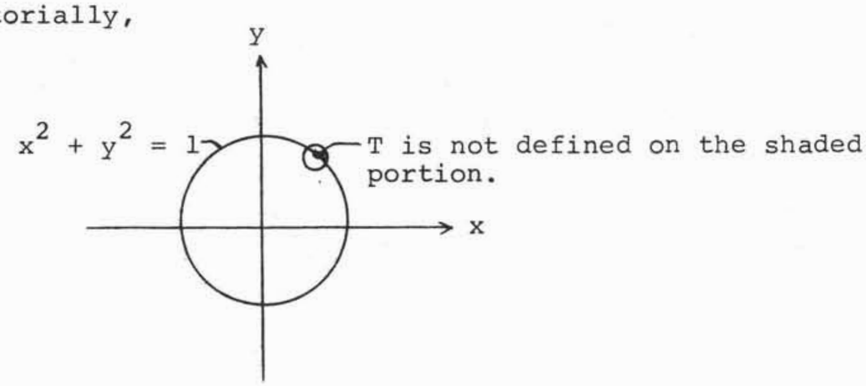
Unit 7: Maxima/Minima for Functions of Several Variables

4.7.1(L)

a. Since the domain of T is the disc $x^2 + y^2$ 1, we first observe that for points (x,y) on the boundary of the disc (i.e., at those points for which $x^2 + y^2 = 1$), we cannot talk about T being continuously differentiable. Namely, any neighborhood of a point on the boundary is not contained in the domain of T. Pictorially,



If we limit our attention to the interior of the disc (i.e., $x^2 + y^2 < 1$) we have that T is a continuously differentiable function of x and y everywhere in the interior. Hence T can attain maximum and/or minimum values in the interior of our plate only at those points (x,y) for which both T_x and $T_y = 0$.

Since $T = x^2 + 2y^2 - x$ (1)

we have that

$T_x = 2x - 1$ (2)

and

$T_y = 4y.$ (3)

From (2) and (3) we see that $T_x = T_y = 0$ if and only if x = 1/2 and y = 0. At this stage we know that $T(\frac{1}{2},0)$ is either a relative maximum for T in the interior of the disc, or a relative minimum, or a saddle point. We next look at points "near" $(\frac{1}{2},0)$ and compute

4.7.1(L) continued

$T(\frac{1}{2} + h, 0 + k) - T(\frac{1}{2},0)$ where h and k are small. (4)

$$T(\frac{1}{2} + h, k) = (\frac{1}{2} + h)^2 + 2k^2 - (\frac{1}{2} + h) = -\frac{1}{4} + h^2 + 2k^2$$

$$T(\frac{1}{2}, 0) = (\frac{1}{2})^2 + 2(0)^2 - \frac{1}{2} = -\frac{1}{4}.$$

Therefore,

$$T(\frac{1}{2} + h, k) - T(\frac{1}{2},0) = (-\frac{1}{4} + h^2 + 2k^2) - (-\frac{1}{4})$$

$$= h^2 + 2k^2. \quad (5)$$

Clearly $h^2 + 2k^2$ is non-negative for all h and k (since neither h^2 nor k^2 can be negative). Moreover $h^2 + 2k^2$ is zero only when $h = k = 0$.

Hence, as long as not both h and k are zero, we see from (5) that $T(\frac{1}{2} + h, k) - T(\frac{1}{2}, 0)$ is positive, so that in any neighborhood of $(\frac{1}{2},0)$. $T(\frac{1}{2},0)$ is a minimum value of T.

In summary, if $T = x^2 + 2y^2 - x$ and dom $T = \{(x,y) = x^2 + y^2 < 1\}$ then T has a relative (in fact, absolute) minimum at $(\frac{1}{2},0)$, and this minimum value of T is $-\frac{1}{4}$.

Note:

In order to be able to check our result, we "cheated" a little and chose a problem which could be worked out rather simply by the use of elementary algebra. Namely, given

$$T(x,y) = x^2 + 2y^2 - x$$

we complete the square to obtain

$$T(x,y) = (x^2 - x) + 2y^2$$

$$= (x^2 - x + \frac{1}{4} - \frac{1}{4}) + 2y^2$$

$$= [(x - \frac{1}{2})^2 - \frac{1}{4}] + 2y^2$$

4.7.1(L) continued

$$= (x - \tfrac{1}{2})^2 + 2y^2 - \tfrac{1}{4} \,. \qquad (6)$$

Equation (6), although equivalent to Equation (1), suggests why the minimum value of $T(x,y)$ is $-\frac{1}{4}$ and occurs at $(\frac{1}{2}, 0)$. In particular, both $(x - \frac{1}{2})^2 + 2y^2$ are non-negative, so (6) tells us that $T(x,y) \geq 0 + 0 - \frac{1}{4} = -\frac{1}{4}$. Moreover, $(x - \frac{1}{2})^2 = 0$ and $2y^2 = 0$ if and only if $x = \frac{1}{2}$ and $y = 0$.

Equation (6), however, tells us much more than this. It tells us that $T(x,y)$ can be made as large as we choose simply by picking both x and y (in fact, we can again talk about the magnitudes of x and y since both variables appear as squares in Equation (6)) to be sufficiently large. That is, if x is large in magnitude $(x - \frac{1}{2})^2$ is a large positive number and if y is large in magnitude $2y^2$ is a large positive number.

This observation leads us to part (b) of this exercise. Notice that when we set T_x and T_y equal to zero, we got the single candidate (1/2,0) which turned out to yield a minimum value for $T(x,y)$. We got no candidate(s) for maximum values.

b. What we propose to illustrate here is a counterpart of the theory of max/min for calculus of a single variable. We mentioned then that if $f(x)$ was continuous on a <u>closed</u> interval (i.e., one which contained its endpoints) then f attained both a maximum and a minimum value in the interval. Thus, if the extreme value did not occur in the interior of the interval (i.e., the open interval) then it occurred at one of the endpoints of the interval.

Without going into the theory of what happens in the case of two independent variables, it turns out that the counterpart of a closed interval is a region which contains its boundary, while a region without its boundary corresponds to an open interval. In other words, the region $x^2 + y^2 < 1$ is the interior of the unit circle. Its boundary is the circle $x^2 + y^2 = 1$. Thus, if we refer to $\{(x,y): x^2 + y^2 < 1\}$ we are talking about an open region. On the other hand, $\{(x,y): x^2 + y^2 \leq 1\}$ is called a closed region. At any rate,

4.7.1(L) continued

the theory says that if $f(x,y)$ is continuous on a closed region R, it attains its maximum and minimum values in R.

Accepting the truth of the theory, we see that since $T(x,y)$ does not attain a maximum value in the interior of the disc $x^2 + y^2 \leq 1$, and since it must attain its maximum somewhere on the disc (because T is continuous on the [closed] disc), it must be that the maximum value is attained on the boundary of the disc. Since the boundary of the disc is $x^2 + y^2 = 1$, it follows that when the domain of T is restricted to the boundary of the plate, T is given by

$$T(x,y) = x^2 + 2(1 - x^2) - x \tag{7}$$

where (7) is obtained by replacing y^2 by $1 - x^2$ in (1). This substitution is permissible because this is how x and y are related in the boundary. Notice that in (7), x and y are no longer independent because (x,y) was chosen to be on the curve (circle) $x^2 + y^2 = 1$. That is, on the boundary

$$T(x,y) = g(x) = x^2 + 2(1 - x^2) - x$$

$$= 2 - x - x^2, \ -1 \leq x \leq 1. \tag{8}$$

From equation (8), we have that

$$\left.\begin{aligned} g'(x) &= -1-2x \\ g''(x) &= -2 \end{aligned}\right\}. \tag{9}$$

Therefore, $g'(x) = 0 \leftrightarrow x = -\frac{1}{2}$; so since $g''(-\frac{1}{2}) = -2 < 0$, we see that $x = -\frac{1}{2}$ corresponds to a maximum value of T.

Since x and y are related by $x^2 + y^2 = 1$, $x = -\frac{1}{2}$ implies that $y^2 = 1 - x^2 = 1 - \frac{1}{4} = \frac{3}{4}$, or

$$y = \pm \frac{1}{2} \sqrt{3} .$$

Then since $g(-\frac{1}{2})$, from (8), equals $2 - (-\frac{1}{2}) - (-\frac{1}{2})^2 = 2 + \frac{1}{2} - \frac{1}{4} = \frac{9}{4}$, we have that:

4.7.1(L) continued

$T(x,y)$ has $\frac{9}{4}$ as its maximum value on the closed disc $x^2 + y^2 \leq 1$, and this value occurs at the boundary points $(-\frac{1}{2}, \frac{1}{2}\sqrt{3})$ and $(-\frac{1}{2}, -\frac{1}{2}\sqrt{3})$.

Notice that equations (9) tell us that $g(x)$ has no minimum on $x^2 + y^2 = 1$ for $-1 \leq x \leq 1$. At the endpoints, we have

$$\left.\begin{aligned} g(1) &= 2 - 1 - 1 = 0 \\ g(-1) &= 2 - (-1) - (-1)^2 = 2 \end{aligned}\right\} \qquad (10)$$

Equations (10) tell us that the minimum value of T <u>on the circle $x^2 + y^2 = 1$</u> is 0 and this occurs when $x = 1$, that is, at the point $(1,0)$. This minimum value exceeds $-\frac{1}{4}$ which we saw was the value of T at $(\frac{1}{2},0)$ in the disc. Thus, the minimum value of T over the whole disc is $-\frac{1}{4}$ at $(\frac{1}{2},0)$.

In summary, if <u>dom $T = \{(x,y): x^2 + y^2 \leq 1\}$</u> then

$$\left.\begin{aligned} T_{max} &= T(-\tfrac{1}{2}, \pm\tfrac{1}{2}\sqrt{3}) = \tfrac{9}{4} \\ T_{min} &= T(\tfrac{1}{2}, 0) = -\tfrac{1}{4} \end{aligned}\right\}$$

<u>Note</u>:

In this particular example, we can express T on $\{(x,y): x^2 + y^2 = 1\}$ very conveniently in terms of polar coordinates. Namely $x = \cos\theta$ and $y = \sin\theta$, $0 \leq \theta \leq 2\pi$. This substitution converts

$$T(x,y) = x^2 + 2y^2 - x$$

into

$$\begin{aligned} T(r,\theta) &= \cos^2\theta + 2\sin^2\theta - \cos\theta \\ &= \cos^2\theta + 2(1 - \cos^2\theta) - \cos\theta \\ &= 2 - \cos\theta - \cos^2\theta \\ &= -(\cos^2\theta + \cos\theta - 2) \end{aligned}$$

4.7.1(L) continued

$= -(\cos^2\theta + \cos\theta + \frac{1}{4} - \frac{9}{4})$

$= -[(\cos\theta + \frac{1}{2})^2 - \frac{9}{4}]$

$$= \frac{9}{4} - (\cos\theta + \frac{1}{2})^2. \qquad (11)$$

Since $(\cos\theta + \frac{1}{2})^2 \geq 0$, we see from (11) that $T(r,\theta) \leq \frac{9}{4}$ with equality holding if and only if $(\cos\theta + \frac{1}{2})^2 = 0$, or $\cos\theta = -\frac{1}{2}$. But $\cos\theta = -\frac{1}{2}$ means $x = -\frac{1}{2}$ since $x = \cos\theta$, while $y = \sin\theta$ then implies that $y = \pm\frac{1}{2}\sqrt{3}$, so that we have an algebraic verification of our previous work.

4.7.2(L)

Our main aim here is to show how messy it can be to compute $f(a + h, b + k) - f(a,b)$ [where $f_x(a,b) = f_y(a,b) = 0$] This, in turn, will serve as motivation for us to explore the existence of more convenient formulae.

We have that

$$f(x,y) = x^3 + y^3 - 9xy + 27. \qquad (1)$$

Hence,

$$\left.\begin{aligned} f_x(x,y) &= 3x^2 - 9y \\ f_y(x,y) &= 3y^2 - 9x \end{aligned}\right\} . \qquad (2)$$

Thus, the requirement that $f_x(x,y) = f_y(x,y) = 0$ yields, from (2), the result that

$$\left.\begin{aligned} y &= \frac{1}{3}x^2 \\ x &= \frac{1}{3}y^2 \end{aligned}\right\} . \qquad (3)$$

Therefore $y = \frac{1}{3}(\frac{1}{3}y^2)^2 = \frac{1}{27}y^4$.

Therefore $y^4 - 27y = 0$.

4.7.2(L) continued

Therefore $y(y^3 - 27) = 0$, so either $y = 0$ or $y = 3$.

From the second equation in (2), $y = 0$ implies $x = 0$, while $y = 3$ implies $x = 3$. Therefore, the only candidates for max/min points are (0,0) and (3,3).

To check (3,3) we must look at $f(3 + h, 3 + k) - f(3,3)$. From (1) we have

$$\begin{aligned} f(3 + h, 3 + k) &= (3 + h)^3 + (3 + k)^3 - 9(3 + h)(3 + k) + 27 \\ &= [3^3 + 3(3)^2h + 3(3)h^2 + h^3] \\ &\quad + [3^3 + 3(3)^2k + 3(3)k^2 + k^3] \\ &\quad - 81 - 27h - 27k - 9hk + 27 \\ &= 9h^2 + h^3 + 9k^2 + k^3 - 9hk . \qquad (4) \end{aligned}$$

$$f(3,3) = 3^3 + 3^3 - 9(3)(3) + 27 = 0. \qquad (5)$$

So, from (4) and (5),

$$f(3 + h, 3 + k) - f(3,3) = 9h^2 + h^3 + 9k^2 + k^3 - 9hk. \qquad (6)$$

Now the algebra problems begin! Notice that since h and k can be either positive or negative, etc., it is not easy to analyze the sign of $9h^2 + h^3 + 9k^2 + k^3 - 9hk$ for all sufficiently small values of h and k.

If we are good at algebra we may notice that

$$\begin{aligned} 9h^2 + h^3 + 9k^2 + k^3 - 9hk &= (h^3 + k^3) + 9(h^2 - hk + k^2) \\ &= (h + k)(h^2 - hk + k^2) \\ &\quad + 9(h^2 - hk + k^2) \\ &= (h^2 - hk + k^2)([h + k] + 9) \end{aligned}$$

4.7.2(L) continued

$$= [(h - \frac{1}{2}k)^2 + \frac{3}{4}k^2](h + k + 9). \tag{9}$$

Now $[(h - \frac{1}{2}k)^2 + \frac{3}{4}k^2] \geq 0$, and equals 0 only if $h = \frac{1}{2}k$ and $k = 0$, which in turn means only when $h = 0$, $k = 0$. Since we are interested in points $(3 + h, 3 + k)$ near $(3,3)$, we exclude the case $h = k = 0$, since then $(3 + h, 3 + k) = (3,3)$. So if not both h and k are zero, the first factor on the right side of (7) is positive. The second factor can be positive, negative, or zero depending on how h and k are chosen - however - we are interested only in what is happening "near" (3,3). That is, we may assume that h and k are as small as we wish (in magnitude) except that at least one must be unequal to zero. Under this condition, $h + k + 9$ may be assumed to be positive.*

In summary, the right side of (7) is positive whenever h and k are sufficiently small. In still other words, combining (6) and (7), $f(3 + h, 3 + k) - f(3,3)$ is positive whenever $(3 + h, 3 + k)$ is sufficiently close to (3,3) [the difference is zero when $h = k = 0$, since then we are computing $f(3,3) - f(3,3)$].

Therefore, $f(x,y)$ has a relative minimum at (3,3) and the value of this minimum is $f(3,3) = 0$.

As for checking (0,0), we must look at $f(h,k) - f(0,0) =$
$$(h^3 + k^3 - 9hk + 27) - 27 = h^3 + k^3 - 9hk. \tag{8}$$

If we let $h = k$ in (8), we obtain $f(h,h) - f(0,0) = 2h^3 - 9h^2 = h^2(2h - 9)$.

For small values of h, $2h - 9$ is negative whole h^2 is positive. Consequently, for small h ($\neq 0$),

$$\underline{f(h,h) - f(0,0) \text{ is negative}}. \tag{9}$$

On the other hand, if we let $k = -h$ in (8), we obtain

$$f(h, -h) - f(0,0) = h^3 + (-h)^3 - 9h(-h) = 9h^2$$

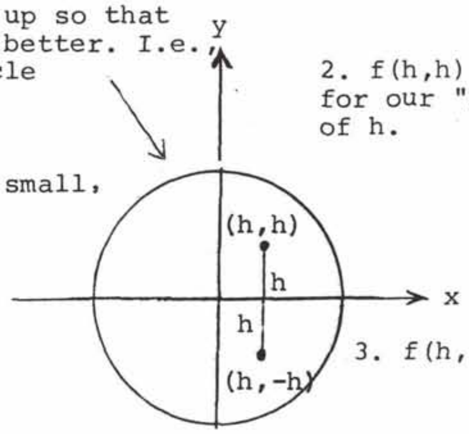
*See note at the end of this exercise.

4.7.2(L) continued

so that if $h \neq 0$,

$$f(h, -h) - f(0,0) = 9h^2 > 0. \quad (10)$$

We may now combine results (9) and (10) to conclude that in <u>any</u> neighborhood of (0,0) there are values of h and k such that $f(h,k) - f(0,0)$ is positive and other values of h and k such that $f(h,t) - f(0,0)$ is negative. Pictorially



Therefore (0,0) is a <u>saddle-point</u> since in <u>every</u> neighborhood of (0,0) we can find points (x,y) such that $f(x,y) > f(0,0)$ and other points (x,y) such that $f(x,y) < f(0,0)$.

Our main aim in solving this problem is to illustrate that even for this rather simple function [i.e., there are certainly more complicated functions from E^2 to E than $f(x,y) = x^3 + y^3 - 9xy + 27$], there was a considerable amount of algebraic manipulation necessary if we were to determine the behaviour of $f(a + h, b + k)$ once we knew the points (a,b) for which $f_x(a,b) = f_y(a,b) = 0$.

What we shall develop in the next exercise is a formula in terms of $f_{xx}(a,b)$, $f_{yy}(a,b)$, and $f_{xy}(a,b)$ that tell us whether f(a,b) is a relative maximum, minimum or saddle point once we know that $f_x(a,b) = f_y(a,b) = 0$. The next exercise actually is a repetition of Section 18.5 in the Thomas text, but with the chunks broken up into smaller pieces.

4.7.2(L) continued

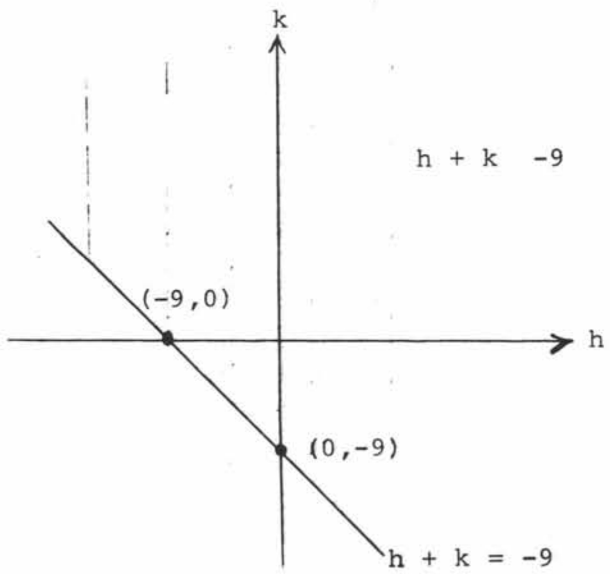
Note:

When we say that $h + k + 9$ is positive when both h and k are sufficiently small, it might be helpful if we were sure that we understood the full meaning of "sufficiently small". For example, if we let $h = -5$ and $k = -6$, $h + k + 9$ is then clearly negative. But for this choice of h and k, the point $(3 + h, 3 + k)$ is $(-2,-3)$ and certainly we probably sense that this point is not "sufficiently close" to $(3,3)$.

There is an interesting geometric way to handle this problem. Namely, since we want $h + k + 9$ to be positive, it follows that

$h + k > -9$.

If we now look at the hk-plane, we see that this is the region which lies above the line $h + k = -9$. That is,

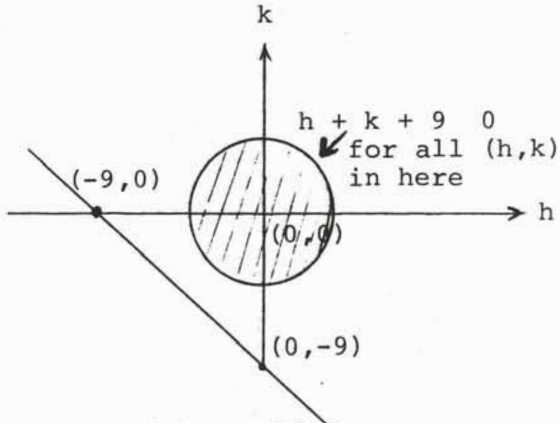


(Figure 1)

With respect to Figure 1, when we talk about small values of h and k, we are referring to a small neighborhood of (0,0) since in this graph (0,0) names $h = 0$, $k = 0$. We may choose this neighborhood any way we want, and provided only that the neighborhood never extends on or below the line $h + k = -9$,

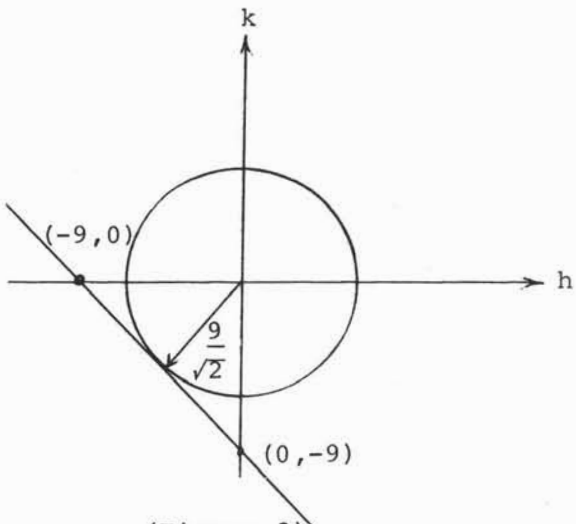
4.7.2(L) continued

we can be sure that h + k + 9 is positive for all points in that neighborhood. For example,



(Figure 2)

Suppose, we want the largest (circular) neighborhood of (0,0) that permits us to conclude that h + k + 9 is positive for every point (h,k) in this neighborhood. We need only compute the perpendicular distance of the origin to the line. That is

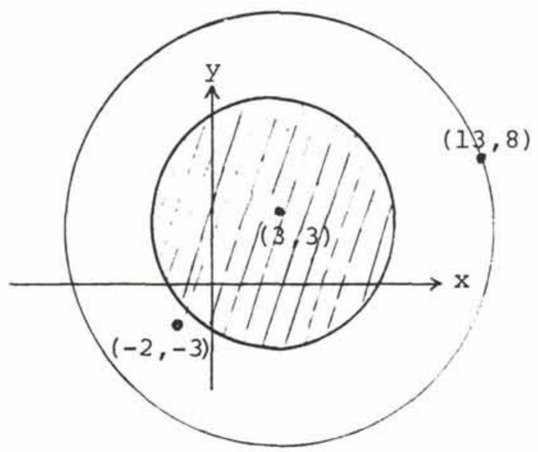


(Figure 3)

In other words, if R is the region $h^2 + k^2 < \frac{81}{2}$ then h + k + 9 is positive for every $(h,k) \varepsilon R$.

4.7.2(L) continued

Returning to the original problem, this means that if we draw a circle in the xy-plane of radius $9\sqrt{2}/2$ with (3,3) as center then $f(x,y) - f(3,3)$ is positive for every point (x,y) in the interior of this circle. Clearly $f(x,y) - f(3,3)$ may be positive for some points (x,y) outside the circle, but $f(x,y) - f(3,3)$ cannot be non-positive inside the circle. Again pictorially,



(Figure 4)

for all (x,y) in the disc $(x - 3)^2 + (y - 3)^2 < \frac{81}{2}$, $f(x,y) - f(3,3)$ is positive.

1. $f(-2,-3) - f(3,3)$ is negative but this is okay since (-2,-3) is outside the neighborhood in question.

2. For example $f(0,0) - f(3,3)$ is positive since $h = k = 9 > 0$ when $h = k = -3$. Notice that (0,0) is in the circle.

There are points (x,y) outside this circle for which $f(x,y) - f(3,3)$ is greater than zero. For example if $h = 10$ and $k = 5$ then $h + k + 9 = 24$. $h = 10$, $k = 5$ corresponds to (13,8). Thus $f(13,8) - f(3,3)$ is positive. However, if we took the circle centered at (3,3) passing through (13,8) then there would be points (x,y) in this circle for which $f(x,y) - f(3,3)$ would not be positive. The easiest way to see this is to note

4.7.2(L) continued

in Figure 1 that the circle centered at (0,0) passing through (10,5) does not lie above the line $h + k = -9$. Those points of the circle lying below the line $h + k = -9$ yield points (h,k) for which $h + k + 9$ is negative. That is, there are points on the circle $h^2 + k^2 = (\sqrt{10^2 + 5^2})^2 = 125$ for which $h + k + 9$ is negative. These points correspond to points (x,y) in the xy-plane which lie in the circle $(x - 3)^2 + (y - 3)^2 = 125$. In particular (-2,-3) lies in this circle and $f(-2,-3) - f(3,3)$ is negative.

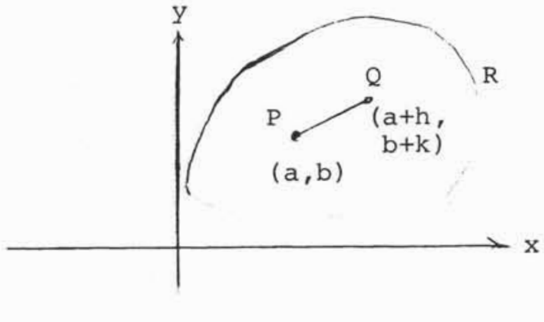
4.7.3(L)

a. Perhaps the hardest part of this problem is trying to figure out where or how we managed to invent the function $F(t) = f(a + ht, b + kt)$, $0 \leq t \leq 1$ other than by saying that it was in the book. The key point is that in a neighborhood of any point (a,b) for which $f_x(a,b) = f_y(a,b) = 0$, we want to look at

$$f(a + h, b + k) - f(a,b). \qquad (1)$$

To tackle this problem we (for the time being) arbitrarily choose an h and k, subject only to the condition that not <u>both</u> h and k equal zero. Otherwise there are no restrictions; h and/or k may be negative and either h or k can be zero provided the other is not.

Pictorially, we have:



(Figure 1)

4.7.3(L) continued

Since we ultimately are interested in points near P(a,b), we draw the line segment $\overline{PQ}$. Since $\overline{PQ} = h\vec{i} + k\vec{j}$ and since (a,b) is a known (given) point, the equation of the line determined by P and Q is

$$\frac{x - a}{h} = \frac{y - b}{k} \quad (= t) \qquad (2)$$

or

$$\left.\begin{aligned} x &= a + ht \\ y &= b + kt \end{aligned}\right\}. \qquad (2')$$

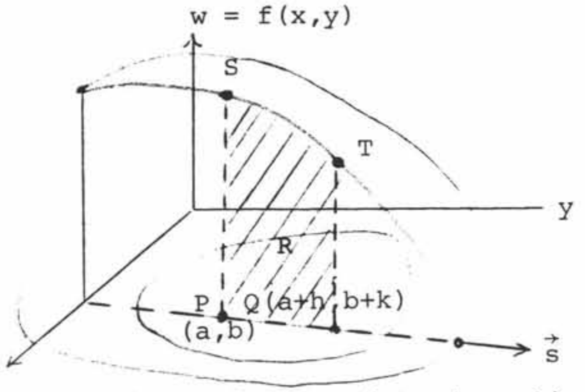
The <u>segment</u> PQ is obtained from (2) and (2') by restricting t to the range $0 \leq t \leq 1$, since when t = 0 (x,y) = (a,b) = P and when t = 1, (x,y) = (a + h, b + k) = Q.

In summary, then

$F(t) = f(a + ht, b + kt), \quad 0 \leq t \leq 1$

defines f(x,y) <u>if</u> the domain of f is restricted to the line segment $\overline{PQ}$. Again pictorially,

(Figure 2)



1. The curve ST consists of the points (x,y,f(x,y)) where (x,y) belongs to the line segment PQ.

2. Another form of this curve is the set of points (a + ht, b + kt, f(a + ht, b + kt)) where $0 \leq t \leq 1$.

4.7.3(L) continued

3. That is, $w = F(t) = f(a + ht, b + kt)$, $0 \leq t \leq 1$, is an equation for the curve ST.

This, then, is where

$$F(t) = f(a + ht, b + kt),\ 0 \leq t \leq 1 \tag{3}$$

comes from. In the "extreme" cases $t = 0$ and $t = 1$, we obtain

$$\left.\begin{aligned} F(0) &= f(a,b) \\ F(1) &= f(a + h, b + k) \end{aligned}\right\} \tag{4}$$

so that from (4) we see that

$$F(1) - F(0) = f(a + h, b + k) - f(a,b). \tag{5}$$

Since we shall be interested in the sign of $f(a + h, b + k) - f(a,b)$ in sufficiently small neighborhoods of those points (a,b) for which $f_x(a,b) = f_y(a,b) = 0$, we see from equation (5) that we may instead study the sign of $F(1) - F(0)$.

b. The fact that $F(t) = f(a + ht, b + kt)$ for $0 \leq t \leq 1$ can be restated as follows in terms of the chain rule:

Let $w = F(t)$ $0 \leq t \leq 1$ then

$$\left.\begin{aligned} w &= f(x,y) \\ x &= a + ht \\ y &= b + kt \end{aligned}\right\}. \tag{6}$$

Since we are assuming that $f(x,y)$ is continuously differentiable at (a,b), we may use the chain rule to conclude

$$F'(t) = \frac{dw}{dt} = f_x(x,y)\frac{dx}{dt} + f_y(x,y)\frac{dy}{dt},\ 0 \leq t \leq 1. \tag{7}$$

Since h and k are arbitrarily chosen constants, we see from (6) that $dx/dt = h$ and $dy/dt = k$. Putting these results into (7) yields

$$F'(t) = f_x(x,y)\,h + f_y(x,y)\,k,\ 0 \leq t \leq 1 \tag{8}$$

4.7.3(L) continued

or writing f in terms of t [from (6)], we have

$$F'(t) = hf_x(a + ht, b + kt) + kf_y(a + ht, b + kt),\ 0 \leq t \leq 1. \tag{8'}$$

In particular we see from (8) [or (8')] that F"(t) exists for all t such that $0 < t < 1$ (as usual we don't talk about differentiability at the end points of a closed interval), while the continuity of f(x,y) in the region R guarantees the continuity of F(t) [that is, if f is continuous throughout R it is, in particular, continuous along any curve which is contained in R].

Thus we may use the <u>mean value theorem for functions of a single variable</u> to conclude that there exists a number t_1, where $0 < t_1 < 1$, such that

$$\frac{F(1) - F(0)}{1 - 0} = F'(t_1)$$

or

$$F(1) - F(0) = F'(t_1). \tag{9}$$

Substituting the results of (5) and (8') into (9), we see that there exists a number t_1, $0 < t_1 < 1$, such that

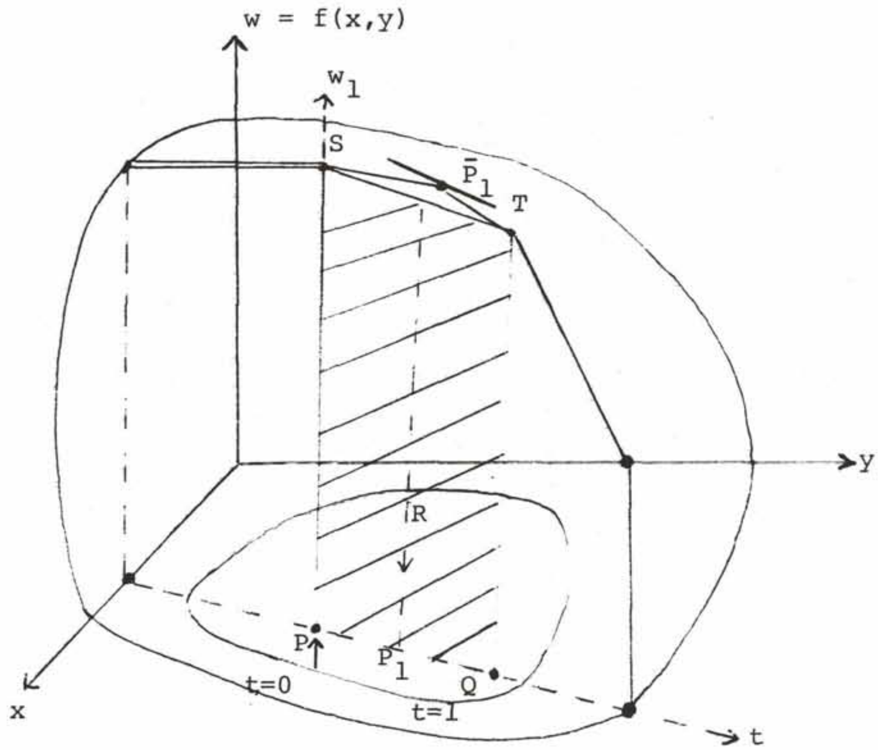
$$f(a + h, b + k) - f(a,b) = hf_x(a + ht_1, b + kt_1) + kf_y(a + ht_1, b + kt_1). \tag{10}$$

Equation (10) is known as the mean value theorem for functions of two independent variables. In summary, if f(x,y) is continuously differentiable in some neighborhood R of (a,b) and (a + h, b + k) is any other point in R, then there exists a number t_1, $0 < t < 1$ such that

$$f(a + h,b + k) - f(a,b) = hf_x(a + ht_1, b + kt_1) + kf_y(a + ht_1, b + kt_1).$$

4.7.3(L) continued

Pictorially, using the same diagram as in Figure 2, this means



(Figure 3)

1. Slope of the line $\overline{ST}$ is equal to $\Delta w/\Delta t = f(a + h, b + k) - f(a,b)/1 - 0 = f(a + h, b + k) - f(a,b)$.

2. In the tw_1-plane there is a point $\bar{P}$ on the curve ST at which the slope of the curve equals the slope of the line (the "ordinary" mean value theorem).

3. The point $\bar{P}_1$ is the image under F of some point P_1 on the line segment PQ.

4. This point P_1 has the name $(a + ht_1, b + kt_1) = F(t_1)$ for some $0 < t_1 < 1$.

5. $F'(t_1) = hf_x(a + ht_1, b + kt_1) + kfy(a + ht, b + kt_1)$ [from equation (8')].

6. Equating the two slopes yields equation (10).

4.7.3(L) continued

c. So far, h and k are arbitrary constants. If we now restrict our attention to sufficiently small values of h and k we have that the points $(a + ht, b + kt)$, $0 < t < 1$ must be near (a,b) since all these points lie on the line segment $\overline{PQ}$ which joins (a,b) to $(a + h, b + k)$. In particular, since f_x and f_y are continuous at (a,b), $f_x(a + ht, b + kt)$ and $f_x(a,b)$ must have the same sign <u>unless</u> $f_x(a,b) = 0$.* Similarly, unless $f_y(a,b) = 0$, we have that $f_y(a + ht, b + kt)$ and $f_y(a,b)$ have the same sign. Consequently, <u>unless</u> <u>both</u> $f_x(a,b) = 0$ <u>and</u> $f_y(a,b) = 0$, we have from (10) that

$$\underline{\text{sign}}\,[f(a + h, b + k) - f(a,b)] = \underline{\text{sign}}\,[hf_x(a,b) + kf_y(a,b)] \quad (11)$$

[For example if $f_x(a,b) = 0$ but $f_y(a,b) \neq 0$, we can make $f_x(a + ht, b + kt)$ as nearly equal in magnitude (although not necessarily in sign) to 0 as we wish, whereupon the sign of $f(a + ht, b + kt) - f(a,b)$ is determined by the sign of $kf_y(a,b)$].

d. Since $f(x,y) = x^3y^5 - y^6 + x^7 - 2xy$, $f_x(x,y) = 3x^2y^5 + 7x^6 - 2y$ and $f_y(x,y) = 5x^3y^4 - 6y^5 - 2x$.

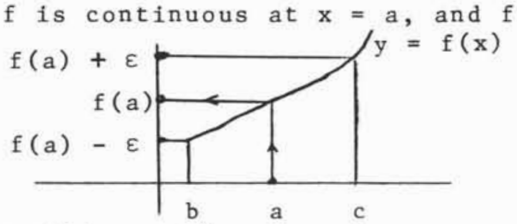
Hence,

$f_x(1,1) = 8$ and $f_y(1,1) = -3$, so from equation (11)

$$\text{sign}[f(1 + h, 1 + k) - f(1,1)] = \text{sign}[8h - 3k]. \quad (12)$$

In particular (1.01, 1.02) is of the form $(1 + h, 1 + k)$ with $h = .01$ and $k = .02$, so (12) becomes

*Graphically if f is continuous at $x = a$, and $f(a) > 0$ we have



1. Pick ε so that $f(a) - \varepsilon > 0$
2. Then $f(x) > 0$ for all $x \in [b,c]$

A similar argument applies if $f(a) < 0$. But if $f(a) = 0$, any neighborhood of $f(a)$ may contain both negative and positive values so we cannot be sure of the sign of $f(x)$ for $x \in [b,c]$. In general we are saying that if f is continuous at $x = a$ then in a sufficiently small neighborhood N of $x = a$, $f(x)$ and $f(a)$ have the same sign for all $x \in N$ except possibly when $f(a) = 0$.

4.7.3(L) continued

$$\text{sign}[f(1.01, 1.02) - f(1,1)] = \text{sign}\ [8(.01) - 3(.02)]$$

$$= \text{sign}\ [0.02], \text{ which is positive.}$$

Since $f(1.01, 1.02) - f(1,1)$ is positive

$\underline{f(1.01, 1.02) > f(1,1)}$.

Notice that equation (11) does not tell us the size of $f(1.01, 1.02)$; it is a very quick way [compared with computing $f(1.01, 1.02)$] for determining whether or not $f(1.01, 1.02)$ exceeds $f(1,1)$.

e. In the event that $f_x(a,b) = f_y(a,b) = 0$, we have already seen that equation (11) does not apply. This is, to say the least, a bit distressing since we are particularly interested in the sign of $[f(a + h, b + k) - f(a,b)]$ when we have a point (a,b) at which $f_x(a,b) = f_y(a,b) = 0$. To overcome this problem we use the extended mean value theorem (or Taylor's Theorem with remainder [for review see Section 18.4, Thomas]), which says that if $F(t)$ together with its first $n = 1$ derivatives are continuous on an interval containing 0 and 1, then there exists a number t_1, $0 > t_1 > 1$ such that

$$F(t) = F(0) + F'(0)t + \ldots + \frac{F^{(n)}(0)t^n}{n!} + \frac{F^{(n+1)}(t_1)t^{n+1}}{(n+1)!}. \qquad (13)$$

Letting $t = 1$ in (13) yields

$$F(1) = F(0) + F'(0) + \ldots + \frac{F^{(n)}(0)}{n!} + \frac{F^{(n+1)}(t_1)}{(n+1)!}$$

or

$$F(1) - F(0) = F'(0) + \ldots + \frac{F^{(n)}(0)}{n!} + \frac{F^{(n+1)}(t_1)}{(n+1)!}, \text{ some } t, \text{ where } 0 < t_1 < 1. \qquad (14)$$

Part (c) was the special case of (14) where $n = 0$. We next try to see what happens when $n = 1$. When $n = 1$, equation (14) tells us that there is a number $0 < t_1 < 1$ such that

4.7.3(L) continued

$$F(1) - F(0) = F'(0) + \frac{F''(t_1)}{2}. \qquad (15)$$

From equation (8) we have already shown that $F'(t) = hf_x + kf_y$ where $x = a + ht$, $y = b + kt$. Using the chain rule again, this time on $F'(t)$, we obtain

$$\begin{aligned} F''(t) &= [F'(t)]_x \frac{dx}{dt} + [F'(t)]_y \frac{dy}{dt} \\ &= h[F'(t)]_x + k[F'(t)]_y \\ &= h \frac{\partial}{\partial x} [hf_x + kf_y] + k\frac{\partial}{\partial y}[hf_x + kf_y] \\ &= h[hf_{xx} + kf_{yx}] + k[hf_{xy} + kf_{yy}] \\ &= h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}. \end{aligned}$$

Putting these results into (15) and recalling that $F(1) - F(0) = f(a + h, b + k) - f(a,b)$, we see:

$$\begin{aligned} f(a + h, b + k) - f(a,b) = {} & hf_x(a,b) + kf_y(a,b) \\ & + \tfrac{1}{2}[h^2 f_{xx}(a + ht_1, b + kt_1) \\ & + 2hkf_{xy}(a + ht_1, b + kt_1) \\ & + k^2 f_{yy}(a + ht_1, b + kt_1)]. \end{aligned}$$

Then, since $f_x(a,b) = f_y(a,b) = 0$, we obtain

$$\begin{aligned} f(a + h, b + k) - f(a,b) = {} & \tfrac{1}{2}[h^2 f_{xx}(a + ht_1, b + kt_1) \\ & + 2hkf_{xy}(a + ht_1, b + kt_1) \\ & + k^2 f_{yy}(a + ht_1, b + kt_1)]. \end{aligned} \qquad (16)$$

We now invoke the continuity of f_{xx}, f_{xy}, and f_{yy} at (a,b) to conclude that if h and k are sufficiently small and if <u>at least one</u> of the numbers $f_{xx}(a,b)$, $f_{xy}(a,b)$, $f_{yy}(a,b)$ is unequal to zero, then the right side of (16) has the same

4.7.3(L) continued

sign as $h^2f_{xx}(a,b) + 2hkf_{xy}(a,b) + k^2f_{yy}(a,b)$ [if $f_{xx}(a,b) = f_{xy}(a,b) = f_{yy}(a,b) = 0$ we would have to try $n = 2$ in Taylor's Theorem with Remainder, etc.].

In other words, if $f_x(a,b) = f_y(a,b) = 0$, then unless $h^2f_{xx}(a,b) + 2hkf_{xy}(a,b) + k^2f_{yy}(a,b) = 0$

$$\text{sign}[f(a + h,\ b + k) - f(a,b)] = \text{sign}\ [h^2f_{xx}(a,b) + 2hkf_{xy}(a,b) + 2hkf_{xy}(a,b) + k^2f_{yy}(a,b)]. \quad (17)$$

[If $h^2f_{xx} + 2hkf_{xy} + k^2f_{yy}(a,b) = 0$ we must look at the term involving $f'''(t)$ etc. just as in the case of a single variable].

f. From the result in part (e), we would like to analyze the sign of $h^2f_{xx}(a,b) + 2hkf_{xy}(a,b) + k^2f_{yy}(a,b)$.

[Keep in mind that h and k are the "unknowns". $f_{xx}(a,b)$, $f_{xy}(a,b)$ and $f_{yy}(a,b)$ are fixed constants determined by f and (a,b)].

Let $g(h,k) = h^2f_{xx}(a,b) + 2hkf_{xy}(a,b) + k^2f_{yy}(a,b)$.

Then

$$g(h,k)f_{xx}(a,b) = h^2f_{xx}^{\ 2}(a,b) + 2hkf_{xy}(a,b)f_{xx}(a,b) + k^2f_{yy}(a,b)f_{xx}(a,b)$$

$$= h^2f_{xx}^{\ 2}(a,b) + 2hkf_{xy}(a,b)f_{xx}(a,b) + k^2f_{xy}^{\ 2}(a,b)^* - k^2f_{xy}^{\ 2}(a,b) + k^2f_{yy}(a,b)f_{xx}(a,b)$$

$$= [hf_{xx}(a,b) + kf_{xy}(a,b)]^2 + k^2[f_{xx}(a,b)f_{yy}(a,b) - f_{xy}^{\ 2}(a,b)]. \quad (18)$$

* Again we have employed the usual trick of completing the square.

4.7.3(L) continued

On the right side of (18) we see that our first term, being a perfect square, cannot be negative. Hence the right side of (18) must be positive as soon as $[f_{xx}(a,b)f_{yy}(a,b) - f_{xy}^2(a,b)]$ is positive.

We now analyze (18) by cases.

<u>Case 1</u>: $f_{xx}(a,b)f_{yy}(a,b) - f_{xy}^2(a,b) > 0$.

In this event the right side of (18) is positive. Namely $[hf_{xx}(a,b) + kf_{yy}(a,b)]^2$, being a square, is non-negative, k^2 is non-negative. Hence if $k \neq 0$ we have

$$\underbrace{[hf_{xx}(a,b) + kf_{yy}(a,b)]^2}_{\geq 0} + \underbrace{k^2}_{>0}\underbrace{[f_{xx}(a,b)f_{yy}(a,b) - f_{xy}^2(a,b)]}_{>0} > 0$$

If $k = 0$ then $h \neq 0$ since $(a + h, b + k) \neq (a,b)$, so in this case the right side of (18) is $[hf_{xx}(a,b)]^2$ which is positive unless $f_{xx}(a,b) = 0$. But in Case 1 $f_{xx}(a,b)$ cannot be zero since in that even $f_{xx}(a,b)f_{yy}(a,b) - f_{xy}^2(a,b)$ equals $-f_{xy}^2(a,b)$ which is non-positive, contrary to our assumption that $f_{xx}(a,b)f_{yy}(a,b) - f_{xy}^2(a,b) > 0$.

So in Case 1 the right side of (18), under any circumstances, must be positive. Hence, so also must the left side of (18) be positive.

Now the sign of $g(h,k)$ $[=h^2f_{xx}(a,b) + 2hkf_{xy}(a,b) + k^2f_{yy}(a,b)]$ is the same as the sign on the left side of (18) (i.e., positive) provided $f_{xx}(a,b)$ is positive; it is negative if $f_{xx}(a,b)$ is negative, and the case $f_{xx}(a,b) = 0$, as we mentioned earlier, cannot occur.

When $g(h,k)$ is always positive, so also is $f(a + h, b + k) - f(a,b)$ [this follows from (17)] and in this case $f(a,b)$ is a minimum. Similarly if $g(h,k)$ is always negative so also is $f(a + h, b + k) - f(a,b)$, so in this case $f(a,b)$ is a maximum. In summary of Case 1, then

4.7.3(L) continued

If $f_x(a,b) = f_y(a,b) = 0$ and $f_{xx}(a,b)f_{yy}(a,b) - f_{xy}^2(a,b) > 0$ then $f(a,b)$ is a maximum if $f_{xx}(a,b) > 0$ and minimum if $f_{xx}(a,b) < 0$ [the possibility that $f_{xx}(a,b) = 0$ is excluded by the premise that $f_{xx}(a,b)f_{yy}(a,b) - f_{xy}^2(a,b) > 0$].

Case 2: $f_{xx}(a,b)f_{yy}(a,b) - f_{xy}^2(a,b) < 0$

Now the last term on the right side of (18) is negative while the first term is positive. Hence the sum may be either positive or negative depending on the magnitudes of the individual terms. For example, with $k = 0$, the right side of (18) is $[hf_{xx}(a,b)]^2$ which is positive. On the other hand the first term on the right side of (18) is zero if

$$\frac{h}{k} = -\frac{f_{xy}(a,b)}{f_{xx}(a,b)}$$

or if

$$\frac{k}{h} = -\frac{f_{xx}(a,b)}{f_{xy}(a,b)}.$$

Not both $f_{xx}(a,b)$ and $f_{xy}(a,b)$ can be zero, also $f_{xx}(a,b)f_{yy}(a,b) - f_{xy}^2(a,b)$ could not be negative. So if we pick h and k to obey this ratio (and notice that holds for small values of h and k) the right side of (lf) is negative.

In other words, when Case 2 applies $f(a + h, b + k) - f(a,b)$ takes on both positive and negative values in any neighborhood of (a,b). That is

If $f_x(a,b) = f_y(a,b) = 0$ and $f_{xx}(a,b)f_{yy}(a,b) - f_{xy}^2(a,b) > 0$ then (a,b) is a saddle point of f.

As a partial check notice that our "intuition" indicates a saddle point if $f_{xx}(a,b)$ and $f_{yy}(a,b)$ have opposite signs.

4.7.3(L) continued

Namely, the "slice" $x = a$ yields a minimum (maximum) at (a,b) but the "slice" $y = b$ yields a maximum (minimum) at (a,b). Notice in this case that $f_{xx}(a,b)f_{yy}(a,b) < 0$ so that $f_{xx}(a,b)f_{yy}(a,b) - f_{xy}^2(a,b) < 0$, as it should be.

A word of caution, however, notice that even though $f_{xx}(a,b)$ and $f_{yy}(a,b)$ may have the same sign [so that $f_{xx}(a,b)f_{yy}(a,b) > 0$] it may happen that $f_{xy}^2(a,b)$ is sufficiently large so that $f_{xx}(a,b)f_{yy}(a,b) - f_{xy}^2(a,b)$ is not positive. Thus, merely knowing the signs of $f_{xx}(a,b)$ and $f_{yy}(a,b)$ is not sufficient to determine the sign of $f(a + h, b + k) - f(a,b)$.

The only remaining possibility is when $f_{xx}(a,b)f_{yy}(a,b) - f_{xy}^2(a,b) = 0$ [of which a special case is $f_{xx}(a,b) = f_{yy}(a,b) = f_{xy}(a,b) = 0$]. In this case

$$\text{sign }\{f_{xx}(a,b)[f(a + h, b + k) - f(a,b)]\} = \text{sign }\{[hf_{xx}(a,b) + kf_{yy}(a,b)]^2\},$$

$$\geq 0$$

$\text{sign}[\Delta f] = [\text{sign } f_{xx}(a,b)]$ unless $hf_{xx}(a,b) + kf_{yy}(a,b) \neq 0$.

Choosing h and k so that $hf_{xx}(a,b) = kf_{yy}(a,b) = 0$ means that we cannot determine the sign of Δf. In summary

If $f_x(a,b) = f_y(a,b) = 0$ and $f_{xx}(a,b)f_{yy}(a,b)-f_{xy}^2(a,b) = 0$ we cannot conclude whether $f(a,b)$ is a maximum or a minimum.

4.7.4 (optional)

a. Letting $F(t) = f(a + ht, b + kt)$, $0 \leq t \leq 1$, we already have that

$$\left.\begin{aligned} F'(t) &= hf_x + kf_y \\ F''(t) &= h^2f_{xx} + 2hkf_{xy} + k^2f_{yy} \end{aligned}\right\}$$ where all partials are computed at $(a + ht, b + kt)$

4.7.4 continued

$$F'''(t) = \frac{\partial(F''(t))}{\partial x}\frac{dx}{dt} + \frac{\partial(F''(t))}{\partial y}\frac{dy}{dt}$$

$$= h\frac{\partial(F''(t))}{\partial x} + k\frac{\partial(F''(t))}{\partial y}$$

$$= h[h^2 f_{xxx} + 2hkf_{xyx} + k^2 f_{yyx}]$$

$$+ k[h^2 f_{xxy} + 2hkf_{xyy} + k^2 f_{yyy}]$$

$$= h^3 f_{xxx} + 3h^2 k f_{xxy}{}^* + 3hk^2 f_{xyy}{}^* + k^3 f_{yyy}$$

Therefore,

$$F'''(1) = h^3 f_{xxx}(a,b) + 3h^2 k f_{xxy}(a,b) + 3hk^2 f_{xyy}(a,b) + k^3 f_{yyy}(a,b).$$

b. Therefore since $f(a + h, b + k) - f(a,b) = F(1) - F(0)$, we have $F(1) = F(0) + F'(0) + \frac{F''(0)}{2!} + \frac{F'''(0)}{3!} + \ldots$

or

$$f(a + h, b + k) - f(a,b) = [hf_x(a,b) + kf_y(a,b)] + \frac{1}{2!}[h^2 f_{xx}(a,b) + 2hkf_{xy}(a,b) + k^2 f_{yy}(a,b)] + \frac{1}{3!}[h^3 f_{xxx}(a,b) + 3h^2 k f_{xxy}(a,b) + 3hk^2 f_{xyy} + k^3 f_{yyy}(a,b)] + \ldots.$$

[Notice that the terms suggest the binomial coefficients obtained by expanding $(h + k)^3$ and each term is multiplied by a mixed partial derivative. The derivative is obtained by

*If all this partial derivatives exist it is customary to write, say, f_{xxy} to denote f_{xyx}, f_{yxx}. In other words the results about mixed partials extends to any order derivative.

4.7.4 continued

differentiating with respect to x a number of terms equal to the exponent of h while we differentiate with respect to y the number of terms equal to the exponent of k. For example, the 4th degree term would be

$$h^4 f_{xxxx}(a,b) + 4h^3 k f_{xxxy}(a,b) + 6h^2k^2 f_{xxyy}(a,b) + 4hk^3 f_{xyyy}(a,b) + k^2 f_{yyyy}(a,b)]$$

where we are assuming, of course, that the necessary continuity is present to allow us to differentiate in the order we please.

c. $f(x,y) = e^x \cos y$

$f_x = e^x \cos y,\ f_y = -e^x \sin y$

$f_x(0,0) = 1,\ f_y(0,0) = 0$

$f_{xx} = e^x \cos y,\ f_{xy} = -e^x \sin y,\ f_{yy} = -e^x \cos y$

$f_{xx}(0,0) = 1,\ f_{xy}(0,0) = 0,\ f_{yy}(0,0) = -1$

$f_{xxx} = e^x \cos y,\ f_{xxy} = -e^x \sin y,\ f_{xyy} = -\ e^x \cos y,\ f_{yyy} = e^x \sin y$

$f_{xxx}(0,0) = 1,\ f_{xxy}(0,0) = 0,\ f_{xyy}(0,0) = -1,\ f_{yyy}(0,0) = 0.$

Therefore,

$$\begin{aligned} f(h,k) - f(0,0) = {} & [f_x(0,0)h + f_y(0,0)k] \\ & + \frac{1}{2!}[h^2 f_{xx}(0,0) + 2hk f_{xy}(0,0) = k^2 f_{yy}(0,0)] \\ & + \frac{1}{3!}[h^3 f_{xxx}(0,0,0) + 3h^2 k f_{xxy}(0,0) + 3hk^2 f_{xyy} \\ & + k^3 f_{yyy}(0,0)]. \end{aligned}$$

Therefore,

$$e^h \cos k - 1 = [1h + 0k] + \frac{[1h^2 + 0(2hk) - 1k^2]}{2!} + \frac{[1h^3 + 3h^2k(0) + 3hk^2(-1) + k^3(0)]}{3!}$$

Therefore,

$$\underline{e^h \cos k = 1 + h + \tfrac{1}{2}h^2 - \tfrac{1}{2}k^2 + \tfrac{1}{6}h^3 - \tfrac{1}{2}hk^2}. \qquad (1)$$

4.7.5

a. We have the surface $w = f(x,y)$ where

$$f(x,y) = x^3 - y^3 - 2xy + 6. \quad (1)$$

Then

$$f_x(x,y) = 3x^2 - 2y \quad (2)$$
$$f_y(x,y) = -3y^2 - 2x \quad (3)$$
$$f_{xy}(x,y) = -2 \quad (4)$$
$$f_{xx}(x,y) = 6x \quad (5)$$
$$f_{yy}(x,y) = -6y. \quad (6)$$

Setting f_x and f_y equal to zero in (2) and (3) yields

$$\left.\begin{array}{l} 3x^2 - 2y = 0 \quad \text{or } y = \frac{3}{2}x^2 \quad (7) \\ -3y^2 - 2x = 0 \quad \text{or } x = -\frac{3}{2}y^2 \quad (8) \end{array}\right\}.$$

Substituting (8) into (7) we obtain

$$y = \tfrac{3}{2}[-\tfrac{3}{2}y^2]^2$$

or

$$27y^4 - 8y = 0.$$

Therefore, $y(27y^3 - 8) = 0$, therefore $y = 0$ or $y = \frac{2}{3}$.

From (8), $y = 0$ implies $x = 0$ while $y = \frac{2}{3}$ implies $x = -\frac{2}{3}$.

Hence the only candidates for max/min points of $f(x,y)$ are $(x,y) = (0,0)$ and $(x,y) = (-\frac{2}{3}, \frac{2}{3})$.

From (4), (5), and (6) we have

$$f_{xy}(0,0) = -2,\ f_{xx}(0,0) = 0,\ f_{yy}(0,0) = 0.$$

Therefore $f_{xx}(0,0)f_{yy}(0,0) - f_{xy}{}^2(0,0) = 0 - (-2)^2$

4.7.5 continued

$= -4$
$< 0.$

Hence by the criterion of Exercise 4.7.3(L), f has a saddle point at (0,0). From (1), $f(0,0) = 6$. Hence the surface $w = f(x,y) = x^3 - y^3 - 2xy + 6$ has a saddle point at (0,0,6).

Also from (4), (5), and (6) we have

$$f_{xy}(-\tfrac{2}{3}, \tfrac{2}{3}) = -2,\ f_{xx}(-\tfrac{2}{3}, \tfrac{2}{3}) = -4,\ f_{yy}(-\tfrac{2}{3}, \tfrac{2}{3}) = -4.$$

Hence

$$f_{xx}(-\tfrac{2}{3}, \tfrac{2}{3})f_{yy}(-\tfrac{2}{3}, \tfrac{2}{3}) - f_{xy}^{\ 2}(-\tfrac{2}{3}, \tfrac{2}{3}) = (-4)(-4)-(-2)^2$$

$$= 16 - 4 > 0.$$

This coupled with the fact that $f_{xx}(-\frac{2}{3},\frac{2}{3}) < 0$, tells us that f has a maximum at $(-\frac{2}{3},\frac{2}{3})$. Since $w = x^3 - y^3 - 2xy + 6$, at $(-\frac{2}{3},\frac{2}{3})$

$$w = \left(\frac{-2}{3}\right)^3 - \left(\frac{2}{3}\right)^3 - 2\left(\frac{-2}{3}\right)\left(\frac{2}{3}\right) + 6$$

$$= \frac{-8}{27} - \frac{8}{27} + \frac{8}{9} + 6$$

$$= 6 + \frac{8}{27} = \frac{170}{27}.$$

Hence, the surface has a relative maximum (high point) at $(-\frac{2}{3},\frac{2}{3},\frac{170}{27})$. [There is no absolute maximum. For example, if we let $y = 0$ (i.e. the intersection of our surface and the xz-plane) we obtain the curve

$$w = x^3 + 6$$

so that w increases without bound as x increases without bound.]

4.7.5 continued

b. With $w = f(x,y) = x^3 + y^3 + 3x^2 - 3y^2 + 8$, we have

$$f_x(x,y) = 3x^2 + 6x \tag{1}$$

$$f_y(x,y) = 3y^2 - 6y \tag{2}$$

$$f_{xy}(x,y) = 0 \tag{3}$$

$$f_{xx}(x,y) = 6x + 6 \tag{4}$$

$$f_{yy}(x,y) = 6y - 6. \tag{5}$$

From (1) and (2)

$$\left.\begin{aligned} f_x(x,y) = 0 \rightarrow x = 0 \text{ or } x = -2 \\ f_y(x,y) = 0 \rightarrow y = 0 \text{ or } y = 2 \end{aligned}\right\} . \tag{6}$$

From (6) we see that $f_x(x,y)$ and $f_y(x,y)$ are simultaneously zero $\leftrightarrow$ $x = 0$ and $y = 0$, or $x = 0$ and $y = 2$, or $x = -2$ and $y = 0$, or $x = -2$ and $y = 2$. That is, the only candidates for max/min points of f are $(0,0)$, $(0,2)$, $(-2,0)$ and $(-2,2)$.

From (3), (4), and (5), we have

$$f_{xx}(0,0)f_{yy}(0,0) - f_{xy}{}^2(0,0) = 6(-6) - 0 = -36 < 0.$$

Therefore, $(0,0)$ is a saddle point of f. Moreover, $f(0,0) = 8$, so therefore (0,0,8) is a saddle point on our surface. $f_{xx}(0,2) f_{yy}(0,2) - f_{xy}{}^2(0,2) = (6)(6) - 0^2 = 36 > 0$, and $f_{xx}(0,2) = 6 > 0$.

4.7.5 continued

Therefore (0,2) is a relative minimum point for f

$f(0,2) = 0^3 + 2^3 + 3(0)^2 - 3(2)^2 + 8 = 4.$

Therefore (0,2,4) is a local low point on our surface

$f_{xx}(-2,0)f_{yy}(-2,0) - f_{xy}^2(-2,0) = (-6)(-6) - 0^2 = 36 > 0$

and

$f_{xx}(-2,0) = -6 < 0.$

Therefore (-2,0) is a relative maximum of f

$f(-2,0) = (-2)^3 + 0^3 + 3(-2)^2 - 3(0)^2 + 8 = 12.$

(-2,0,12) is a relative (local) high point on our surface.

$f_{xx}(-2,2)f_{yy}(-2,2) - f_{xy}^2(-2,2) = (-6)(6) - 0 = -36 < 0$

$f(-2,2) = -8 + 8 + 12 - 12 + 8 = 8$

Therefore (-2,2,8) is a saddle point on our surface.

4.7.6(L)

The main aim of this exercise is to help you see how our theorems involving the chain rule and implicit functions may be used to simplify the computations which arise in the solution of max/min problems.

We first solve the problem using "brute force". Letting x and y denote the dimensions of the base of the box, and z its height, we have that

$$xy + 2xz + 2yz = 108 \quad (1)$$

4.7.6 continued

and subject to this constraint we wish to maximize the function f where

$$f(x,y,z) = xyz. \tag{2}$$

We may solve equation (1) explicitly for z in terms of x and y (being careful to observe that in more complicated problems, we might by necessity have to use implicit differentiation since it will not always be possible to solve for z explicitly in terms of x and y) to obtain:

$$z = \frac{108 - xy}{2(x + y)} \tag{3}$$

provided, of course, that $x + y \neq 0$. [If $x + y = 0$, then equation (1) would become

$$-x^2 = 108$$

which is an obvious contradiction since $-x^2 \leq 0$.]

In other words, then, if equation (1) is satisfied, equation (3) holds. That is, equations (1) and (3) are equivalent.

If we now replace z in equation (2) by its value in equation (3) we obtain

$$f(x,y,z) = \frac{108xy - x^2y^2}{2(x + y)},\ x + y \neq 0 \tag{4}$$

where f(x,y,z) is now a function of x and y alone, since, from (3), z is a function of x and y. [Paralleling the notation of the lecture, f(x,y,z) as defined by equation (4) will be denoted by h(x,y).]

That is

4.7.6(L) continued

$$h(x,y) = \frac{108xy - x^2y^2}{2(x + y)}, \; x + y \neq 0 \tag{4'}$$

and hence, the maximum (or minimum) values of h occur when $h_x(x,y)$ and $h_y(x,y)$ are simultaneously zero.

From equation (4') we have that

$$h_x(x,y) = \frac{2(x + y)(108y - 2xy^2) - (108xy - x^2y^2)2}{[2(x + y)]^2},$$

so that

$$h_x(x,y) = 0 \leftrightarrow 2(x + y)(108y - 2xy^2) - (108xy - x^2y^2)2 = 0$$

$$\leftrightarrow 2y[(x + y)(108 - 2xy) - (108x - x^2y)] = 0$$

$$\leftrightarrow 2y[108y - x^2y - 2xy^2] = 0$$

$$\leftrightarrow 2y^2[108 - x^2y - 2xy] = 0. \tag{5}$$

Therefore, from equation (5) we have that $h_x(x,y) = 0$ if and only if either $y = 0$ or $108 - x^2 - 2xy = 0$. Clearly $y = 0$ corresponds to a minimum capacity, so it must be that

$$108 - x^2 - 2xy = 0. \tag{6}$$

Since equation (4') is symmetric in x and y, we may use the result of equation (6) to conclude that $h_y(x,y) = 0 \leftrightarrow$

$$108 - y^2 - 2yx = 0. \tag{7}$$

Equating (6) and (7) yields

$$108 - x^2 - 2xy = 108 - y^2 - 2xy$$

or

$$x^2 = y^2$$

4.7.6(L) continued

and since x and y are non-negative, it follows that $x = y$. With $x = y$, equation (6) [or (7)] becomes $108 - 3x^2 = 0$ or $x = 6$.

Therefore $y = 6$, and from equation (3), $z = \frac{108 - 36}{24} = 3$. Consequently the box has a 6" by 6" base and a height of 3".

Note #1

If we solve this problem using the format of our lecture we have

$$f(x,y,z) = xyz$$

and

$$g(x,y,z) = 0 \text{ where } g(x,y,z) = xy + 2xz + 2yz - 108. \qquad (8)$$

Assuming that equation (8) determines z as a differentiable function of x and y, say $z = k(x,y)$, we have

$$h(x,y) = \begin{cases} f(x,y,z) \\ z = k(x,y) \end{cases}.$$

Hence by the chain rule

$$\left.\begin{aligned} h_x &= f_x + f_z \frac{\partial z}{\partial x} \\ h_y &= f_y + f_z \frac{\partial z}{\partial y} \end{aligned}\right\} \qquad (9)$$

while $g(x,y,z) = 0$

$$\left.\begin{aligned} g_x + g_z \frac{\partial z}{\partial x} &= 0 \\ g_y + g_z \frac{\partial z}{\partial y} &= 0 \end{aligned}\right\} \qquad (10)$$

Using the facts that $f(x,y,z) = xyz$ and $g(x,y,z) = xy + 2xz + 2yz - 108$, equations (9) and (10) become

4.7.6(L) continued

$$\left.\begin{aligned} h_x &= yz + xy\frac{\partial z}{\partial x} \\ h_y &= xz + xy\frac{\partial z}{\partial y} \end{aligned}\right\} \qquad (9')$$

and

$$\left.\begin{aligned} y + 2z + (2x + 2y)\frac{\partial z}{\partial x} &= 0 \\ x + 2z + (2x + 2y)\frac{\partial z}{\partial y} &= 0 \end{aligned}\right\} \qquad (10')$$

so that

$$\frac{\partial z}{\partial x} = \frac{-(y + 2z)}{2(x + y)}$$

$$\frac{z}{y} = \frac{-(x + 2z)}{2(x + y)} .$$

Accordingly $h_x = h_y = 0$ then become

$$\left.\begin{aligned} yz - \frac{xy(y + 2z)}{2(x + y)} &= 0 \\ xz - \frac{xy(x + 2z)}{2(x + y)} &= 0 \end{aligned}\right\} \qquad (11)$$

or

$$\left.\begin{aligned} 2(x + y)yz - xy(y + 2z) &= 0 \\ 2(x + y)xz - xy(x + 2z) &= 0 \end{aligned}\right\}$$

or

$$\left.\begin{aligned} 2y^2z - xy^2 &= 0 \\ 2x^2z - x^2y &= 0 \end{aligned}\right\}$$

or

$$\left.\begin{aligned} y^2(2x - x) &= 0 \\ x^2(2z - y) &= 0 \end{aligned}\right\} \qquad (12)$$

4.7.6(L) continued

and since $x = 0$ or $y = 0$ implies that $h(x,y)$ is minimum, equation (12) tells us that $2z - x = 2z - y = 0$; or $x = y = 2z$ whereupon we obtain $x = 0$, $y = 6$, $z = 3$ from equation (1).

Notice that this method is the implicit equivalent of the technique used in the solution of this exercise.

There is a rather nice "trick" that is sometimes helpful in simplifying this type of problem. The "trick" is known as the method of Lagrange Multipliers.

Note #2. Lagrange Multipliers

In Note #1 we were seeking max/min values for $f(x,y,z)$ subject to the constraint that $g(x,y,z) = 0$. From equations (9) and (10) we could conclude that

$$\begin{aligned} h_x &= f_x + f_z\left(-\frac{g_x}{g_z}\right) \\ h_y &= f_y + f_z\left(-\frac{g_y}{g_z}\right) \end{aligned} \tag{13}$$

provided $g_z \neq 0$.

If we now let $h_x = h_y = 0$ in equation (13) it follows that

$$f_x = g_x\left[\frac{f_z}{g_z}\right] \tag{14}$$

$$f_y = g_y\left[\frac{f_z}{g_z}\right] . \tag{15}$$

added to which we have as a triviality,

$$f_z = g_z\left[\frac{f_z}{g_z}\right]. \tag{16}$$

In n-tuple notation, equations (14), (15), and (16) may be combined to yield

$$(f_x, f_y, fz) = \left[\frac{f_z}{g_z}\right](g_x, g_y, g_z). \tag{17}$$

In other words the points $\underline{a}(a_1, a_2, a_3)$ at which $f(x,y,z)$ can have maximum or minimum values, subject to the constraint

4.7.6(L) continued

$g(x,y,z) = 0$ must satisfy

$$[f_x(\underline{a}),\ f_y(\underline{a}),\ f_z(\underline{a})] = \frac{f_z(\underline{a})}{g_z(\underline{a})}[g_x(\underline{a}),\ g_y(\underline{a}),\ g_z(\underline{a})] \qquad (18)$$

or

$$\vec{\nabla}f(\underline{a}) = \lambda\vec{\nabla}g(\underline{a}) \text{ where } \lambda = \frac{f_z(g)}{g_z(\underline{a})}. \qquad (18')$$

(There is a nice geometric interpretation of (18') which is discussed in detail in Section 15.11 of Thomas. It should be noted, however, as we shall soon see, that our approach works for any number of variables while the geometric interpretation does not apply for more than three variables)

If we now use hindsight and begin with our result in (14'), we may introduce the concept of a Lagrange Multiplier as follows:

To maximize or minimize $f(x,y,z)$ subject to the constraint $g(x,y,z) = 0$, construct the function $K(x,y,z,\lambda) = f(x,y,z) - \lambda g(x,y,z)$ and find values of x,y,z, and for which k_x, k_y, k_z, and are simultaneously zero.

(I.e., $k_x = 0$ says $f_x - \lambda g_x = 0$ or $f_x = \lambda g_x$, etc. while $k_\lambda = 0$ simply repeats the constraint that $-g(x,y,z) = 0$ [which is the same as $g(x,y,z) = 0$])

More generally, suppose we want to maximize or minimize $f(x_1,\ldots,x_n)$ subject to the constraint that $g(x_1,\ldots,x_n) = 0$. Our previous theory works precisely as before. Namely, assuming that $g(x_1,\ldots,x_n) = 0$ defines x_n as a continuously differentiable function of $x_1,\ldots,x_n$ (and this will happen provided $g_{x_n} \neq 0$), say, $x_n = \phi(x_1,\ldots,x_{n-1})$.

4.7.6(L) continued

Then, subject of this constraint, our function to be maximized (or minimized) is

$$f(x_1,\ldots,x_{n-1}, \phi(x_1\ldots,x_{n-1})) = h(x_1,\ldots,x_{n-1})$$

whence

$$\left.\begin{aligned} h_{x_1} &= f_{x_1} + f_{x_n}\frac{\partial\phi}{\partial x_1} \\ &\vdots \\ h_{x_{n-1}} &= f_{x_{n-1}} + f_{x_n}\frac{\partial\phi}{\partial x_{n-1}} \end{aligned}\right\} \qquad (19)$$

and we then find $\frac{\partial\phi}{\partial x_1},\ldots,\frac{\partial\phi}{\partial x_{n-1}}$ by differentiating

$$g(x_1,x_2,\ldots,x_{n-1}, (x_1,\ldots,x_{n-1})) = 0$$

to obtain

$$\left.\begin{aligned} g_{x_1} + g_{x_n}\frac{\partial\phi}{\partial x_1} &= 0 \\ &\vdots \\ g_{x_{n-1}} + g_{x_n}\frac{\partial\phi}{\partial x_{n-1}} &= 0 \end{aligned}\right\} \qquad (20)$$

so that

$$\frac{\partial\phi}{\partial x_1} = -\frac{g_{x_1}}{g_{x_n}},\ldots,\frac{\partial\phi}{\partial x_{n-1}} = -\frac{g_{x_{n-1}}}{g_{x_n}} \qquad (21)$$

Using (15) and (17), the conditions that $h_{x_1} = \ldots = h_{x_n} = 0$

become

$$\left.\begin{aligned} f_{x_1} + f_{x_n}\left(-\frac{g_{x_1}}{g_{x_n}}\right) &= 0 \\ f_{x_{n-1}} + f_{x_n}\left(-\frac{g_{x_{n-1}}}{g_{x_n}}\right) &= 0 \end{aligned}\right\} \text{ These come from (19) and (21).}$$

4.7.6(L) continued

$$f_{x_n} + f_{x_n}\left(-\frac{g_{x_n}}{g_{x_n}}\right) = 0 \ \}\ \text{This is just an identity.}$$

Therefore,

$$f_{x_1} = \left(\frac{f_{x_n}}{g_{x_n}}\right) g_{x_1}$$

$$f_{x_n} = \left(\frac{f_{x_n}}{g_{x_n}}\right) g_{x_n}.$$

So letting $\lambda = \dfrac{f_{x_n}}{g_{x_n}}$ and using n-tuple notation we have

$$(f_{x_1},\ldots,f_{x_n}) = \lambda(g_{x_1},\ldots,g_{x_n})$$

or

$$\vec{\nabla} f = \lambda \vec{\nabla} g$$

just as before [equation (18')].

Lagrange multipliers sometimes simplify our computations, but not always.

Applying this technique to the present exercise we have

$$\begin{aligned} k(s,y,z,\lambda) &= f(x,y,z) - \lambda g(x,y,z) \\ &= xyz - \lambda(xy + 2xz + 2yz - 108). \end{aligned}$$

Hence

$$\left.\begin{aligned} k_x &= 0 \rightarrow yz - \lambda(y + 2z) = 0 \\ k_y &= 0 \rightarrow xz - \lambda(x + 2z) = 0 \\ k_z &= 0 \rightarrow xy - \lambda(2x + 2y) = 0 \end{aligned}\right\} \quad (22)$$

The system of equations (22) is relatively easy to solve. For example we may multiply the first equation by x and the second by y to obtain

4.7.6(L) continued

$xyz - x(y + 2z) = 0 = xyz - y(x + 2z)$

or

$x(y + 2z) = y(x + 2z).$

So that with $\lambda \neq 0$,

$x(y + 2z) = y(x + 2z)$

or

$2xz = 2yz$

whereupon

$x = y.$

Similarly, multiplying the last equation of (22) by z, we obtain $xyz - \lambda z(2x + 2y) = 0$ and comparing this with $xyz - \lambda x(y + 2z) = 0$ yields $x = 2z$.

Notice, at least in this example, that the method of Lagrange Multipliers was less computationally involved than the more direct methods.

4.7.7

Let

$$f(x,y,z,w) = xyzw \tag{1}$$

[We use x,y,z,w, rather than x_1,x_2,x_3,x_4, so that our notation for partial derivatives will not be too combersome.]

Our constraint is that

$$x + y + z + w = \text{constant}. \tag{2}$$

Letting $g(x,y,z,w) = x + y + z + w$, we have that $g_w \equiv 1$, so w is a continuously differentiable function of x,y, and z

4.7.7 continued

everywhere.

Now from (2)

$$1 + \frac{\partial w}{\partial x} = 0$$

$$1 + \frac{\partial w}{\partial y} = 0$$

$$1 + \frac{\partial w}{\partial t} = 0$$

so that

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial y} = \frac{\partial w}{\partial z} = -1. \qquad (3)$$

Now letting $h(x,y,z) = f(x,y,z,w(x,y,z)) = xyz[w(x,y,z)]$,

$$h_x = xyzw_x + yzw$$

$$h_y = xyzw_y + xzw$$

$$h_z = xyzw_z + xyw$$

so from (3),

$$\left.\begin{aligned} h_x &= -xyz + yzw \\ h_y &= -xyz + xzw \\ h_z &= -xyz + xyw \end{aligned}\right\} \qquad (4)$$

Since x,y,z, or w = 0 imply that xyzw is not a maximum, we may assume that x,y,z, and w are each unequal to zero, whereupon when we let $h_x = h_y = h_z = 0$ (4) becomes

4.7.7 continued

$w = x$
$w = y$
$w = z$

or

$x = y = z = w$, as required.

4.7.8(L)

We wish to find the extreme values of $f(x,y,z) = x^2 + y^2 + z^2$ subject to the two constraints

$$x^2 + 2y^2 + z^2 = 1 \tag{1}$$

and

$$x + y = 1. \tag{2}$$

That is, we let $S = \{(x,y,z) = x^2 + 2y^2 + z^2 = 1 \text{ and } x + y = 1\}$

We then want the greatest and least values of $x^2 + y^2 + z^2$ subject to the condition that $(x,y,z) \varepsilon S$.

One approach is by direct substitution. From (2), if $(x,y,z) \varepsilon S$ then $y = 1 - x$. Hence the points of S have the form $(x_1, 1-x, z)$. Now from (1) any point (x,y,z), in S satisfies $x^2 + 2y^2 + z^2 = 1$; hence if $(x, 1-x, z)$ S it follows that

$$x^2 + 2(1 - x)^2 + z^2 = 1$$

or

$$z^2 = 4x - 1 - 3x^2 \tag{3}$$

or

$$z = \pm\sqrt{4x - 1 - 3x^2}\ . \tag{3'}$$

4.7.8(L) continued

Hence the points of S have the form

$(x,\ 1 - x,\ \pm\sqrt{4x - 1 - 3x^2}$.

So if we restrict our attention to S,

$$f(x,y,z) = x^2 + y^2 + z^2$$

$$= x^2 + (1-x)^2 + (4x - 1 - 3x^2)$$

$$= g(x).$$

Now

$$g'(x) = 2x - 2(1 - x) + 4 - 6x$$

$$= -2x + 2$$

$$g''(x) = -2.$$

Hence g(x) has a maximum when $x = 1$. When $x = 1$, then $1 - x = 0$, and $4x - 1 - 3x^2 = 0$, so the point $(x,\ 1 - x,\ \pm\sqrt{4x - 1 - 3x^2}$ of S which is furthest from the origin is

(1,0,0).

How about the point of S which is nearest form the origin? Since f is continuously differentiable on S, it must assume its minimum somewhere on S.

But $g'(x) = 0 \leftrightarrow x = 1$ yielded a maximum. Hence the minimum must occur at an end point of the interval upon which g is defined. What limits the domain of g? Well, since $(x,\ 1-x,\ \pm\sqrt{4x - 1 - 3x^2})$ is to be real, in particular $4x - 1 - 3x^2$ must be non-negative. Thus,

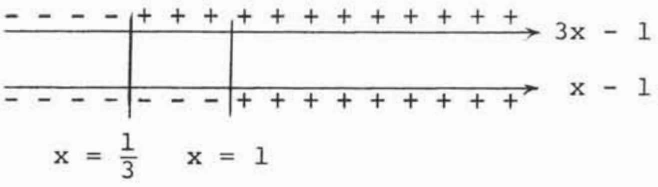
$$4x - 1 - 3x^2 \geq 0 \quad \leftrightarrow$$

$$3x^2 - 4x + 1 \leq 0 \quad \leftrightarrow$$

4.7.8(L) continued

$(3x - 1)(x - 1) \leq 0$

and this means that $(3x - 1)$ and $(x - 1)$ must either be zero or else have opposite signs



In other words, g is defined on $[\frac{1}{3}, 1]$ and hence assumes its minimum either when $x = \frac{1}{3}$ or $x = 1$. But we have already seen that $x = 1$ yields a maximum for g. Therefore g assumes its minimum when $x = \frac{1}{3}$. When $x = \frac{1}{3}$, $1 - x = \frac{2}{3}$, and $\pm\sqrt{4x-1-3x^2} = \pm\sqrt{\frac{4}{3} - 1 - \frac{1}{3}} = 0$. Hence the point of S, $(x, 1 - x, \pm\sqrt{4x-1-3x^2})$, which is nearest the origin is $(\frac{1}{3}, \frac{2}{3}, 0)$.

A second approach is through partial derivatives. We assume that our two constraints are consistent and independent so that we may view $x^2 + y^2 + z^2$ as a function of x alone (this happens provided $\left|\frac{\partial(g,h)}{\partial(y,z)}\right| \neq 0$, and this was disclosed in our treatment of functional dependence).

We then have

$$g(x) = f(x, y(x), z(x)) = x^2 + [y(x)]^2 + [z(x)]^2$$

Hence

$$g'(x) = 2x + 2y\frac{dy}{dx} + 2z\frac{dz}{dx}. \qquad (4)$$

Ultimately we shall solve $g'(x) = 0$, but this involves knowing $\frac{dy}{dx}$ and $\frac{dz}{dx}$ explicitly in terms of x.

From (2) we have that

$$1 + \frac{dy}{dx} = 0$$

4.7.8(L) continued

or

$$\frac{dy}{dx} = -1. \qquad (5)$$

From (1), we have that

$$2x + 4y\frac{dy}{dx} + 2z\frac{dz}{dx} = 0 \qquad (6)$$

and putting (5) into (6) we obtain

$$2x - 4y + 2z\frac{dz}{dx} = 0$$

whence

$$\frac{dz}{dx} = \frac{2y - x}{z} \qquad (7)$$

provided, of course, that $z \neq 0$.

(If $z = 0$, then $x^2 + 2y^2 + z^2 = x^2 + 2y^2$; hence $z = 0$ implies $x^2 + 2y^2 = 1$. This, coupled with $x + y = 1$, or $x = 1 - y$, means that

$$(1 - y)^2 + 2y^2 = 1$$

or

$$1 - 2y + 3y^2 = 1$$

or

$$y(2 - 3y) = 0$$

or

$$y = 0 \text{ or } y = \frac{2}{3}. \qquad (8)$$

What (8) tells us is that (7) is undefined (division by zero) whenever $y = 0$ or $y = \frac{2}{3}$. Thus, the points corresponding to these y-values must be checked separately. [Using hindsight,

4.7.8(L) continued

our first method showed that $y = 0$, that is the point $(1,0,0)$ was furthest from the origin, while $y = \frac{2}{3}$ corresponds to $(\frac{1}{3}, \frac{2}{3}, 0)$ which was nearest to the origin.]

Assuming that $z \neq 0$, the substitution of (5) and (7) into (4) yields

$$g'(x) = 2x - 2y + 2z \frac{(2y - x)}{z}$$

$$= 2[x - y + 2y - x].$$

Therefore,

$$g'(x) = 2y$$

$$g''(x) = \frac{d(2y)}{dy} \frac{dy}{dx}$$

$$= 2(-1)$$

$$= -2.$$

Therefore $g(x)$ is maximum when $y = 0$ (i.e., when $x = 1$ since $x + y = 1$) which agrees with the result of our first method.

To find where g is minimum we must check where g' does not exist and, as we have seen, this is when $z = 0$ whereupon $y = \frac{2}{3}$ and $x = \frac{1}{3}$. Thus,

$$(\tfrac{1}{3}, \tfrac{2}{3}, 0)$$

is closest to the origin.

Quiz

1. (a) To invert

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 7 & 9 \end{bmatrix}$$

we use the row-reduced augmented matrix technique. Namely,

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 7 & 0 & 1 & 0 \\ 3 & 7 & 9 & 0 & 0 & 1 \end{array}\right] \sim$$

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & -3 & 0 & 1 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & 0 & 1 \\ 0 & 1 & 1 & -2 & 1 & 0 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 0 & 3 & 7 & 0 & -2 \\ 0 & 1 & 0 & -3 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{bmatrix} \sim$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & -3 & 1 \\ 0 & 1 & 0 & -3 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right] . \qquad (1)$$

From the second half of (1), we conclude that if

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 7 & 9 \end{bmatrix}$$

then

1. continued

$$A^{-1} = \begin{bmatrix} 4 & -3 & 1 \\ -3 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

<u>Check</u>

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 7 & 9 \end{bmatrix}\begin{bmatrix} 4 & -3 & 1 \\ -3 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 4 - 6 + 3 & -3 + 0 + 3 & 1 + 2 - 3 \\ 8 - 15 + 7 & -6 + 0 + 7 & 2 + 5 - 7 \\ 12 - 21 + 9 & -9 + 0 + 9 & 3 + 7 - 9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3.$$

(b) $AX = B$ implies that

$$A^{-1}(AX) = A^{-1}B$$

or

$$(A^{-1}A)X = A^{-1}B$$

or

$$I_3X = A^{-1}B.$$

Hence,

$$X = A^{-1}B. \qquad (2)$$

From part (a),

$$A^{-1} = \begin{bmatrix} 4 & -3 & 1 \\ -3 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

and we are given that

1. continued

$$B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 6 & 7 \end{bmatrix}.$$

Putting these results into (2) yields

$$X = \begin{bmatrix} 4 & -3 & 1 \\ -3 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 6 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 8 - 12 + 6 & 12 - 15 + 7 \\ -6 + 0 + 6 & -9 + 0 + 7 \\ 2 + 4 - 6 & 3 + 5 + 7 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 4 \\ 0 & -2 \\ 0 & 1 \end{bmatrix}$$

<u>Check</u>

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 7 & 9 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 0 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 + 0 + 0 & 4 - 4 + 3 \\ 4 + 0 + 0 & 8 - 10 + 7 \\ 6 + 0 + 0 & 12 - 14 + 9 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 6 & 7 \end{bmatrix}$$

2. We may row-reduce

$$\begin{array}{cccc|cccc} x_1 & x_2 & x_3 & x_4 & b_1 & b_2 & b_3 & b_4 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 3 & 3 & 0 & 1 & 0 & 0 \\ 2 & 3 & 4 & 4 & 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}$$

to obtain

2. continued

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & -2 & 0 & 1 & 0 \\ 0 & -1 & -2 & -2 & -3 & 0 & 0 & 1 \end{bmatrix} \sim$$

$$\begin{matrix} x_1 & x_2 & x_3 & x_4 & b_1 & b_2 & b_3 & b_4 \end{matrix}$$
$$\begin{bmatrix} 1 & 0 & -1 & -1 & 2 & -1 & 0 & 0 \\ 0 & 1 & 2 & 2 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -5 & 0 & 1 & 1 \end{bmatrix}. \qquad (1)$$

The 3^{rd} row of (1) tells us that

$$-b_1 - b_2 + b_3 = 0$$

or

$$\underline{b_3 = b_1 + b_2} \qquad (2)$$

while the 4^{th} row of (1) tells us that

$$-5b_1 + b_3 + b_4 = 0$$

or

$$b_4 = 5b_1 - b_3. \qquad (3)$$

Putting (2) into (3) yields

$$b_4 = 5b_1 - (b_1 + b_2)$$

or

$$b_4 = 4b_1 - b_2. \qquad (4)$$

2. continued

In other words, unless $b_3 = b_1 + b_2$ and $b_4 = 4b_1 - b_2$, the given system has no solutions. In particular, with $b_1 = b_2 = b_3 = b_4 = 1$ the system has no solutions since then $b_3 \neq b_1 + b_2$ (and $b_4 \neq 4b_1 - b_2$).

(b) If $b_1 = b_2 = 1$, then b_3 must equal 2 (i.e., $b_1 + b_2$) and b_4 must equal 3 (i.e., $4b_1 - b_2$).

In this event, all solutions are determined by the first two rows of (1).

Namely, we must have

$$x_1 - x_3 - x_4 = 2b_1 - b_2$$

and

$$x_2 + 2x_3 + 2x_4 = -b_1 + b_2$$

or, since $b_1 = b_2 = 1$,

$$\left.\begin{array}{l} x_1 = 1 + x_3 + x_4 \\ \text{and} \\ x_2 = -2(x_3 + x_4) \end{array}\right\} \qquad (5)$$

From (5), we see that we may pick, for example, x_3 and x_4 at random whereupon x_1 is then given by $1 + x_3 + x_4$ and x_2 by $-2(x_3 + x_4)$.

3. Written as a linear system, we have

$$\underline{f}(x,y,z) = (u,v,w)$$

where

3. continued

$$\left.\begin{aligned} u &= 2x + 3y + 3z \\ v &= x + 2y + 2z \\ w &= 2x + y + z \end{aligned}\right\}$$

Using our row-reduced matrix technique, we have

$$\begin{matrix} (x & y & z & u & v & w) \end{matrix}$$

$$\begin{bmatrix} 2 & 3 & 3 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 2 & 2 & 0 & 1 & 0 \\ 2 & 3 & 3 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 2 & 2 & 0 & 1 & 0 \\ 0 & -1 & -1 & 1 & -2 & 0 \\ 0 & -3 & -3 & 0 & -2 & 1 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 0 & 0 & 2 & -3 & 0 \\ 0 & -1 & -1 & 1 & -2 & 0 \\ 0 & 0 & 0 & -3 & 4 & 1 \end{bmatrix} . \qquad (1)$$

$$\begin{matrix} x & y & z & u & v & w \end{matrix}$$

The last row of (1) tells us that

$-3u + 4v + w = 0.$

That is, the image of f is the plane (in uvw-space) $w = 3u - 4v$.
Since a plane is 2-dimensional, it is obvious that f is not an onto map.

4. (a) Letting

$$\left.\begin{aligned} u &= x^3 - 3xy^2 \\ v &= 3x^2y - y^3 \end{aligned}\right\} \tag{1}$$

we want x and y such that u = 2.001 and v = 11.001.

From (1), we have

$$\left.\begin{aligned} du &= (3x^2 - 3y^2)dx - 6xy\,dy \\ dv &= 6xy\,dx + (3x^2 - 3y^2)dy \end{aligned}\right\} \tag{2}$$

so that from (2)

$$(3x^2 - 3y^2)du = (3x^2 - 3y^2)^2 dx + (3x^2 - 3y^2)(-6xy\,dy)$$

$$6xy\,dv = (6xy)^2 dx + (3x^2 - 3y^2)(6xy\,dy)$$

or

$$(3x^2 - 3y^2)du + 6xy\,dv = (9x^4 + 18x^2y^2 + 9y^4)dx$$

$$= 9(x^2 + y^2)^2 dx$$

or

$$dx = \frac{x^2 - y^2}{3(x^2 + y^2)^2}\,du + \frac{2xy}{3(x^2 + y^2)^2}\,dv. \tag{3}$$

Returning to (2), we also find that

$$-6xy\,du = -6xy(3x^2 - 3y^2)dx + 36x^2y^2\,dy$$

$$(3x^2 - 3y^2)dv = 6xy(3x^2 - 3y^2)dx + (3x^2 - 3y^2)^2 dy$$

or

$$-6xy\,du + 3(x^2 - y^2)dv = 9(x^2 + y^2)^2 dy$$

4. continued

so that

$$dy = \frac{-2xy\ du}{3(x^2 + y^2)^2} + \frac{(x^2 - y^2)}{3(x^2 + y^2)^2}\ dv. \qquad (4)$$

Utilizing the approximation that $\Delta x \approx dx$ $(= \Delta x_{tan})$, $\Delta y \approx dy$ $(= \Delta y_{tan})$, we may combine (3) and (4) with $x = 2$, $y = 1$, $du = dv = 0.001$ to obtain

$$dx = \frac{3}{3(5)^2}(0.001) + \frac{4}{3(5)^2}(0.001)$$

$$= \frac{.007}{75}$$

or

$$\Delta x \approx 0.000093$$

and

$$dy = \frac{-4(0.001)}{75} + \frac{3(0.001)}{75}$$

$$= \frac{-0.001}{75}$$

or

$$\Delta y \approx -0.000013.$$

Hence, the required point is approximately

$$(2 + 0.000093,\ 1 - 0.000013),$$

or

$$(2.000093,\ 0.999987).$$

4. continued

In summary,

$$(2.000093)^3 - 3(2.000093)(0.999987)^2 \approx 2.001$$

and

$$3(2.000093)^2(0.999987) - (0.999987)^3 \approx 11.001.$$

5. We have

$$f(x,y) = x^3 + y^2 - 6xy + 6x + 3y + 1.$$

Hence,

$$\left.\begin{aligned} f_x(x,y) &= 3x^2 - 6y + 6 = 3(x^2 - 2y + 2) \\ f_y(x,y) &= 2y - 6x + 3 \end{aligned}\right\} . \qquad (1)$$

For max/min values, we wish to solve

$$\left.\begin{aligned} f_x &= 0 \\ f_y &= 0 \end{aligned}\right\} ,$$

so by (1), we have

$$\left.\begin{aligned} x^2 - 2y + 2 &= 0 \\ 2y - 6x + 3 &= 0 \end{aligned}\right\}$$

or

$$\left.\begin{aligned} x^2 - 2y + 2 &= 0 \\ 2y &= 6x - 3 \end{aligned}\right\} . \qquad (2)$$

Therefore,

$$x^2 - 6x + 5 = 0$$

5. continued

or

$x = 1$, or $x = 5$.

Looking at the second equation in (2), we see that if $x = 1$, then $2y = 6 - 3$ or $y = \frac{3}{2}$; and when $x = 5$, $2y = 30 - 3$ so $y = \frac{27}{2}$. Hence, our candidates are the points $(1,\frac{3}{2})$ and $(5,\frac{27}{2})$.

Noticing that

$$f(1,\tfrac{3}{2}) = 1^3 + \left(\frac{3}{2}\right)^2 - 6(1)(\tfrac{3}{2}) + 6(1) + 3(\tfrac{3}{2}) + 1$$

$$= 1 + \frac{9}{4} - 9 + 6 + \frac{9}{2} + 1$$

$$= \frac{23}{4} = 5\tfrac{3}{4} \qquad (3)$$

and

$$f(5,\tfrac{27}{2}) = 5^3 + \left(\frac{27}{2}\right)^2 - 6(5)(\tfrac{27}{2}) + 6(5) + 3(\tfrac{27}{2}) + 1$$

$$= 125 + \frac{729}{4} - 405 + 30 + \frac{81}{2} + 1$$

$$= -\frac{105}{4}. \qquad (4)$$

Comparing (3) and (4), it appears clear that f attains a minimum at $x = 5$, $y = \frac{27}{2}$ and that this maximum value is $-\frac{105}{4}$.

We should be a bit suspicious about $f(1,\frac{3}{2})$ being a maximum since, for example

$$f(2,0) = 21 > 5\tfrac{3}{4} = f(1,\tfrac{3}{2}).$$

To play it safe, we observe that

5. continued

$f_{xx} = 6x$

$f_{yy} = 2$

$f_{xy} = -6.$

Hence,

$$f_{xx}\, f_{yy} - f_{xy}^{\,2} = 12x - 36$$

and

$$12x - 36 \begin{cases} < 0 \text{ if } x = 1 \\ > 0 \text{ if } x = 5 \end{cases}$$

Hence, $x = 1$ corresponds to a saddle point while $x = 5$ corresponds to a minimum (not a maximum since $f_{xx}(5,\frac{27}{2}) = 30 > 0 \rightarrow$ "holding water").

In other words, $x^3 + y^2 - 6xy + 6x + 3y + 1$ is at least as great as $-\frac{105}{4}$, but it can be made as great as we wish by appropriate choices of x and y.