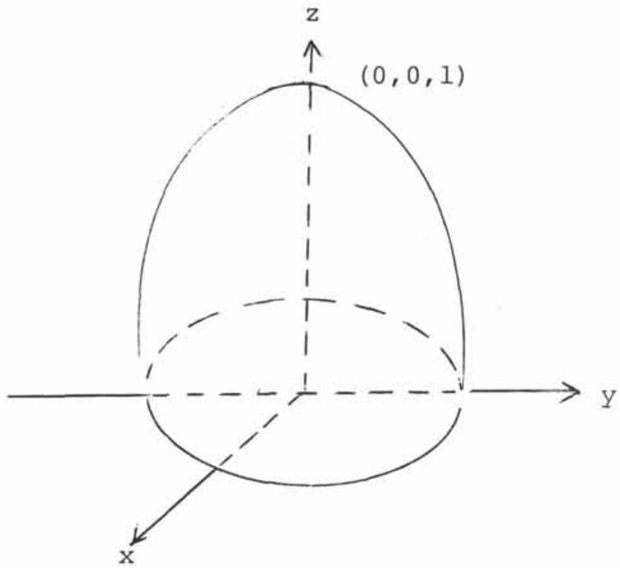
Unit 6: Surface Area

5.6.1(L)



S projects onto the circular disc $x^2 + y^2 \leq 1$ in the xy-plane. (We get this information by letting $z = 0$ in $z = 1 - x^2 - y^2$). Now, a normal vector to S at $(x_o, y_o, 1 - x_o^2 - y_o^2)$ is given by

$$\frac{\partial z}{\partial x}\vec{i} + \frac{\partial z}{\partial y}\vec{j} - \vec{k}.$$

(If this is hazy review Section 15.3 of the text), or

$$-2x\vec{i} - 2y\vec{j} - \vec{k},$$

so that

$$\frac{2x\vec{i} + 2y\vec{j} + \vec{k}}{\sqrt{4x^2 + 4y^2 + 1}}$$

is a unit normal, $\vec{u}_n$.

Hence an element of surface area in terms of dx and dy is given by

5.6.1(L) continued

$$\frac{dy\ dx}{|\vec{u}_n \cdot \vec{k}|} = \frac{dy\ dx}{\frac{(2x\vec{i} + 2y\vec{j} + \vec{k})}{\sqrt{4x^2 + 4y^2 + 1}} \cdot \vec{k}}$$

$$= \sqrt{4x^2 + 4y^2 + 1}\ dy\ dx.$$

Thus, the surface area of S is given by

$$\iint_{x^2 + y^2 \le 1} \sqrt{4x^2 + 4y^2 + 1}\ dy\ dx. \qquad (1)$$

The form of this integral suggests polar coordinates, in which case the region $\{(x,y): x^2 + y^2 \le 1\}$ is written as $\{(r,\theta): 0 \le r \le 1, 0 \le \theta \le 2\pi\}$. So, written in polar coordinates formula (1) becomes

$$\int_0^{2\pi}\int_0^1 \sqrt{4r^2 + 1}\ r dr\ d\theta$$

$$= \int_0^{2\pi} \frac{1}{12} (4r^2 + 1)^{\frac{3}{2}} \Big|_{r=0}^{1} d\theta$$

$$= \frac{1}{12} \int_0^{2\pi} (5^{\frac{3}{2}} - 1^{\frac{3}{2}})\ d\theta$$

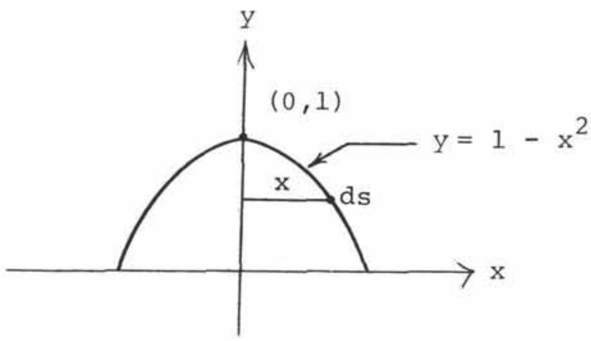
$$= \frac{2\pi}{12} [5\sqrt{5} - 1]$$

$$= \frac{\pi}{6} (5\sqrt{5} - 1).$$

5.6.1(L) continued

We have elected to start off with this exercise because the region S happens to be a surface of revolution. Consequently we may check our answer by the technique used for finding surface area in Part 1.

Namely, notice that the surface S may be obtained by revolving the portion of the parabola; $y = 1 - x^2$, $y \geq 0$, about the y-axis. That is



In other words the surface area should also be given by

$$\int_{y=0}^{1} 2\pi x \; ds = \int_0^1 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy. \qquad (2)$$

Since $y = 1 - x^2$, it follows that $\frac{dy}{dx} = -2x$, so that $\frac{dx}{dy} = -\frac{1}{2x}$ and $\left(\frac{dx}{dy}\right)^2 = \frac{1}{4x^2}$. Hence

$$2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}$$

$$= 2\pi x \sqrt{1 + \frac{1}{4x^2}}$$

$$= 2\pi x \frac{\sqrt{4x^2 + 1}}{2x}$$

$$= \sqrt{4x^2 + 1}$$

and this, in turn is $\pi\sqrt{5 - 4y}$ since $y = 1 - x^2$ implies that $x^2 = 1 - y$, or $4x^2 + 1 = 4(1 - y) + 1 = 5 - 4y$.

5.6.1(L) continued

Putting this information into (2) yields that the required area is

$$\int_0^1 \sqrt{5 - 4y}\, dy$$

$$= -\frac{\pi}{6}(5 - 4y)^{\frac{3}{2}} \Big|_{y=0}^{1}$$

$$= -\frac{\pi}{6}[1 - 5^{\frac{3}{2}}]$$

$$= \frac{\pi}{6}(5^{\frac{3}{2}} - 1) = \frac{\pi}{6}(5\sqrt{5} - 1)$$

which checks with our previous result.

5.6.2

Pictorially we have

$z = 1$ meets $2z = x^2 + y^2$ in the circle $2 = x^2 + y^2$ (i.e., let $z = 1$ in $2z = x^2 + y^2$).

So our surface projects onto the region $x^2 + y^2 \leq 2$ in the xy-plane.

Our element of surface area is

$$\frac{dy\ dx}{|\vec{u}_n \cdot \vec{k}|},$$

and $\vec{u}_n$ is given by

5.6.2 continued

$$\pm \frac{[\frac{\partial z}{\partial x}\vec{i} + \frac{\partial z}{\partial y}\vec{j} - \vec{k}]}{\sqrt{(\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2 + 1}} \qquad (1)$$

and since, $z = \frac{1}{2}(x^2 + y^2)$ it follows that $\frac{\partial z}{\partial x} = x$ and $\frac{\partial z}{\partial y} = y$. Consequently (1) becomes

$$\pm \frac{x\vec{i} + y\vec{j} - \vec{k}}{\sqrt{x^2 + y^2 + 1}},$$

whereupon

$$|\vec{u}_n \cdot \vec{k}| = \frac{1}{\sqrt{x^2 + y^2 + 1}}.$$

Hence, the surface area is

$$\iint_{x^2 + y^2 \le 2} \sqrt{x^2 + y^2 + 1}\, dy\, dx$$

and introducing polar coordinates yields

$$\int_0^{2\pi}\int_0^{\sqrt{2}} \sqrt{r^2 + 1}\, r dr\, d\theta$$

$$= \int_0^{2\pi} \frac{1}{3}(r^2 + 1)^{\frac{3}{2}} \Big|_{r=0}^{\sqrt{2}} d\theta$$

$$= \frac{2\pi}{3}[(2 + 1)^{\frac{3}{2}} - (0 + 1)^{\frac{3}{2}}]$$

$$= \frac{2\pi}{3}[3\sqrt{3} - 1].$$

5.6.3

Since the plane is cut by the cylinder $x^2 + y^2 = 1$ it is clear that the cut portion of the plane projects onto the region $x^2 + y^2 \le 1$ in the xy-plane.

A normal vector to the plane $x + y + z = 1$ is $\vec{i} + \vec{j} + \vec{k}$. Consequently $\vec{u}_n \cdot \vec{k}$, in this case, is $1/\sqrt{3}$.

Thus, our desired area is given by

$$\iint_{x^2+y^2 \le 1} \frac{dy\,dx}{|\vec{u}_n \cdot \vec{k}|}$$

$$= \iint_{x^2+y^2 \le 1} \sqrt{3}\,dy\,dx$$

$$= \int_0^{2\pi}\int_0^1 \sqrt{3}\,r\,dr\,d\theta$$

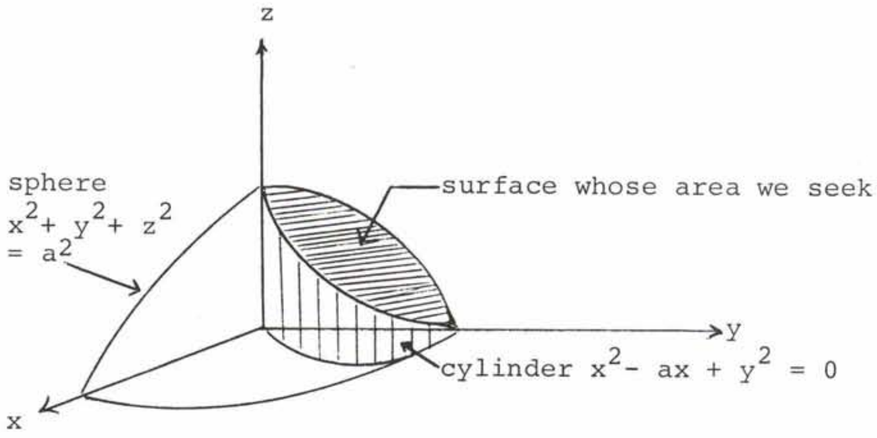
$$= \int_0^{2\pi} \frac{\sqrt{3}}{2}\, r^2 \Big|_{r=0}^{1} d\theta$$

$$= \frac{\sqrt{3}}{2}\int_0^{2\pi} d\theta$$

$$= \sqrt{3}\,\pi$$

5.6.4

Our diagram, in the first octant is given by



The surface we seek has, by symmetry, four times the area of the region shown above.

At any rate we have that the surface area in question is given by

$$4 \iint \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}^{\,*}\, dy\, dx \qquad (1)$$

$$\left.\begin{array}{l} x^2 - ax + y^2 \le 0 \\ \text{or} \\ (x - \frac{a}{2})^2 + y^2 \le \frac{a^2}{4} \end{array}\right\}$$

*Here we have combined a few steps and rewrote $\frac{dy\ dx}{|\vec{u}_n \cdot \vec{k}|}$ as $\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\ dy\ dx$ directly.

5.6.4 continued

$$4 \int_0^a \int_0^{\sqrt{ax-x}} \sqrt{\frac{x^2 + y^2}{a^2 - x^2 - y^2} + 1}\ dy\ dx. \qquad (2)$$

Since

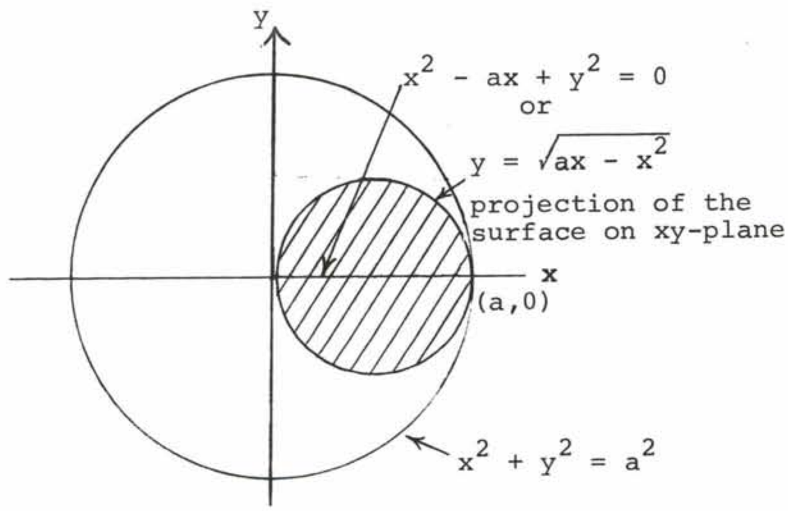
(i) $z = + \sqrt{a^2 - x^2 - y^2}$ $\quad \frac{\partial z}{\partial x} = \frac{-x}{\sqrt{a^2 - x^2 - y^2}}$ and $\frac{\partial z}{\partial y} = \frac{-y}{\sqrt{a^2 - x^2 - y^2}}$

and

$$\frac{\partial z}{\partial y} = \frac{-y}{\sqrt{a^2 - x^2 - y^2}}$$

and

(ii)



Simplifying (2) yields

$$4 \int_0^a \int_0^{\sqrt{ax-x^2}} \frac{a\ dy\ dx}{\sqrt{a^2 - x^2 - y^2}} \qquad (3)$$

and again, polar coordinates are suggested.

since the polar form of $x^2 - ax + y^2 = 0$ is given by

5.6.4 continued

$r^2 - ar\cos\theta = 0$ or $r = a\cos\theta$, formula (3) becomes

$$4\int_0^{\frac{\pi}{2}}\int_0^{a\cos\theta}\frac{ar\,dr\,d\theta}{\sqrt{a^2 - r^2}}$$

$$= 4a\int_0^{\frac{\pi}{2}} -\sqrt{a^2 - r^2}\,\Bigg|_{r=0}^{a\cos\theta}\,d\theta$$

$$= 4a\int_0^{\frac{\pi}{2}}[-\sqrt{a^2 - a^2\cos^2\theta} - (-\sqrt{a^2})]\,d\theta$$

$$= 4a^2\int_0^{\frac{\pi}{2}}(1 - \sin\theta)\,d\theta$$

$$= 4a^2[\theta + \cos\theta\,\Bigg|_{\theta=0}^{\frac{\pi}{2}}$$

$$= 4a^2[\tfrac{\pi}{2} - 1].$$

5.6.5(L)

Here we see a more "practical" need for surface area in terms of the mass of a "shell".

Letting $d\sigma$ denote an element of area on the hemispherical shell, we have that the mass of that element is $\rho d\sigma$, so that the mass is

$$\iint_\sigma \rho\, d\sigma\ . \qquad (1)$$

5.6.5(L) continued

Now $\rho(x,y,z) = kz = k\sqrt{1 - x^2 - y^2}$ at each point on the hemisphere, and $d\sigma$ is, as usual,

$$\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\ dy\ dx$$

$$= \frac{dy\ dx}{\sqrt{1 - x^2 - y^2}}. \qquad (2)$$

(just as in the previous exercise with a = 1).

Hence, putting (2) into (1) yields that the required mass is

$$\iint\limits_{x^2 + y^2 \leq 1} k\ dy\ dx$$

$$= \int_0^{2\pi}\int_{r=0}^{1} kr\ dr d\theta$$

$$= \int_0^{2\pi} \frac{1}{2} kr^2 \Big|_0^1 d\theta$$

$$= \pi k.$$

The main aim of this exercise is to emphasize that there are problems in which it is crucial that we replace plane regions by more general surfaces - quite apart from simply wanting to find surface areas.

5.6.6 (optional)

The main object of this exercise is to emphasize how we compute $\vec{u}_n$ when the surface S is given in the implicit form

$$g(x,y,z) = 0. \quad (1)$$

In this case we have already learned that $\vec{\nabla}g$ is normal to S, provided of course that we are at a point on S for which $\vec{\nabla}g \neq 0$. Thus, as long as $\vec{\nabla}g \neq 0$, $\vec{\nabla}g/|\vec{\nabla}g|$ may be used as the expression for u_n; and in this case

$$\frac{dy\ \ dx}{|\vec{u}_n \cdot \vec{k}|}$$

becomes

$$\frac{|\vec{\nabla} g\ |}{\vec{\nabla}g \cdot \vec{k}}\ dy\ dx. \quad (2)$$

The only way that (2) can get us in trouble is if $\vec{\nabla}_g \cdot \vec{k} = 0$, but were this to happen it would mean that $g_z = 0$, in which case $g(x,y,z) = 0$ does not define z as a differentiable function of x and y (recall in our discussion in Block 4 we pointed out that if $g(x,y,z)$ was continuously differentiable then $g(x,y,z) = 0$ defined z implicitly as a differentiable function of x and y <u>if and only if</u> $g_z \neq 0$).

Applying this discussion to the present exercise, we have that $x^2 + y^2 + z^2 = a^2$. Hence S may be written as $g(x,y,z) \equiv 0$ where $g(x,y,z) = x^2 + y^2 + z^2 - a^2$. Then $g_x = 2x$, $g_y = 2y$, and $g_z = 2z$; so that $\vec{\nabla}g = 2xi + 2yj + 2zk$ and $\vec{\nabla}g \cdot \vec{k} = 2z$.

Hence from formula (2), we obtain that the surface area in question is given by

$$\iint_{x^2+ y^2 \leq a^2} \sqrt{\frac{4x^2 + 4y^2 + 4z^2}{2z}}\ dy\ dx \quad (2)$$

and since $x^2 + y^2 + z^2 = a^2$ on S, formula (3) becomes

5.6.6 continued

$$\iint_{x^2+y^2\le 1} \frac{a\,dy\,dx}{z}$$

$$= \iint_{x^2+y^2\le a^2} \frac{a\,dy\,dx}{\sqrt{a^2-x^2-y^2}} \quad . \qquad (4)$$

A quick check now reveals that our integrand in (4) checks with that of formula (3) in Exercise 5.6.4.

To be sure, in this problem we did ultimately solve for z explicitly in terms of x and y, but our main point was to show that the technique for finding surface areas applies to surfaces of the form $g(x,y,z) = 0$.

5.6.7 (optional)

a. The main aim of this exercise is to generalize the procedure for finding surface area to surfaces of the general parametric form

$$\left.\begin{aligned} x &= f(u,v) \\ y &= g(u,v) \\ z &= h(u,v) \end{aligned}\right\} \begin{aligned} a \le u \le b \\ c \le v \le d \end{aligned} \qquad (1)$$

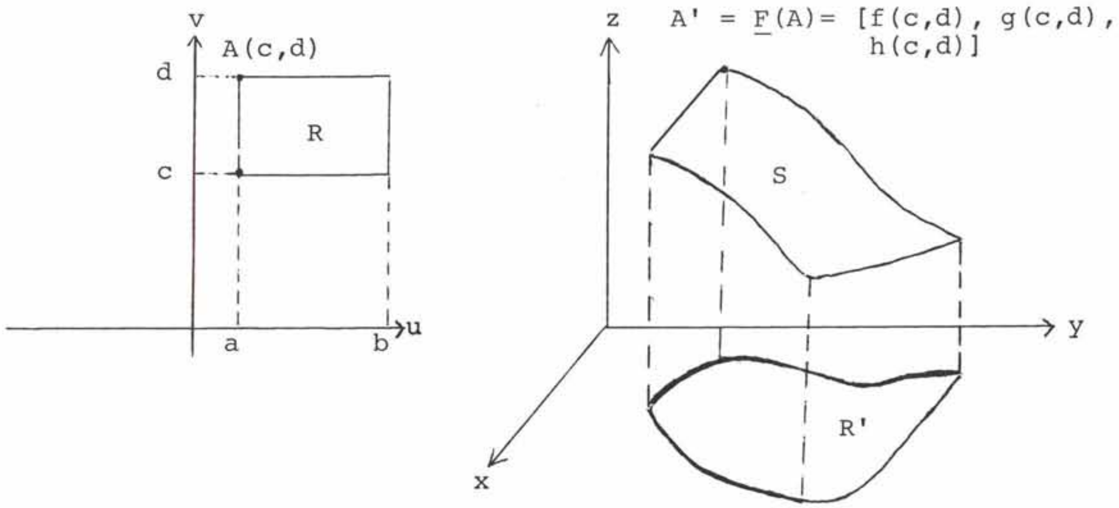
where f, g, and h are differentiable functions of u and v.

The key point is that we may view equation (1) as a mapping of the rectangular region $R = \{(u,v): a \le u \le b,\ c \le v \le d\}$ in E^2 (i.e., 2-space) onto the surface S defined by (1) in 3-space, E^3. (See note at the end of this exercise).

Analytically, this mapping is no more difficult to define than it was to define a similar mapping from E^2 into E^2, such as when we discussed mappings of the xy-plane into the uv-plane. From a geometric point of view, however, things are a bit tougher, but only because it is more difficult to draw pictures in E^3 than it was to draw them in E^2.

5.6.7 continued

At any rate, from a pictorial point of view, we have that equation (1) may be viewed as:



F is defined on R by F(u,v) = [f(u,v),g(u,v), h(u,v)] where (u,v) ε R.

(Figure 1)

With reference to Figure 1, what we can now conclude is that the surface area of S is given by

$$\iint_{R'} \frac{dy\ dx}{|\vec{u}_n \cdot \vec{k}|} \qquad (2)$$

where u_n is the (outer) unit normal to the surface S at the point (x,y,z).

The problem with formula (2) is that it expresses things in terms of x and y (z is a function of x and y on S) while the

5.6.7 continued

equation of S is given in terms of u and v.* Thus, somehow or other, we must use the chain rule or its equivalent to express (2) in terms of u and v.

We prefer to avoid the direct use of any method which requires that we express u and v in terms of x and y (or inversely which requires that we express x and y in terms of u and v). For one thing, such an approach may lead to difficult computations in some cases (i.e., it might be difficult to invert the equations). For another thing, in some abstract situations, we might not know how u and v are related to x and y, but only that S is determined by a pair of parameters u and v. Thirdly, from a purely philosophical point of view, we feel that any attempt to convert everything into x and y coordinates gives the impression that the Cartesian coordinate system is vital to the concept of surface area; and consequently we prefer an approach that is self-contained in terms of u and v.

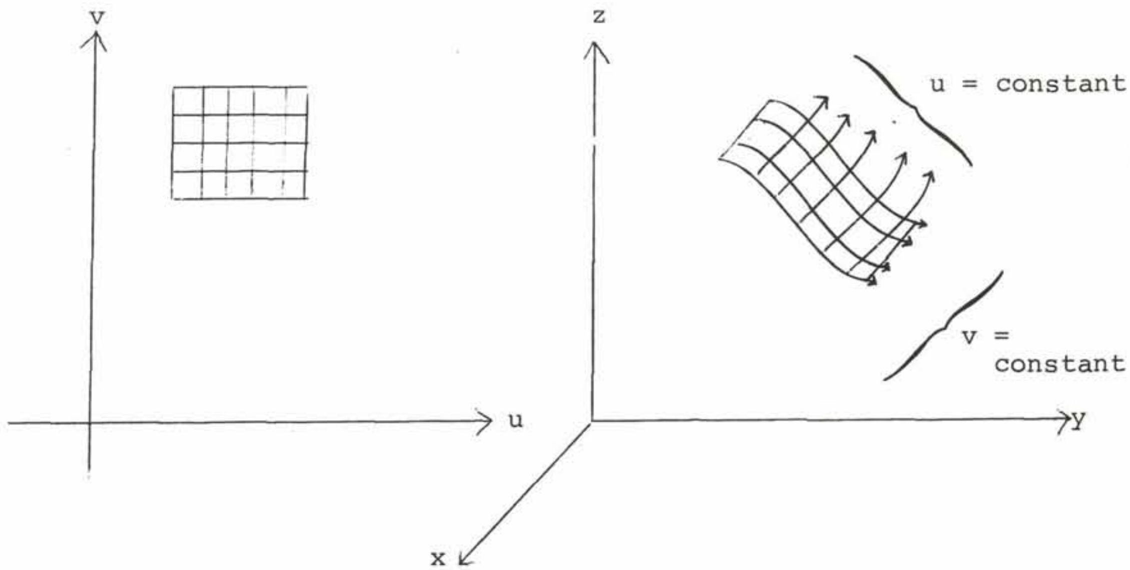
To carry out our objective what we do is look to see how the surface S evolves from the region R. Assuming, as usual, that our mapping is single-values, we may view S as being made up of the images of elemental rectangular regions of R. That is, we partition R into rectangles by the grid of lines u = constant and v = constant, and we then look at S as being made up of the union of the images of these rectangles. Pictorially,

*As a special case, notice that when u = x and v = y (i.e., in the case that z = h(x,y), which is the case treated in the text), R and R' are the same region, whereupon formula (2) becomes

$$\iint_R \frac{dy\,dx}{|\vec{u}_n \cdot \vec{k}|}$$

and the xy-plane and uv-plane coincide point by point. Thus, in this case, formula (2) is the usual recipe for computing the surface area of S when S is given in the form z = h(x,y).

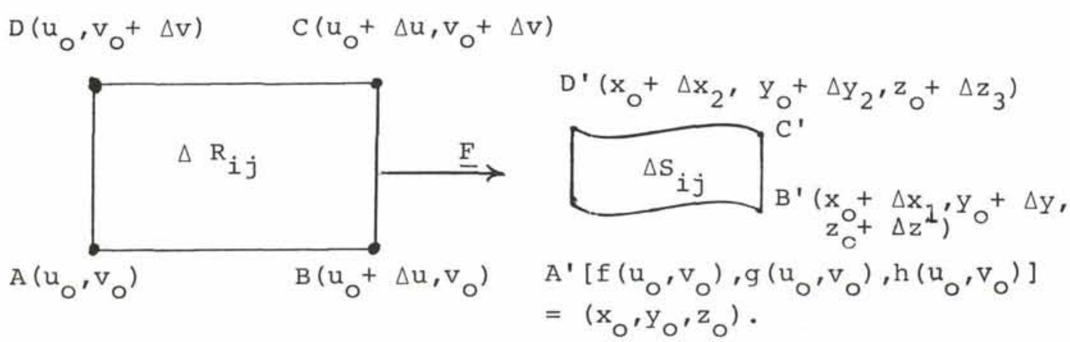
5.6.7 continued



(Figure 2)

We now compute the scaling factor necessary to convert the area of a rectangle in the uv-plane into the area of its image on S.

Again, pictorially

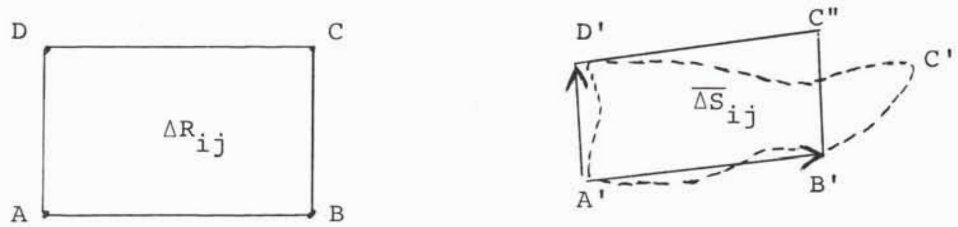


(Figure 3)

The crucial step, as usual in such discussions, now involves our being able to replace the region ΔS_{ij} by the parallelogram determined by, say, A', B', and D', which we can do provided ΔR_{ij} is sufficiently small.

5.6.7 continued

With reference to Figure 3, then what we are doing is replacing ΔS_{ij} by $\overline{\Delta S}_{ij}$, where



(Figure 4)

[Notice that in going from Figure 3 to Figure 4 we are again invoking local linearity. In effect, we are saying that locally (i.e. on small domains, ΔR_{ij}) $\underline{F}$ may be replaced by its linearization (i.e., the one which maps ΔR_{ij} onto the parallelogram A'B'C"D')]

At any rate, since we assume that ΔS_{ij} may be replaced by $\overline{\Delta S}_{ij}$, we are "home free" because it is easy to compute the area of $\overline{\Delta S}_{ij}$ in terms of the area of ΔR_{ij}.

In particular,

$$\overrightarrow{A'B'} = [f(u_o + \Delta u, v_o), g(u_o + \Delta u, v_o)] - [f(u_o,v_o), g(u_o,v_o), h(u_o,v_o)]$$

$$= [f(u_o + \Delta u,v_o) - f(u_o,v_o), g(u_o + \Delta u, v_o) - g(u_o, v_o), h(u_o + \Delta u, v_o) - h(u_o, v_o)]$$

$$= [\frac{f(u_o + \Delta u, v_o) - f(u_o, v_o)}{\Delta u}, \frac{g(u_o + \Delta u, v_o) - g(u_o, v_o)}{\Delta u}, \frac{h(u_o + \Delta u, v_o) - h(u_o, v_o)}{\Delta v}]\Delta u$$

and since on the straight line Δ'B', the average rate of

5.6.7 continued

change change equals instantaneous rate of change we have that

$$A'B' = [f_u(u_o,v_o),\ g_u(u_o,v_o),\ h_u(u_o,v_o)]\ \Delta u$$

$$= \vec{F}_u(u_o,v_o)\ \Delta u.$$

Similarly

$$\vec{A'D'} = \vec{F}_v(u_o,v_o)\ \Delta v.$$

Then, since the area of $\Delta\vec{S}_{ij}$ is given by $|\vec{A'B'} \times \vec{A'D'}|$ we have that the area of $\Delta\vec{S}_{ij}$ is

$$|\vec{F}_u(u_o,v_o) \times \vec{F}_v(u_o,v_o)|\ \Delta u\ \Delta v$$

$$= |\vec{F}_u(u_o,v_o) \times \vec{F}_v(u_o,v_o)|\ \Delta A_{R_{ij}}\ .$$

If we now sum over the entire partition and take limits we see that the surface area of S is given by

$$\iint_R |\vec{F}_u \times \vec{F}_v|\ dA_R$$

where

$$S = \{[f(u,v),\ g(u,v),\ h(u,v)]:\ (u,v)\varepsilon R\}$$

and

$$\vec{F}: R \to S \text{ with } \vec{F}(u,v) = [f(u,v),\ g(u,v),\ h(u,v)].$$

b. In the case that S is given by $z = f(x,y)$ we have

$x = u$
$y = v$
$z = f(u,v).$

Therefore,

$\vec{F}(u,v) = (u,v,f(u,v)$

5.6.7 continued

and

$F_u = (1,0,f_u)$

$F_v = (0,1,f_v)$.

Hence,

$$\vec{F}_u \times \vec{F}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & f_u \\ 0 & 1 & f_v \end{vmatrix}$$

$$= - f_u\vec{i} - f_v\vec{j} + \vec{k}$$

$$= - f_x\vec{i} - f_y\vec{j} + \vec{k} \quad \text{(since } x = u, \text{ and } y = v)$$

Consequently

$$|\vec{F}_u \times \vec{F}_v| = \sqrt{f_x^2 + f_y^2 + 1}\ ,$$

and the formula established in the previous exercise yields that the surface area of S is

$$\iint_R \sqrt{1 + f_x^2 + f_y^2}\ dA_R$$

which agrees with our previous result.

A Note on Arc Length and Surface Area

When we define a curve in parametric form (either a plane curve or a more general space curve) we are very close to having a non-geometric definition of a curve. For example, suppose we say that the curve C is defined by

$$\left.\begin{array}{l} x = t^2 \\ y = t^4 + 1 \\ z = t^6 + t^2 \end{array}\right\} \qquad 0 \leq t \leq 1 \tag{1}$$

(Since $t^4 = x^2$ and $t^6 = x^3$, this space curve is the intersection of the two surfaces $y = x^2 + 1$ and $z = x^3 + x$.)

What equation (1) says from a purely analytical point of view is that if we let R denote the closed interval [0,1] and if we define $\underline{F}$ on R by $\underline{F}(t) = (t^2, t^4 + 1, t^6 + 2t)$ for each $t \in R$ then the curve C is precisely $\underline{F}(R)$; that is, the image of R under the mapping $\underline{F}$.

Notice that this interpretation lends itself to any number of dimensions. In particular, we define an n-dimensional space curve to be any continuous mapping from E into E^n. The usual geometric notion of a space curve is then viewed as a continuous mapping from E into E^3, and a plane curve as a mapping from E into E^2, Moreover, we say that the curve is smooth if the mapping is differentiable.

In a similar way one defines an n-dimensional surface as a continuous mapping from E^2 into E^n. In particular, this agrees with the usual parametric definition of a surface in 3-space. Namely when we say that S is the surface defined by

$x = f(u,v)$
$y = g(u,v)$
$z = h(u,v)$

we are saying that we may view S as the image of $\underline{F}: E^2 \rightarrow E^3$ where $\underline{F}$ is defined by

$\underline{F}(u,v) = (f(u,v), g(u,v), h(u,v))$

for each $(u,v) \in E^2$.

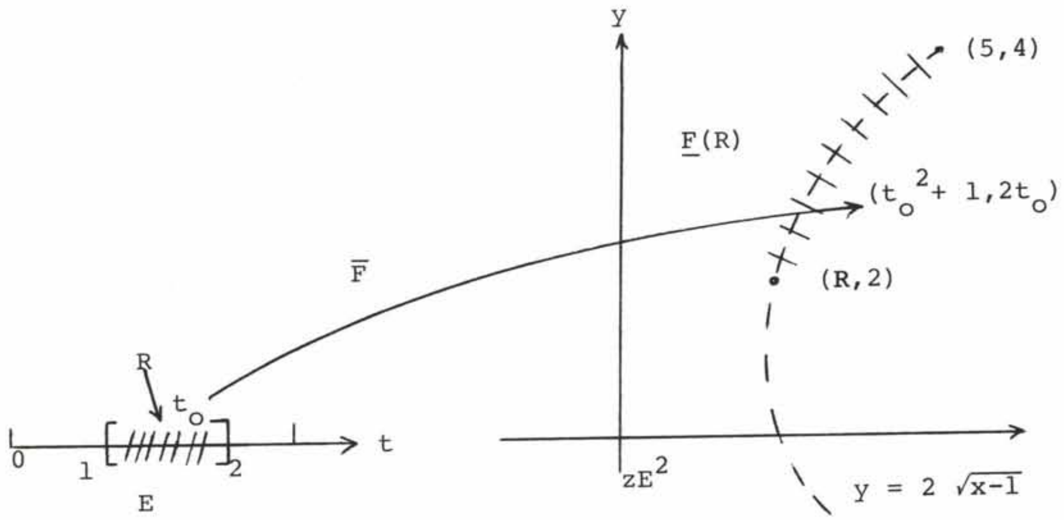
Again, we say that the surface is smooth if $\underline{F}$ is differentiable.

The point that we would like to make in this note is that arc length may be viewed in the same way that surface area was discussed in this exercise; and perhaps by revisiting arc length at this time, it will become easier to follow the technique developed in this exercise (since the theory is the same but the diagrams involve one less dimension).

Rather than look at the general abstract case, let us pick a particular curve C. Suppose that C is defined by

$$\left.\begin{array}{l} x = t^2 + 1 \\ y = 2t \end{array}\right\} \quad 1 \le t \le 2.$$

Then the curve C is the image of the interval [1,2] under the mapping $\underline{F}$ defined by $\underline{F}(t) = (t^2 + 1, 2t)$. Pictorially, we may view $\underline{F}$ as

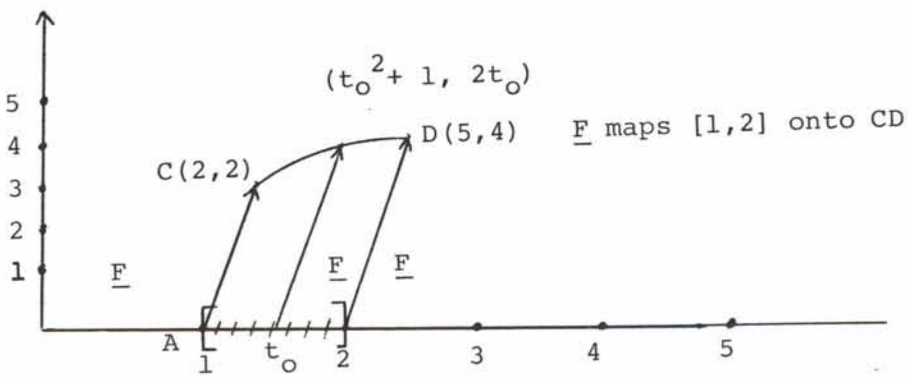


$$\underline{F}(R) = \{(x,y): x = t^2 + 1,\ y = 2t,\ 1 \le t \le 2\}$$

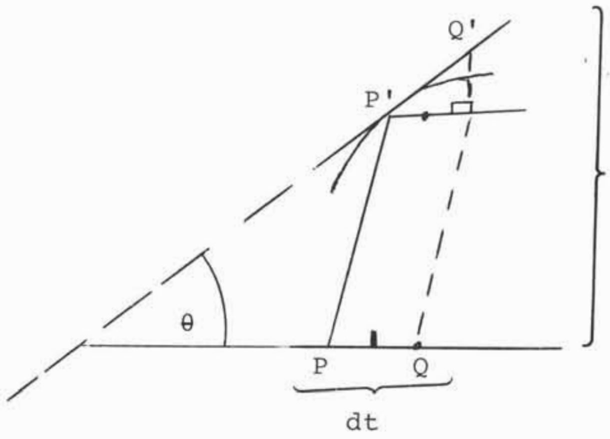
$$= \{(x,y): y = 2\sqrt{x-1},\ 1 \le t \le 2\}$$

$$= \{(x,y): y = 2\sqrt{x-1},\ 2 \le x \le 5\}$$

In fact, the entire mapping may be viewed in the xy-plane. Namely



Notice that a "small" segment of AB maps onto a "small" segment of CD and the scaling factor is the slope of CD at some point on the small segment



$\underline{F}(\overline{PQ}) \approx P'Q'$

Therefore,

$|\underline{F}(\overline{PQ})| \approx \overline{PQ} \sec\theta$

$\tan\theta = \frac{dy}{dx} \rightarrow$

$\sec\theta = \sqrt{1 + (\frac{dy}{dx})^2} \rightarrow$

$\underline{F}(R) = \int_1^2 \sqrt{1 + (\frac{dy}{dx})^2}\, dt$

In other words, we could have interpreted arc length solely in terms of $\underline{F}(R)$, etc. It is in precisely this same way that the parametric form for a surface can be used to develop the corresponding formulas for surface area.

5.6.8 (optional)

a. In the case of polar coordinates we have that S is given by:

$x = r\cos\theta$*
$y = r\sin\theta$
$z = f(r,\theta)$

and hence,

$\vec{F}(r,\theta) = (r\cos\theta, r\sin\theta, f(r,\theta)$.

Therefore,

$\vec{F}_r = (\cos\theta, \sin\theta, f_r)$
$\vec{F}_\theta = (-r\sin\theta, r\cos\theta, f_\theta)$

* The symbols u and v are not important, but if one wished to adhere to the notation of the previous exercise we would write

$$\begin{cases} x = u\cos v \\ y = u\sin v \\ z = f(u,v) \end{cases}.$$

5.6.8 continued

and

$$F_r \times F_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & f_r \\ -r\sin\theta & r\cos\theta & f_\theta \end{vmatrix}$$

$$= \vec{i}(f_\theta \sin\theta - rf_r \cos\theta) - \vec{j}(f_\theta\cos\theta + rf_r\sin\theta) + \vec{k}(\underbrace{r\cos^2\theta + r\sin^2\theta}_{r})$$

Therefore,

$$|\vec{F}_r \times \vec{F}_\theta|^2 = (f_\theta\sin\theta - rf_r\cos\theta)^2 + (f_\theta\cos\theta + rf_r\sin\theta)^2 + r^2$$

$$= f_\theta^{\ 2}\sin^2\theta - 2rf_rf_\theta\sin\theta\cos\theta + r^2f_r^{\ 2}\cos^2\theta + f_\theta^{\ 2}\cos^2\theta + 2rf_rf_\theta\sin\theta\cos\theta + r^2f_r^{\ 2}\sin^2\theta + r^2$$

$$= f_\theta^{\ 2} + r^2f_r^{\ 2} + r^2.$$

Thus, the required surface area is

$$\iint_R |\vec{F}_r \times \vec{F}_\theta|\ dA_R$$

$$= \iint_R \sqrt{f_\theta^{\ 2} + r^2f_r^{\ 2} + r^2}\ dr\ d\theta^* \qquad (1)$$

b. For a plane region S, z = 0. That is, $f(r,\theta) \equiv 0$. In this case $f_\theta = f_r = 0$, so that $\sqrt{f_\theta^{\ 2} + r^2f_r^{\ 2} + r^2} = r$, so that (1) becomes

$$\iint_R r\,dr\ d\theta$$

---

*Notice here that $dA_R = dr\ d\theta$ not $r\,dr\,d\theta$ since the region R is in the uv-plane (or in our present context, the $r\theta$-plane, not the xy-plane.

5.6.8 continued

which agrees with the usual formula for finding (plane) area in polar coordinates.

c. If $f_\theta = 0$ (which, by the way, is always true for a region of revolution about the z-axis since the height of an element remains constant through the revolution) then formula (1) becomes

$$\iint_R \sqrt{r^2 f_r^{\,2} + r^2}\, dr\, d\theta$$

or

$$\iint_R \sqrt{1 + f_r^{\,2}}\; r dr\, d\theta. \qquad (2)$$

d. In this case our region R is defined by

$$R = \{(r,\theta):\ r = \frac{\theta}{2},\ 0 \leq \theta \leq \pi\}$$

and our surface S is given by

$$z = + \sqrt{4 - x^2 - y^2}$$

or

$$z = \sqrt{4 - r^2}\ .$$

In terms of formula (1) this means that $f(r,\theta) = \sqrt{4 - r^2}$, so that $f_\theta = 0$ and $f_r = -r/\sqrt{4 - r^2}$.

Since $f_\theta = 0$, formula (2) applies and we see that the surface area is given by

$$\iint_R \sqrt{1 + \frac{r^2}{4 - r^2}}\; r dr\, d\theta$$

$$= \int_0^{\pi}\int_0^{\frac{\theta}{2}} \frac{2r}{\sqrt{4 - r^2}}\, dr\, d\theta$$

5.6.8 continued

$$= \int_0^{\pi} -2 \sqrt{4 - r^2} \Bigg|_{r=0}^{\frac{\theta}{2}} d\theta$$

$$= -2 \int_0^{\pi} \left(\sqrt{4 - \left(\frac{\theta}{2}\right)^2} - \sqrt{4 - 0} \right) d\theta$$

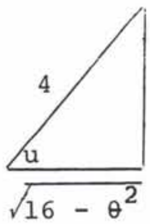
$$= -2 \int_0^{\pi} \left(\frac{\sqrt{16 - \theta^2}}{2} - 2 \right) d\theta$$

$$= \int_0^{\pi} (4 - \sqrt{16 - \theta^2} \; d\theta$$

$$= 4\pi - \int_0^{\pi} \sqrt{16 - \theta^2} \, d\theta$$

$$= 4\pi - 16 \int_0^{\sin^{-1}\frac{\pi}{4}} \cos^2 u \, du$$

(where $u = \sin^{-1} \frac{\theta}{4}$; i.e.,



)

$$= 4\pi - 8 \int_0^{\sin^{-1}\frac{\pi}{4}} (1 + \cos 2u) du$$

$$= 4\pi - 8 \left[u + \frac{1}{2} \sin 2u \right|_{u=0}^{\sin^{-1}\frac{\pi}{4}}$$

5.6.8 continued

$$=4\pi - 8[\sin^{-1}\frac{\pi}{4} + \frac{\sqrt{16-\pi^2}}{16}]$$

$$= 4\pi - 8\sin^{-1}\frac{\pi}{4} - \frac{\pi\sqrt{16-\pi^2}}{2}$$

$$\approx 4(3{,}14) - 8\sin^{-1}(0.785) - 1.57\sqrt{16-(3.14)^2}$$

$$\approx 12.56 - 8(.91) - 1.57\sqrt{6.14}$$

$$\approx 1.4$$