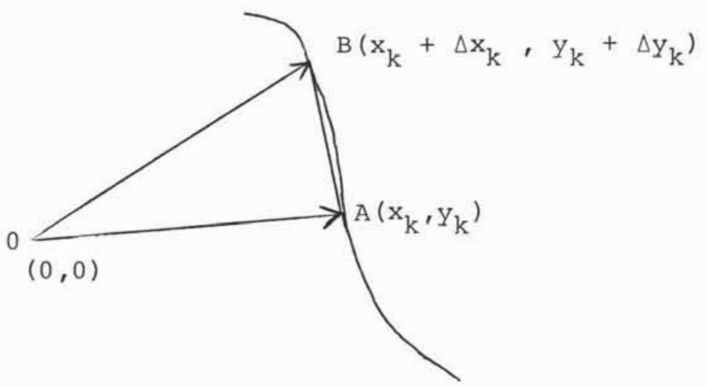
$$= 2A_R.$$

Hence, as asserted

$$A_R = \frac{1}{2} \oint_c -ydx + xdy.$$

b. The main aim of this part of the exercise is to show that while Green's Theorem was helpful for solving part (a) we could have obtained the same result without it.

Without trying to be extremely rigorous here, the key idea is that we now pick a triangle as our basic element of area. Thus,



*Notice that having $N(x,y) = x$ and $M(x,y) = -y$ is quite convenient but unnecessary. What seems to be the crucial thing is that $N_x - M_y$ is a constant.

5.8.4(L) continued

Now from our knowledge of vector geometry, the area of

$$AOB = \frac{1}{2}\left|\vec{OB} \times \vec{OA}\right|$$

$$= \frac{1}{2}\left|[(x_k + \Delta x_k)\vec{i} + (y_k + \Delta y_k)\vec{j}] \times [x_k\vec{i} + y_k\vec{j}]\right|$$

$$= \frac{1}{2}\left|(x_k + \Delta x_k)\, y_k\, (\vec{i} \times \vec{j}) + (y_k + \Delta y_k)x_k(\vec{j} \times \vec{i})\right|$$

$$= \frac{1}{2}\left|[(x_k + \Delta x_k)y_k - (y_k + \Delta y_k)x_k](\vec{i} \times \vec{j})\right|$$

$$= \frac{1}{2}\left|y_k\Delta x_k - x_k\Delta y_k\right|. \qquad (1)$$

Thus, our element of area written in differential form is

$$\left|\frac{1}{2}(ydx - xdy)\right|$$

which checks with the form in part (a).

Again our main aim in part (b) is to show that one could have arrived at

$$A_R = \frac{1}{2}\oint_C - ydx + xdy$$

without recourse to Green's Theorem, even though Green's Theorem is convenient.

5.8.5

Parametrically the ellipse C is given by

$$\left.\begin{aligned} x &= a\cos t \\ y &= b\sin t \end{aligned}\right\} \quad 0 \le t \le 2\pi \qquad (1)$$

5.8.5 continued

Hence by the previous exercise

$$A_R = \frac{1}{2} \oint_C - ydx + xdy$$

$$= \frac{1}{2} \int_C^{2\pi} \left(-y \frac{dx}{dt} + x \frac{dy}{dt}\right) dt,$$

or by (1),

$$A_R = \frac{1}{2} \int_0^{2\pi} [-b \sin t(-a \sin t) + a \cos t(b \cos t)]dt$$

$$= \frac{1}{2} \int_0^{2\pi} ab(\sin^2 t + \cos^2 t)dt$$

$$= \frac{1}{2} \int_0^{2\pi} ab\ dt$$

$$= \frac{1}{2} [2\ \pi ab]$$

$$= \pi ab.$$

[Had we elected to solve this problem without line integrals we would have had to evaluate

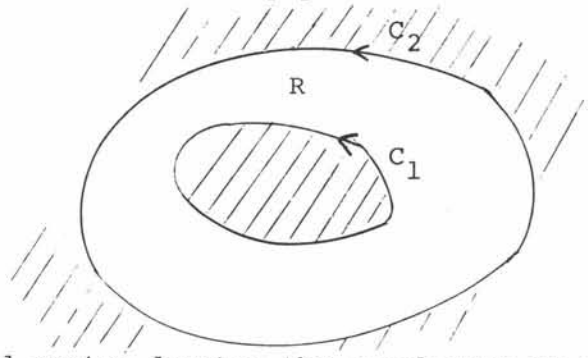
$$4 \int_0^a \frac{b}{a} \sqrt{a^2 - x^2}\ dx$$

which is certainly far from an overwhelming task, but our main aim in this exercise is to emphasize the formula

$$\oint_C - ydx + xdy = 2\iint_R dA_R\ ]$$

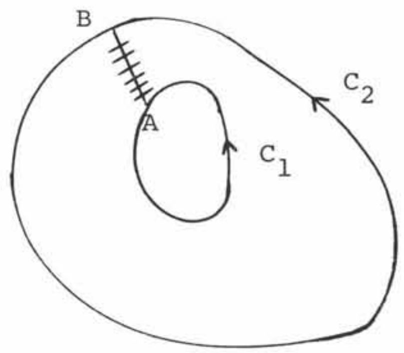
5.8.6

Recall that a connected region R is simply connected (by definition) if and only if its complement is connected. Clearly, then, the region R below is not simply connected.



The shaded region denotes the complement of R and since this shaded region consists of two disjoint pieces it is not connected. Hence, R is not simply-connected.

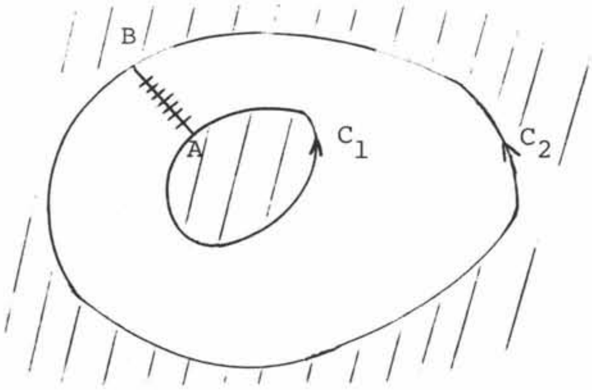
Suppose now we slit R by deleting the line AB.



Let R_1 denote the region obtained when the line AB is deleted from R. Certainly R and R_1 are different regions since the points on AB belong to R but not to R_1. In fact, $R_1 \subsetneq R$.

The point is that R_1 is simply connected because its complement is connected. Namely, the complement of R_1, since AB is part of the complement, is the shaded region.

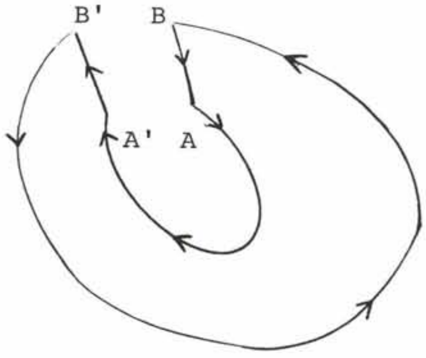
5.8.6 continued



The portion AB connects the 2 pieces which were not connected when we were dealing with the complement of R.

The key point is that we may think in terms of oriented boundaries for simply-connected regions quite easily. For example with regard to R_1 we may start at B follow C_2 until we return to B, then proceed along AB to A, whereupon we follow C_1 in the opposite sense (since oriented boundaries require that we move in the direction that keeps the enclosed region on our left) until we return to A, and then we close our boundary by returning to B along AB.

To see this more clearly let us exaggerate our region R_1 by pretending that AB has thickness. That is, view R_1 as



keeping in mind that A = A' and B = B' (i.e., A and A' coincide as do B and B'). The arrows indicate the oriented boundary.

Now since Mdx + Ndy is exact in the region R_1 as well as on its boundary, we conclude that

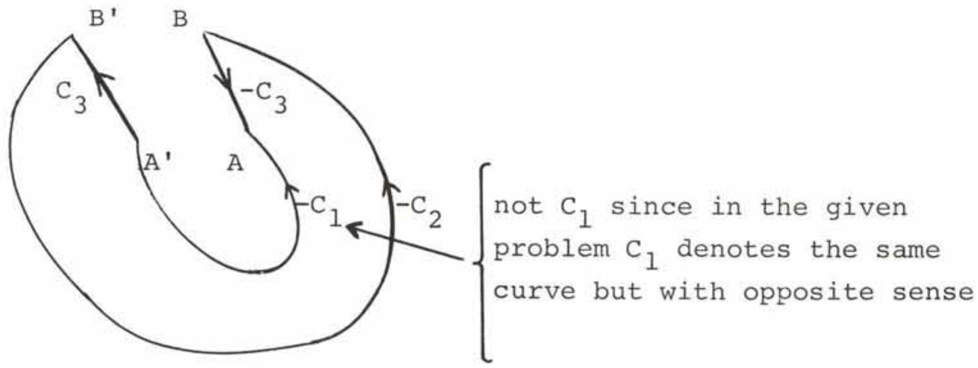
$$\oint_C M dx + N dy = 0 \tag{1}$$

5.8.6 continued

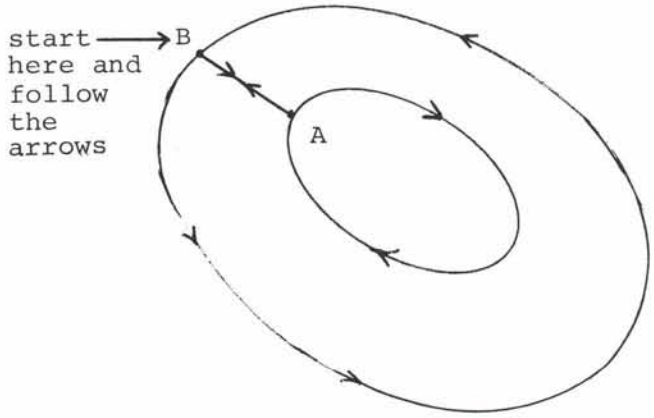
where C denotes the boundary of R_1 (not R).

If we let C_3 denote the directed segment AB then BA is denoted by $-C_3$ and we have that $C = C_1 \cup (-C_3) \cup (-C_2) \cup C_3$.

that is



In the "true" diagram we are saying that



In any event we have

$$\oint_C Mdx + Ndy = \int_{C_1} Mdx + Ndy + \int_{-C_3} Mdx + Ndy + \int_{-C_2} Mdx + Ndy$$

$$+ \int_{C_3} Mdx + Ndy. \qquad (2)$$

5.8.6 continued

The key structural property of line integrals which we now invoke is

$$\int_{c} Mdx + Ndy = - \int_{-c} Mdx + Ndy.$$

In particular

$$\int_{c_3} Mdx + Ndy = - \int_{-c_3} Mdx + Ndy$$

so that

$$\int_{c_3} Mdx + Ndy + \int_{-c_3} Mdx + Ndy = 0^*$$

and

$$\int_{-c_2} Mdx + Ndy = - \int_{c_2} Mdx + Ndy.$$

Combining these facts with (1) and (2) we obtain

$$0 = \oint_{c} Mdx + Ndy = \int_{c_1} Mdx + Ndy - \int_{c_2} Mdx + Ndy,$$

whence

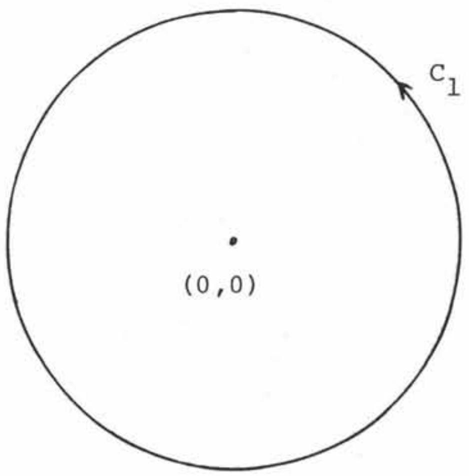
$$\int_{c_1} Mdx + Ndy = \int_{c_2} Mdx + Ndy.$$

*This is the crucial step that allows us to replace R by R_1 for even though $R \neq R_1$ we are integrating around the boundary and C_3 is the only place where the boundaries of R and R_1 are different. The fact that this contribution to the line integral vanishes is what allows us to conclude that no error is introduced when R_1 is used to replace R.

5.8.7(L)

The previous exercise supplies us with a very powerful result about exact differentials and line integrals (and this will come again later in the context of complex variables). With reference to the present exercise, we observe that the given line integral would be particularly convenient to evaluate had our oriented curve been, for example, the circle of radius 1 centered at the origin.

In this event we would have



$$C_1: \left.\begin{array}{l} x = \cos t \\ y = \sin t \end{array}\right\} 0 \leq t \leq 2\pi$$

Hence:

$$\frac{dx}{dt} = -\sin t,\ \frac{dy}{dt} = \cos t$$

$$x^2 + y^2 = 1$$

Then

$$\oint_{C_1} \frac{-ydx + xdy}{x^2 + y^2} = \int_0^{2\pi} (-y\frac{dx}{dt} + x\frac{dy}{dt}\ dt$$

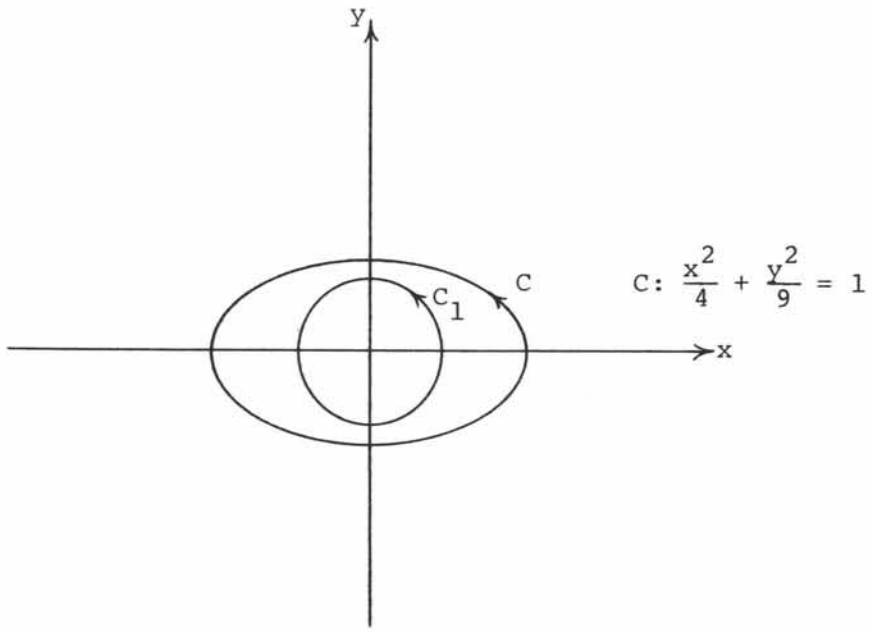
$$= \int_0^{2\pi} (\sin^2 t + \cos^2 t)dt$$

$$= \int_0^{2\pi} dt$$

$$= 2\pi\ .$$

5.8.7(L) continued

The trouble is that we do not want the integral around C_1 but rather around C, where



but, according to the previous exercise, since

$$\frac{-\,ydx + xdy}{x^2 + y^2}$$

is exact on and between the curves C_1 and C [recall that

$$\frac{-\,ydx}{x^2 + y^2} + \frac{-\,ydx}{x^2 + y^2}$$

is exact everywhere but at (0,0)], we may conclude that

$$\oint_c \frac{-\,ydx + xdy}{x^2 + y^2} = \oint_{c_1} \frac{-\,ydx + xdy}{x^2 + y^2}.$$

5.8.7(L) continued

Consequently,

$$\oint_C \frac{-\,ydx + xdy}{x^2 + y^2} = 2\pi$$

while we shall not belabor the point here, it might be worthwhile for you to try to compute

$$\oint_C \frac{-\,ydx + xdy}{x^2 + y^2}$$

directly. For example you might represent C by

$$\left.\begin{array}{l} x = a\cos t \\ y = b\sin t \end{array}\right\} 0 \le t \le 2$$

in which case

$$\oint_C \frac{-\,ydx + xdy}{x^2 + y^2} = \int_0^{2\pi} \frac{-ab\cos^2 t - ab\sin^2 t}{a^2\cos^2 t + b^2\sin^2 t}\,dt$$

$$= -\,ab\int_0^{2\pi} \frac{dt}{a^2\cos^2 t + b^2\sin^2 t} \qquad (1)$$

and the complexity of the integrand in (1) should make it clear why the method used in this exercise is desirable.

5.8.8

a. $M(x,y) = \dfrac{-\,y}{(x-2)^2 + y^2}$

$N(x,y) = \dfrac{x-2}{(x-2)^2 + y^2}$.

Therefore,

5.8.8 continued

$$\frac{\partial N}{\partial x} = \frac{[(x - 2)^2 + y^2](1) - (x - 2)[2(x - 2)]}{\left[(x - 2)^2 + y^2\right]^2}$$

$$= \frac{y^2 - (x - 2)^2}{[(x - 2)^2 + y^2]^2} \qquad (1)$$

$$\frac{\partial M}{\partial y} = \frac{[(x - 2)^2 + y^2](-1) - (-y)[2y]}{[(x - 2)^2 + y^2]^2}$$

$$= \frac{y^2 - (x - 2)^2}{[(x - 2)^2 + y^2]^2}. \qquad (2)$$

Comparing (1) and (2) we see that $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$ except when $x = 2$ and $y = 0$ (since then neither (1) nor (2) is well-defined in the sense that both are $\frac{0}{0}$ forms).

In summary then

$$\frac{-ydx + (x - 2)dy}{(x - 2)^2 + y^2}$$

is exact in any region R which does not contain the point (2,0).

b. Since the region enclosed by C_1 (including the boundary C_1 itself) does not contain (2,0), we obtain from Green's Theorem, using the fact that

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0, \quad \oint_{C_1} \frac{-ydx + (x - z)\,dy}{(x - 2)^2 + y^2} = 0. \qquad (3)$$

c. We cannot use Green's Theorem to evaluate

$$\oint_{C_2} \frac{-ydx + (x - 2)dy}{(x - 2)^2 + y^2}$$

5.8.8 continued

since (2,0) is included in the region enclosed by C_2. We could try to evaluate the given integral by "brute force" letting C_2 be written parametrically as

$$\left.\begin{aligned} x &= 4\cos t \\ y &= 4\sin t \end{aligned}\right\} \; 0 \leq t \leq 2$$

and we could then "hack out" the given integral. (Feel free to try this approach and see how you make out.)

On the other hand, we can take advantage of the technique of the previous exercise and observe that the oriented curve C_3 centered at the "trouble spot" (2,0) with radius, say, 1 is enclosed by C_2 and our integrand is exact in the region between C_2 and C_3 as well as on the boundaries of the curve. Hence

$$\oint_{C_2} \frac{-\,ydx + (x-2)dy}{(x-2)^2 + y^2} = \oint_{C_3} \frac{-\,ydx + (x-2)dy}{(x-2)^2 + y^2} \,. \qquad (4)$$

The reason for choosing C_3 is that it simplifies the integrand. For example, the Cartesian equation for C_3 is $(x-2)^2 + y^2 = 1$ in which case

$$\oint_{C_3} \frac{-\,ydx + (x-2)dy}{(x-2)^2 + y^2}$$

immediately simplifies to

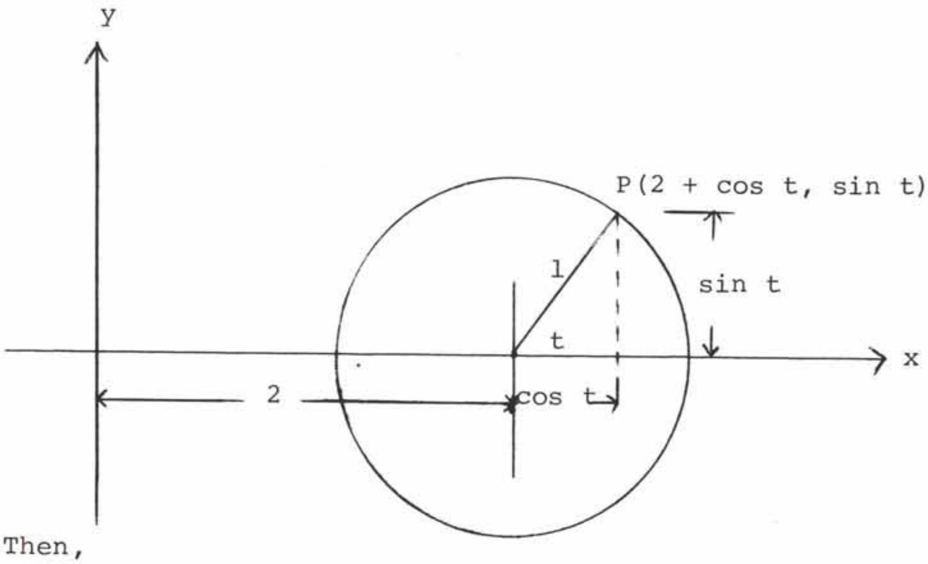
$$\oint_{C_3} -\,ydx + (x-2)dy \,. \qquad (5)$$

To evaluate (5) without too much mess, the polar representation for C_3 should be used. Namely

$$C_3\!: \left.\begin{aligned} x &= 2 + \cos t \\ y &= \sin t \end{aligned}\right\} \; 0 \leq t \leq 2 \,.$$

5.8.8 continued

Pictorially,



Then,

$dx = - \sin t\, dt$

$dy = \cos t\, dt$

$x - 2 = \cos t$

and we have

$$\int_{c_3} - ydx + (x - 2)dy = \int_0^{2\pi} [(- \sin t)(- \sin t\, dt) + \cos t(\cos t\, dt)]$$

$$= \int_0^{2\pi} [\sin^2 t + \cos^2 t]\, dt$$

$$= \int_0^{2\pi} dt$$

$$= 2\pi \ . \qquad (6)$$

Combining the results of (4), (5), and (6) we have

$$\oint_{c_2} \frac{- ydx + (x - 2)dy}{(x - 2)^2 + y^2} = 2\pi \ . \qquad (7)$$

5.8.8 continued

Comparing (3) and (7) we see that

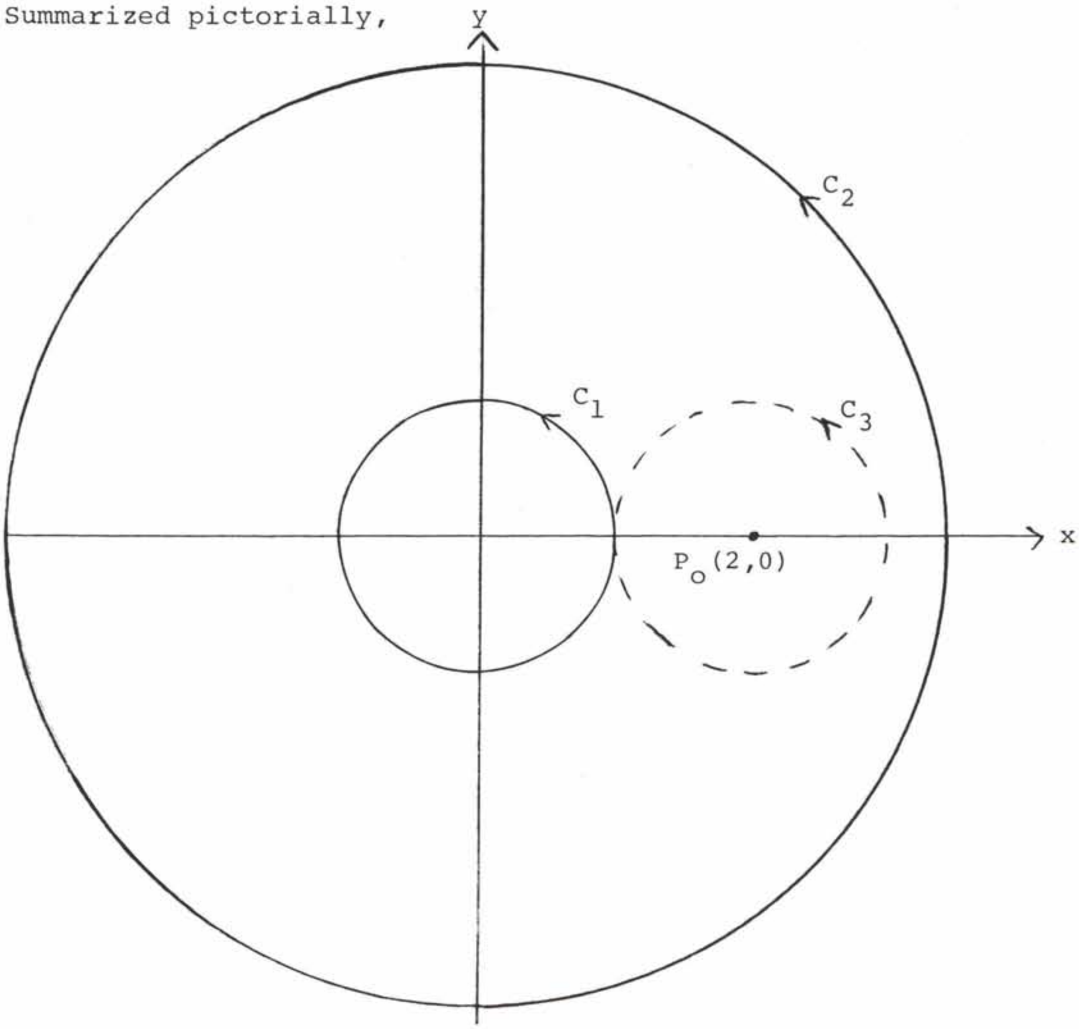
$$\oint_{c_2} \frac{-\,ydx + (x - 2)dy}{(x - 2)^2 + y^2} \neq \oint_{c_2} \frac{-\,ydx + (x - z)dy}{(x - 2)^2 + y^2}$$

d. This does not contradict the result established in Exercise 5.8.6 because

$$\frac{-\,ydx + (x - 2)dy}{(x - 2)^2 + y^2}$$

is not exact at <u>each</u> point in the region enclosed between C_1 and C_2.

Summarized pictorially,



5.8.8 continued

P_o is not enclosed between C_2 and C_3, and P_o is the only point at which our integrand is not exact.

Hence,

$$\oint_{C_2} = \oint_{C_3} = 2\pi.$$

P_o is enclosed between C_1 and C_2 and therefore $\oint_{C_1}$ need not equal $\oint_{C_2}$.

5.8.9

We have from Green's Theorem that

$$\oint_C - u_y u dx + u_x u dy = \iint_R \left[\frac{\partial(u_x u)}{\partial x} - \frac{\partial(-u_y u)}{\partial y}\right] dA_R$$

$$= \iint_R (u_x u_x + uu_{xx} + u_y u_y + uu_{yy}) dA_R$$

$$= \iint_R [(u_x^2 + u_y^2) + u(u_{xx} + u_{yy})] dA_R. \qquad (1)$$

Now, since $u = v - w$ and $v \equiv w$ on C, we have that on C , $u \equiv 0$ and hence

$$\oint_C - u_y u dx + u_x u dy = \oint_C u[-u_y dx + u_x dy]$$

$$= \oint_C 0[-u_y dx + u_x dy]$$

$$= 0.$$

5.8.9 continued

Moreover, since $v_{xx} + v_{yy} \equiv 0$ and $w_{xx} + w_{yy} \equiv 0$ in R it follows that $u_{xx} + u_{yy} = (v_{xx} - w_{xx}) + (v_{yy} - w_{yy})$

$$= (v_{xx} + v_{yy}) - (w_{xx} + w_{yy})$$

$$= 0 + 0$$

$$= 0 \text{ in R.}$$

Substituting these results into (1) yields

$$0 = \oint_C - u_y u dx + u_x u dy = \iint_R (u_x{}^2 + u_y{}^2) dA_R. \qquad (2)$$

Since $u_x{}^2 + u_y{}^2 \geq 0$ (i.e., the sum of real squares is non-negative) then

$$\iint_R (u_x{}^2 + u_y{}^2) dA_R \geq 0 \text{ and equals } 0 \leftrightarrow u_x{}^2 + u_y{}^2 \equiv 0.$$

Hence from (2) we conclude that $u_x{}^2 + u_y{}^2 \equiv 0$ or, in turn,

$$u_x \equiv u_y \equiv 0. \qquad (3)$$

Since $u_x \equiv u_y = 0$ implies u = constant we know that u = v - w = constant. But $v \equiv w$ on C means that u = v - w = 0 on C, and since u is a constant the fact that it is 0 on C means that it is C throughout R.

Thus v = w throughout R as asserted. In other words Green's Theorem gives us a proof that if C is the boundary of R and if

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \equiv 0$$

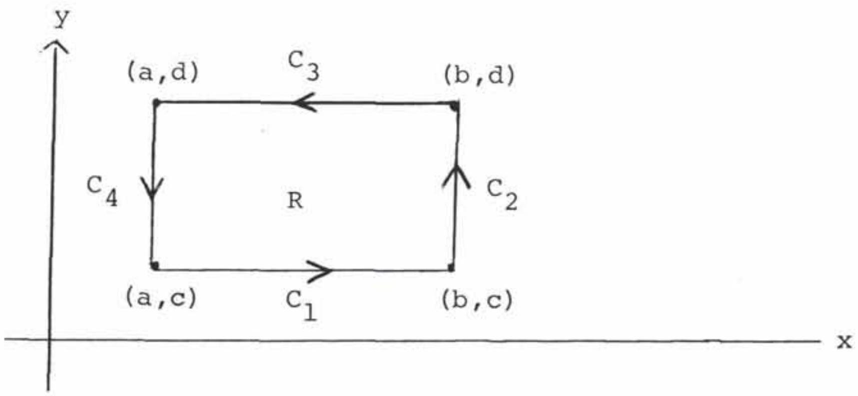
in R then w is uniquely determined once we know its behaviour on the boundary, C, of R.

5.8.10 (optional)

a. We are given

$$\int_C M(x,y)dx + N(x,y)dy$$

where $C = C_1 \cup C_2 \cup C_3 \cup C_4$, and



Parametrically we have

C_1: $y = c$, $a \leq x \leq b$ (therefore, $dy = 0$)
C_2: $x = b$, $c \leq y \leq d$ (therefore, $dx = 0$)
C_3: $y = d$, x varies from b to a (therefore, $dy = 0$)
C_4: $x = a$, y varies from d to c (therefore, $dx = 0$)

Hence

$$\int_{C_1} M(x,y)dx + N(x,y)dy = \int_a^b M(x,c)dx + N(x,c)0$$

$$= \int_a^b M(x,c)dx^* \qquad (1)$$

*Since c is a given constant (as are a,b, and d) $\int_a^b M(x,c)dx$ is an ordinary Riemann (definite) integral.

5.8.10 continued

Similarly

$$\int_{C_2} M(x,y)dx + N(x,y)dy = \int_c^d M(b,y)0 + N(b,y)dy$$

$$= \int_c^d N(b,y)dy \qquad (2)$$

$$\int_{C_3} M(x,y)dx + N(x,y)dy = \int_b^a M(x,d)dx + N(x,d)0$$

$$= \int_b^a M(x,d)dx$$

$$= -\int_a^b M(x,d)dx^* \qquad (3)$$

$$\int_{C_4} M(x,y)dx + N(x,y)dy = \int_d^c M(a,y)0 + N(a,y)dy$$

$$= \int_d^c N(a,y)dy$$

$$= -\int_c^d N(a,y)dy \qquad (4)$$

Thus, utilizing results (1), (2), (3), and (4) we have

$$\oint_C Mdx + Ndy = \sum_{i=1}^{4} \int_{C_i} Mdx + Ndy$$

*All we are doing here is using the usual fact about the definite integral that

$\int_a^b f(x)dx = -\int_a^b f(x)dx.$

As will be clear soon we use the form $\int_a^b$ rather than $\int_b^a$ so that (1) and (3) may be combined more conveniently.

5.8.10 continued

$$= \int_a^b M(x,c)\,dx + \int_c^d N(b,y)\,dy - \int_a^b M(x,d)\,dx - \int_c^d N(a,y)\,dy$$

$$= \int_c^d [N(b,y) - N(a,y)]dy + \int_a^b [M(x,c) - M(x,d)]dx. \qquad (5)$$

Result (5) is self-contained in its own right, but since our aim is to identify

$$\oint_C Mdx + Ndy$$

with a double integral we observe that

$$N(b,y) - N(a,y) = \int_{x=a}^{b} \frac{\partial N(x,y)}{\partial x} dx \qquad (6)$$

[i.e., $\int \frac{\partial N}{\partial x} dx = N + g(y)$ so that $\int_a^b \frac{\partial N}{\partial x} dx = N(b,y) + g(y) - [N(a,y) + g(y)] = N(b,y) - N(a,y)$]

and

$$M(x,c) - M(x,d) = \int_{y=d}^{c} \frac{\partial M(x,y)}{\partial y} dy$$

$$= -\int_c^d \frac{\partial M(x,y)}{\partial y} dy. \qquad (7)$$

Putting the results of (6) and (7) into (5) we obtain

$$\oint Mdx + Ndy = \int_c^d \left[\int_a^b \frac{\partial N}{\partial x} dx\right]dy + \int_a^b \left[- \int_c^d \frac{\partial M}{\partial y} dy\right]dx, \qquad (8)$$

and if we now assume that $\frac{\partial N}{\partial x}$ and $\frac{\partial M}{\partial y}$ are "sufficiently well behaved" to justify changing the order of integration, we have

$$\oint Mdx + Ndy = \int_c^d \int_a^b \frac{\partial N}{\partial x} dxdy - \int_a^b \int_c^d \frac{\partial M}{\partial y} dxdy$$

5.8.10 continued

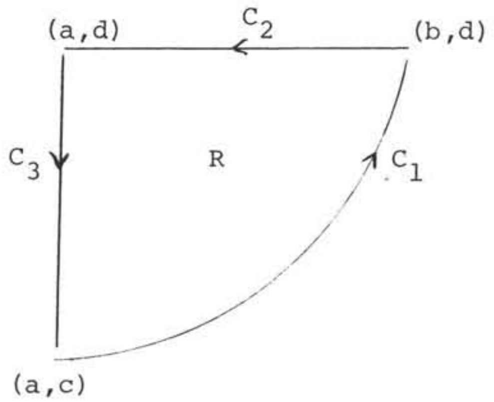
$$= \int_c^d \int_a^b \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$$

$$= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA_R$$

and this establishes Green's Theorem for the special case in which R is a rectangle with its sides parallel to the coordinate axes.

b. We have

$C = C_1 \cup C_2 \cup C_3$



Parametrically we have

$$\left.\begin{array}{ll} C_1: & y = f(x) \quad a \leq x \leq b \\ C_2: & y = d, \text{ x varies from b to a*} \\ C_3: & x = a, \text{ y varies from d to c.} \end{array}\right\} \qquad (9)$$

---

*Since $b > a$ we cannot write $b \leq x \leq a$ so we say "x varies from b to a". Now the "purist" would prefer to use only mathematical symbols and for this reason one finds that C_2 (or any curve of this nature) is often written in the parametric form

$$\left.\begin{array}{l} y = d \\ x = b - t \end{array}\right\} \quad 0 \leq t \leq b - a.$$

Notice in this form as t varies from 0 to $b - a$, x varies from $b - 0$ (= b) to $x = b - (b - a) = a$, and we obtain the same result as in our verbal form. Since we feel that "x varies from b to a" is sufficiently precise for our purposes, we have not resorted to the more rigorous parametric form.

5.8.10 continued

In any event

$$\oint_C Mdx + Ndy = \int_C Mdx + \int_C Ndy^*$$

$$= \int_{C_1} Mdx + \int_{C_2} Mdx + \int_{C_3} Mdx + \int_{C_1} Ndy + \int_{C_2} Ndy$$

$$+ \int_{C_3} Ndy,$$

but since dy = 0 on C_2 (i.e., y is constant on C_2) and dx = 0 on C_3, we have

$$\oint_C Mdx + Ndy = \int_{C_1} Mdx + \int_{C_2} Mdx + \int_{C_1} Ndy + \int_{C_3} Ndy. \qquad (10)$$

From (9)

$$\int_{C_1} Mdx = \int_a^b M(x,f(x)dx$$

and

$$\int_{C_2} Mdx = \int_b^a M(x,d)dx$$

$$= -\int_a^b M(x,d)dx$$

so that

$$\int_{C_1} Mdx + \int_{C_2} Mdx = \int_a^b M(x,f(x))dx - \int_a^b M(x,d)dx$$

$$= \int_a^b [M(x,f(x)) - M(x,d)]dx$$

$$= \int_a^b [\int_d^{f(x)} \frac{\partial M(x,y)}{\partial y} dy]dx$$

*We split the line integral this way to take advantage of those portions of C on which dx = 0 or dy = 0.

5.8.10 continued

$$= -\int_a^b \int_{f(x)}^d \frac{\partial M(x,y)}{\partial y}\,dydx$$

$$= -\iint_R \frac{\partial M}{\partial y}\,dA_R. \qquad (11)$$

In a similar way we evaluate

$$\int_{c_1} Ndy + \int_{c_3} Ndy$$

but to take advantage of the fact that dx is absent from these integrals, we rewrite C_1 parametrically as

$$x = f^{-1}(y),\ c \leq y \leq d$$

and this is possible because we have restricted f to being monotorically increasing (hence, 1-1).

We then obtain

$$\int_{c_1} Ndy = \int_a^d N(f^{-1}(y),y)\,dy$$

$$\int_{c_3} Ndy = \int_d^c N(a,y)\,dy$$

$$= -\int_c^d N(a,y)\,dy.$$

Hence,

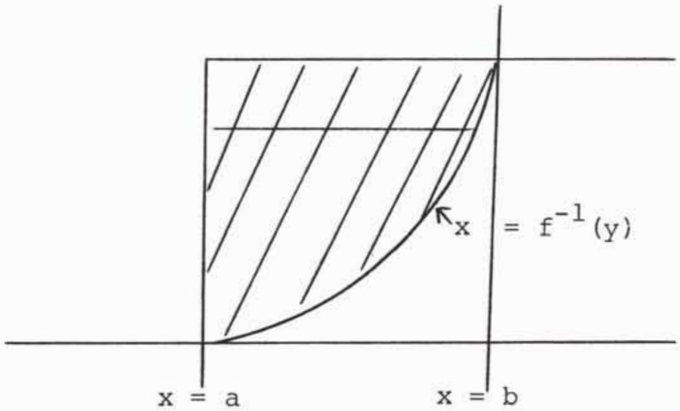
$$\int_{c_1} Ndy + \int_{c_3} Ndy = \int_c^d [N(f^{-1}(y),y) - N(a,y)]\,dy$$

$$= \int_c^d [\int_a^{f^{-1}(y)} \frac{\partial N(x,y)}{\partial x}\,dx]\,dy$$

$$= \iint_R \frac{\partial N}{\partial x}\,dA_R.$$

5.8.10 continued

[i.e.,



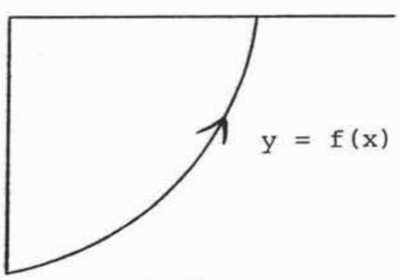
]

Combining (11) and (12) with (10) we have

$$\int_C Mdx + Ndy = -\iint_R \frac{\partial M}{\partial y}\, dA_R + \iint_R \frac{\partial N}{\partial x}\, dA_R$$

$$= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA_R$$

which proves Green's Theorem for regions of the form



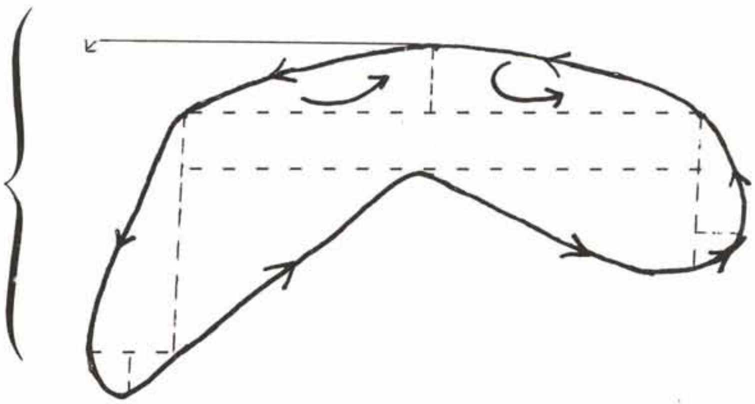
The key point is that every line integral of the form

$$\oint_C Mdx + Ndy$$

may be viewed as a sum of integrals around regions as described in (a) and (b).

For example,

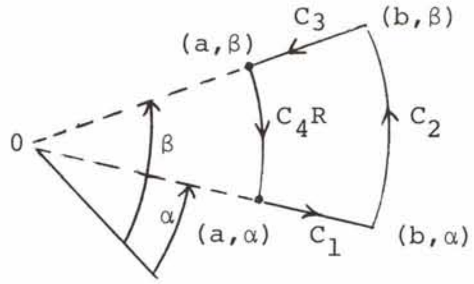
5.8.10 continued



On the dotted common boundaries the integrals cancel since each boundary is traversed twice, with opposite orientations and this establishes the proof of Green's Theorem in general.

5.8.11 (optional)

a.



$$\int_C Mdr + Nd\theta = \int_{C_1} Mdr + Nd\theta + \int_{C_2} Mdr + Nd\theta + \int_{C_3} Mdr + Nd\theta + \int_{C_4} Mdr + Nd\theta$$

$$= \int_{C_1} Mdr + \int_{C_2} Mdr + \int_{C_3} Mdr + \int_{C_4} Mdr + \int_{C_1} Nd\theta + \int_{C_2} Nd\theta + \int_{C_3} Nd\theta + \int_{C_4} Nd\theta . \qquad (1)$$

Now (polar) parametric forms for C_1, C_2, C_3 and C_4 are:

5.8.11 (continued)

C_1: $\theta = \alpha$, $a \leq r \leq b$, (therefore, $d\theta = 0$)
C_2: $r = b$, $\alpha \leq \theta \leq b$, (therefore, $dr = 0$)
C_3: $\theta = \beta$, r varies from b to a, (therefore $d\theta = 0$) (2)
C_4: $r = a$, θ varies from β to α, ($dr = 0$)

Using the results of (2) in (1) we have

$$\oint_c M(r,\theta)dr + N(r,\theta)d\theta = \int_{c_1} M(r,\theta)dr + \int_{c_3} M(r,\theta)\, dr + \int_{c_2} N(r,\theta)d\theta + \int_{c_4} N(r,\theta)d\theta$$

or

$$\oint_c M(r,\theta)dr + N(r,\theta)d\theta = \int_a^b M(r,\alpha)dr + \int_b^a M(r,\beta)dr + \int_\alpha^\beta N(b,\theta)d\theta + \int_\alpha^\beta N(a,\theta)d\theta$$

$$= \int_a^b M(r,\alpha)dr - \int_a^b M(r,\beta)dr + \int_\alpha^\beta N(b,\theta)d\theta - \int_\alpha^\beta N(a,\theta)d\theta$$

$$= \int_a^b [M(r,\alpha) - M(r,\beta)]dr + \int_\alpha^\beta [N(b,\theta) - N(a,\theta)]d\theta$$

$$= - \int_a^b [\int_\alpha^\beta \frac{\partial M(r,\theta)}{\partial\theta} d\theta]dr + \int_\alpha^\beta \int_a^b \frac{\partial N(r,\theta)}{\partial r} dr]d\theta$$

$$= - \int_a^b \int_\alpha^\beta \frac{\partial M(r,\theta)}{\partial\theta} d\theta dr + \int_\alpha^\beta \int_a^b \frac{\partial N(r,\theta)}{\partial r} dr d\theta. \qquad (2)$$

5.8.11 continued

Under the assumptions that both $\frac{\partial M}{\partial \theta}$ and $\frac{\partial N}{\partial r}$ are (piecewise-) continuous the integrals in (2) may have their order of integration reversed (and since the limits of integration are constants, this merely involves reversing the integrals), so that

$$\oint_C Mdr + Nd\theta = \int_\alpha^\beta \int_a^b \frac{\partial N}{\partial r}\, drd\theta - \int_\alpha^\beta \int_a^b \frac{\partial M}{\partial \theta}\, drd\theta$$

$$= \int_\alpha^\beta \int_a^b \left(\frac{\partial N}{\partial r} - \frac{\partial M}{\partial \theta} \right) drd\theta. \tag{3}$$

Equation (3) is Green's Theorem in polar coordinates. The subtlety in (3) lies in the fact that the element of area in polar coordinates is not $drd\theta$ but rather $rdrd\theta$.

Thus, if we wish to rewrite (3) as an integral involving dA_R we must say

$$\oint_C Mdr + Nd\theta = \int_\alpha^\beta \int_a^b \frac{1}{r} \left(\frac{\partial N}{\partial r} - \frac{\partial M}{\partial \theta} \right) rdrd\theta \tag{4}$$

where we obtained (4) from (3) just by multiplying and dividing the integrand on the right by r.

Since the limits of integration on the right side of (4) define the region R, equation (4) becomes

$$\oint_C Mdr + Nd\theta = \iint_R \frac{1}{r} \left(\frac{\partial N}{\partial r} - \frac{\partial M}{\partial \theta} \right) dA_R. \tag{5}$$

A comparison of equations (3) and (5) seems to make the following generalization plausible: If we are using u and v as coordinates where

$$\left.\begin{array}{l} u = u(x,y) \\ v = v(x,y) \end{array}\right\}$$

is invertible and if C enclosed the coordinate rectangle bounded between the pairs of curves u = a, u = b and v = c, v = d (where

5.8.11 continued

a <b, c < d) then mimicking the procedure of the last two exercises

$$\oint_C M(u,v)du + N(u,v)dv = \int_c^d \int_{u=a}^b \left(\frac{\partial N}{\partial u} - \frac{\partial M}{\partial v}\right) dudv. \qquad (6)$$

However the element of area in uv-coordinates is not dudv but rather

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv.$$

Hence, to write equation (6) in a form which emphasizes dA_R we have

$$\oint_C Mdu + Ndv = \iint_R \left(\frac{\partial N}{\partial u} - \frac{\partial M}{\partial v}\right) \left|\frac{\partial(u,v)}{\partial(x,y)}\right| \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv$$

$$= \iint_R \left|\frac{\partial(u,v)}{\partial(x,y)}\right| \left(\frac{\partial N}{\partial u} - \frac{\partial M}{\partial v}\right) dA_R. \qquad (7)$$

Now what is sure is that much of our work with line integrals is restricted to Cartesian coordinates whenever possible since other coordinate systems are much more unwieldly. Yet there are time when we are forced to use other coordinate systems. In these cases it is obvious that our basic formulas must be stated in such a way that the concepts are stated correctly, even though the form which expresses the concept might well vary with the coordinate system.

In any event,much of the work done in vector analysis (and several of these topics are discussed in Chapter 17 of the text) involves stating Green's Theorem (especially with regard to such important physical applications as work and fluid flow) in a form that does not depend on the particular coordinate system being used in the plane. Analogous results hold in 3-space where we find that Stoke's Theorem is the analog of Green's Theorem, and reference is also made to the Divergence Theorem - results that we do not wish to discuss in our course since in many respects they become a complete course in themselves.

5.8.11 continued

The point is that Green's Theorem in vector language is written in the form

$$\oint_C \vec{F} \cdot d\vec{s} = \iint_R (\vec{\nabla} \times \vec{F}) \cdot d\vec{A}_R \qquad (8)$$

where

$d\vec{s} = ds\vec{u}$ ($\vec{u}$ is a unit tangent vector to C)

and

$d\vec{A}_R = dA_R\vec{k}$ ($\vec{k}$ the usual unit vector perpendicular to the xy-plane). [Notice, by the way, that even when other coordinate systems are used to describe the xy-plane, $\vec{k}$ is still the unit vector perpendicular to the plane.]

$\vec{\nabla} \times \vec{F}$ is called the <u>curl of $\vec{F}$</u> and in Cartesian Coordinates

$$\vec{\nabla} = \vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}$$

so that by definition the operator ∇ in Cartesian coordinates is defined by

$$\vec{\nabla}f = \vec{i}\,\frac{\partial f}{\partial x} + \vec{j}\,\frac{\partial f}{\partial y} + \vec{k}\,\frac{\partial f}{\partial z} \quad \text{(the gradient of f)}$$

$$\vec{\nabla} \cdot \vec{F} = \left(\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}\right) \cdot (\vec{i}f_1 + \vec{j}f_2 + \vec{k}f_3) \text{ where } F = \vec{i}f_1 + \vec{j}f_2 + \vec{k}f_3$$

$$= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} \quad \text{(the divergence of } \vec{F}\text{)}$$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) + \vec{j}\left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right) + \vec{k}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

(the curl of $\vec{F}$).

5.8.11 continued

Note #1

Do not confuse f_x and f_1, etc. f_x refers to the partial derivative of a scalar function $f(x,y,z)$ while f_1 refers to the $\vec{1}$-component of a vector function $\vec{F}(x,y,z)$.

Note #2

In polar coordinates, for example, notice that the gradient of $f(r,\theta)$ is not $f_r\vec{u}_r + f_\theta\vec{u}_\theta(+ f_z\vec{u}_z)$ (where $\vec{u}_z = \vec{k}$). Rather we saw in Block 3 that in this case $\vec{\nabla}f = f_r\vec{u}_r + \frac{1}{r} f_\theta\vec{u}_\theta(+ f_z\vec{k})$.

Thus, while the definition of $\vec{\nabla}f$ is independent of any coordinate system (i.e., $\vec{\nabla}f$ is defined by the identity $\frac{df}{ds} = \vec{\nabla}f \cdot \vec{u}_s$) its form does depend on the coordinate system.

In summary, the, returning to (8) the general form of Green's Theorem in vector form is

$$\oint_C (\vec{F} \cdot \vec{u})ds = \iint_R [(\vec{\nabla} \times \vec{F}) \cdot \vec{k}]dA_R.$$

Further pursuit of this topic is left to the individual interest of the student.

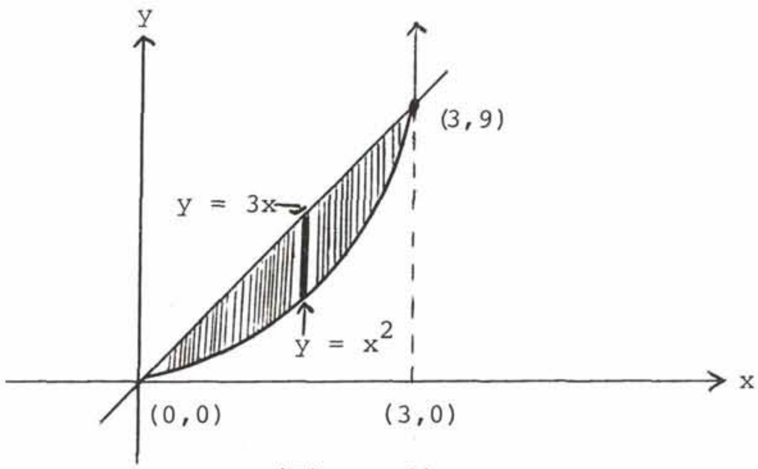
Quiz

1. The notation

$$\int_0^3 \int_{x^2}^{3x}$$

indicates the region in which $0 \le x \le 3$ and for each such x, y varies from the curve $y = x^2$ to the curve $y = 3x$.

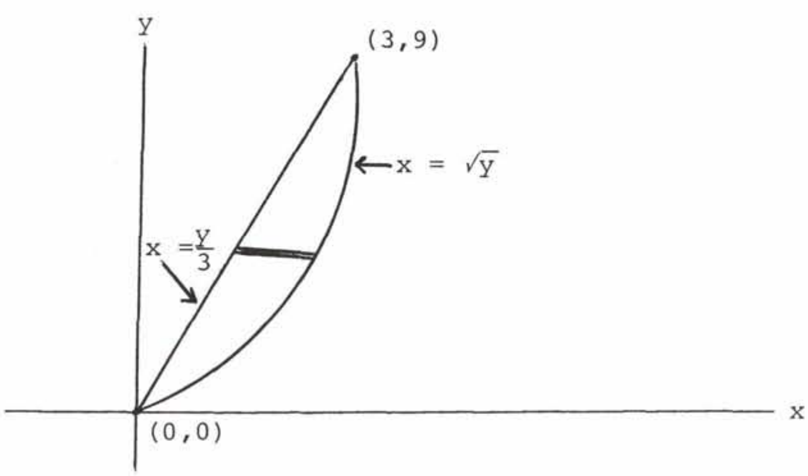
Thus,



(Figure 1)

To interchange the order of integration, we rewrite Figure 1 using a horizontal element of area.

That is,



(Figure 2)

1. continued

From Figure 2 we see that y may be any number such that $0 \leq y \leq 9$; and that for a fixed value of y, x varies from $\frac{y}{3}$ to $\sqrt{y}$.

Hence,

$$\int_0^3 \int_{x^2}^{3x} xy^2 dy\, dx = \int_0^9 \int_{\frac{y}{3}}^{\sqrt{y}} xy^2 dx\, dy.$$

Check:

$$\int_0^3 \int_{x^2}^{3x} xy^2 dy\, dx = \int_0^3 \left[\int_{y=x^2}^{3x} xy^2 dy \right] dx$$

$$= \int_0^3 \frac{1}{3} xy^3 \Big|_{y=x^2}^{3x} dx$$

$$= \int_0^3 \left[\frac{1}{3} x(3x)^3 - \frac{1}{3} x(x^2)^3 \right] dx$$

$$= \frac{1}{3} \int_0^3 (27x^4 - x^7) dx$$

$$= \frac{1}{3} \left[\frac{27x^5}{5} - \frac{x^8}{8} \right]_{x=0}^{3}$$

$$= \frac{1}{3} \left[\frac{27(3)^5}{5} - \frac{3^8}{8} \right]$$

$$= \frac{3^5}{3} \left[\frac{27}{5} - \frac{3^3}{8} \right]$$

$$= 3^4 \left(\frac{3^3}{5} - \frac{3^3}{8} \right)$$

$$= 3^7 \left(\frac{1}{5} - \frac{1}{8} \right)$$

$$= 3^7 \left(\frac{3}{40} \right)$$

$$= \frac{3^8}{40}.$$

1. continued

$$\int_0^9 \int_{\frac{y}{3}}^{\sqrt{y}} xy^2 dx\, dy = \int_0^9 [\int_{x=\frac{y}{3}}^{\sqrt{y}} xy^2 dx] dy$$

$$= \int_0^9 \frac{1}{2} x^2 y^2 \Big|_{x=\frac{y}{3}}^{\sqrt{y}} dy$$

$$= \int_0^9 [\frac{1}{2}(\sqrt{y})^2 y^2 - \frac{1}{2}(\frac{y}{3})^2 y^2] dy$$

$$= \frac{1}{2}\int_0^9 (y^3 - \frac{y^4}{9}) dy$$

$$= \frac{1}{2}[\frac{1}{4}y^4 - \frac{y^5}{45} \Big|_{y=0}^{9}$$

$$= \frac{1}{2}[\frac{9^4}{4} - \frac{9^5}{45}]$$

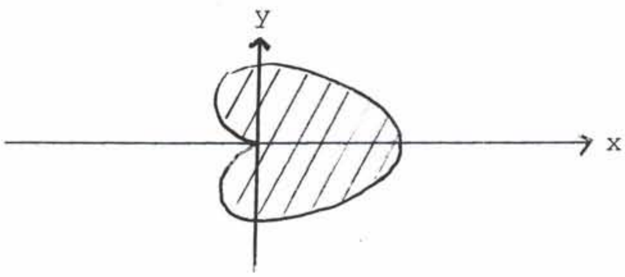
$$= \frac{9^4}{2}[\frac{1}{4} - \frac{9}{45}]$$

$$= \frac{9^4}{2}[\frac{1}{20}]$$

$$= \frac{9^4}{40}$$

$$= \frac{3^8}{40}.$$

2. (a) Our region R is given by



2. continued

and is defined analytically by

$R = \{(r, \theta): 0 \leq \theta \leq 2\pi$ and for a fixed θ, $0 \leq r \leq 1 + \cos\theta\}$

We also know that the density of R at any point is equal to the distance of that point from the origin; and using polar coordinates this distance is denoted by r.

Hence, the mass is given by

$$\iint_R \rho\, dA_R = \int_0^{2\pi} \int_0^{1+\cos\theta} r(r\,dr\,d\theta)$$

$$= \int_0^{2\pi} \left[\int_0^{1+\cos\theta} r^2 dr \right] d\theta$$

$$= \int_0^{2\pi} \frac{1}{3} (1 + \cos\theta)^3 d\theta$$

$$= \frac{1}{3} \int_0^{2\pi} (1 + 3\cos\theta + 3\cos^2\theta + \cos^3\theta)\, d\theta$$

$$= \frac{1}{3} \int_0^{2\pi} [1 + 3\cos\theta + \frac{3}{2}(1 + \cos 2\theta) + \cos\theta(1 - \sin^2\theta)]\, d\theta$$

$$= \frac{1}{3} \int_0^{2\pi} (\frac{5}{2} + 4\cos\theta + \frac{3}{2}\cos 2\theta - \sin^2\theta \cos\theta)\, d\theta$$

$$= \frac{1}{3} \left[\frac{5\theta}{2} + 4\sin\theta + \frac{3}{4}\sin 2\theta - \frac{1}{3}\sin^3\theta \right|_{\theta=0}^{2\pi}$$

$$= \frac{1}{3} \left[\frac{5(2\pi)}{2} \right]$$

$$= \frac{5\pi}{3}.$$

2. continued

(b) The volume is given by

$$\iint_R \sqrt{x^2 + y^2}\,dy\,dx,$$

which in polar coordinates is

$$\iint_R r(r\,dr\,d\theta),$$

and this is the same integral as in part (a). In other words, part (b) is simply another interpretation of the integral in (a); consequently, the answer to (b) is also $\frac{5\pi}{2}$.

3. (a) By linearity the fact that (2,-1) maps into (1,0) implies that

$$(2u, -u) \text{ maps into } (u,0) \tag{1}$$

while (-3,2) maps into (0,1) implies

$$(-3v, 2v) \text{ maps into } (0,v). \tag{2}$$

Again by linearity, (1) and (2) imply that

$$(2u, -u) + (-3v, 2v) \text{ maps into } (u,0) + (0,v);$$

or

$$(2u - 3v, -u + 2v) \text{ maps into } (u,v). \tag{3}$$

Thus, if (x,y) maps into (u,v), it follows from (3) that

$$\left.\begin{array}{l} x = 2u - 3v \\ y = -u + 2v \end{array}\right\} \tag{4}$$

Solving (4) for u and v in terms of x and y yields

3. continued

$$\left.\begin{aligned} x &= \ \ 2u - 3v \\ 2y &= -2u + 4v \end{aligned}\right\} \text{ or } v = x + 2y$$

$$\left.\begin{aligned} 2x &= \ \ 4u - 6v \\ 3y &= -3u + 6v \end{aligned}\right\} \text{ or } u = 2x + 3y$$

In other words, our mapping is defined by

$$u = 2x + 3y$$
$$v = \ x + 2y.$$

(b) Letting $u = 2x + 3y$ and $v = x + 2y$, we have that

$$\iint_R e^{2x+3y}\cos(x + 2y)\,dA_R = \iint_{S=\underline{f}(R)} e^u \cos v \frac{\partial(x,y)}{\partial(u,v)}\,dA_S$$

$$= \iint_S e^u \cos v \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} dv\,du.$$

Hence, by (4),

$$\iint_R e^{2x+3y}\cos(x + 2y)\,dA_R = \int_0^1\int_0^1 e^u \cos v \begin{vmatrix} 2 & -3 \\ -1 & 2 \end{vmatrix} dv\,du$$

$$= \int_0^1\int_0^1 e^u \cos v\,dv\,du$$

$$= \left[\int_0^1 \cos v\,dv\right]\left[\int_0^1 e^u du\right]$$

$$= \sin 1(e^1 - e^o)$$

$$= (e - 1)\sin 1.$$

4. By symmetry we may assume that the base of the cylinder lies in the xy-plane and that the cylinder is cut from above by the hemisphere

$$z = +\sqrt{4a^2 - x^2 - y^2},$$

and then double the resulting solution.

Since our region R in the xy-plane is the circle of radius α centered at the origin and since our "top" is the surface

$$z = \sqrt{4a^2 - x^2 - y^2}$$

We have that the volume in question is given by

$$\iint_R \sqrt{4a^2 - x^2 - y^2}\, dA_R$$

and, if we elect to use polar coordinates, this integral becomes

$$\int_0^{2\pi}\int_0^a \sqrt{4a^2 - r^2}\, r dr d\theta;$$

whereupon the answer to this exercise is given by

$$2\int_0^{2\pi}\int_0^a \sqrt{4a^2 - r^2}\; r dr d\theta$$

$$= 2\int_0^{2\pi}\left[\int_0^a \sqrt{4a^2 - r^2}\; r dr\right] d\theta$$

$$= 2\int_0^{2\pi}\left[-\tfrac{1}{3}(4a^2 - r^2)^{\frac{3}{2}}\right|_{r=0}^{a} d\theta$$

$$= -\frac{2}{3}\int_0^{2\pi}\left[(4a^2 - a^2)^{\frac{3}{2}} - (4a^2 - 0)^{\frac{3}{2}}\right] d\theta$$

4. continued

$$= \frac{2}{3} \int_0^{2\pi} [(4a^2)^{\frac{3}{2}} - (3a^2)^{\frac{3}{2}}]d\theta$$

$$= \frac{2}{3} [(4a^2)^{\frac{3}{2}} - (3a^2)^{\frac{3}{2}}]2\pi$$

$$= \frac{4\pi}{3} [8a^3 - 3^{\frac{3}{2}} a^3]$$

$$= \frac{4\pi a^3}{3} (8 - 3\sqrt{3}).$$

5. (a) Since

$$\frac{\partial(2xy)}{\partial y} = \frac{\partial(x^2 + \cos y)}{\partial x} = 2x,$$

it follows that $2xy\,dx + (x^2 + \cos y)dy$ is exact. In particular,

$$f_x = 2xy \rightarrow f = x^2y + g(y) \rightarrow f_y = x^2 + g'(y).$$

Then, since f_y must also equal $x^2 + \cos y$, we obtain

$$x^2 + g'(y) = x^2 + \cos y$$

or

$$g'(y) = \cos y.$$

Hence,

$$g(y) = \sin y + c,$$

whereupon

$$f = x^2y + g(y) \rightarrow$$
$$f = x^2y + \sin y + c.$$

That is,

5. continued

$$d(x^2y + \sin y + c) = 2xydx + (x^2 + \cos y)dy. \quad (1)$$

(b) Since $2xydx + (x^2 + \cos y)dy$ is exact,

$$\int_{(0,0)}^{(1,1)} 2xydx + (x^2 + \cos y)dy \quad (2)$$

doesn't depend on the path which joins (0,0) to (1,1). In particular, using (1), (2) becomes

$$\int_{(0,0)}^{(1,1)} d(x^2y + \sin y + c)$$

$$= (x^2y + \sin y + c) \Big|_{(0,0)}^{(1,1)}$$

$$= (1 + \sin 1 + c) - (0 + 0 + c)$$

$$= 1 + \sin 1$$

and this answer applies to any path which joins (0,0) to (1,1).

6. (a) The path c_1 may be expressed parametrically as

$$\left.\begin{array}{l} y = t \\ x = t \end{array}\right\} \text{ t varies from 0 to 1.}$$

Hence, $y = x = t$ and $dy = dx = dt$; therefore,

$$\int_{c_1} (\sin x - y^3)dx + (x^3 - e^y)dy$$

$$= \int_0^1 (\sin t - t^3)dt + (t^3 - e^t)dt$$

6. continued

$$= \int_0^1 (\sin t - e^t)\,dt$$

$$= -\cos t - e^t \Big|_{t=0}^{1}$$

$$= (-\cos 1 - e^1) - (-\cos 0 - e^0)$$

$$= -\cos 1 - e + 2.$$

(b) c_2 is given, for example, by

$$\left.\begin{array}{l} y = t^2 \\ x = t \end{array}\right\} \text{ t varies from 0 to 1.}$$

Hence, $dx = dt$ and $dy = 2t\,dt$. Consequently,

$$\int_{c_2} (\sin x - y^3)\,dx + (x^3 - e^y)\,dy$$

$$= \int_0^1 (\sin t - t^6)\,dt + (t^3 - e^{t^2})\,2t\,dt$$

$$= \int_0^1 (\sin t - t^6 + 2t^4 - 2te^{t^2})\,dt$$

$$= -\cos t - \frac{1}{7}t^7 + \frac{2}{5}t^5 - e^{t^2} \Big|_{t=0}^{1}$$

$$= (-\cos 1 - \frac{1}{7} + \frac{2}{5} - e^1) - (-\cos 0 - e^0)$$

$$= -\cos 1 + \frac{9}{35} - e + 2.$$

(c) The answers to (a) and (b) are different which implies that

$$(\sin x - y^3)\,dx + (x^3 - e^y)\,dy \qquad (1)$$

6. continued

is not exact. Had (1) been exact, we would have to obtain the same answer for (6) as we did for (a) since then the integral does not depend on the path which joins the given points (0,0) and (1,1).

7. By Green's Theorem, we know that

$$\oint_c Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA_R \qquad (1)$$

where R is the region enclosed by c.

In this particular example,

$$M[= M(x,y)] = \sin x - y^3$$
$$N = x^3 - e^y. \qquad (2)$$

Hence,

$$\frac{\partial M}{\partial y} = -3y^2$$

and

$$\frac{\partial N}{\partial x} = 3x^2.$$

Consequently,

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 3x^2 - (-3y^2) = 3(x^2 + y^2). \qquad (3)$$

Moreover, since c is the circle of radius 1 centered at the origin, R is the region defined by

$$R = \{(x,y): x^2 + y^2 \leq 1\} \qquad (4)$$

or, in polar coordinates,

$$R = \{(v,\theta): 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi\}. \qquad (5)$$

7. continued

Thus (1) becomes

$$\oint_C (\sin x - y^3)dx + (x^3 - e^y)dy$$

$$= \iint_R 3(x^2 + y^2)dA_R,$$

which, in turn, becomes by the use of polar coordinates

$$\int_0^{2\pi} \int_0^1 3r^2(rdr\ d\theta)$$

$$= \int_0^{2\pi} [\int_0^1 3r^3 dr] d\theta$$

$$= \int_0^{2\pi} \frac{3}{4} r^4 \Big|_{r=0}^{1} d\theta$$

$$= \int_0^{2\pi} \frac{3}{4} d\theta$$

$$= \frac{3}{4} (2\pi)$$

$$= \frac{3\pi}{2} .$$