Unit 6: Surface Area

1. Overview: (Actually an introduction to Section 16.9 of the Text)

In Part 1 of our course, we discussed briefly the problem of finding areas of surfaces of revolution. In the same way that the problem of finding volumes was more general than finding volumes of revolution (i.e., not all solids are obtained by revolving a plane region), the problem of finding surface area is more difficult than what may have been indicated in our study of surfaces of revolution. Our aim in this unit is to broaden our concept of surface area (and as an optional exercise we will show that the treatment of surfaces of revolution is a special case of the more general theory).

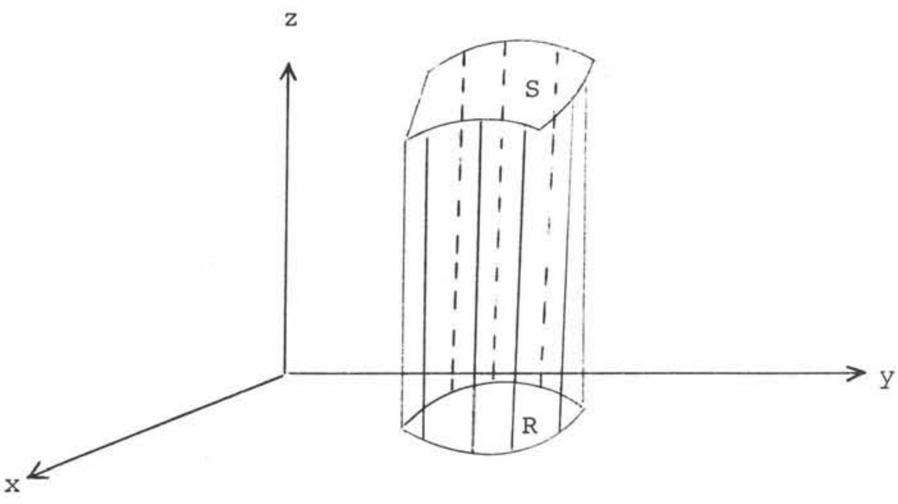
Recall that our study of volume and area in Part 1 was easier to handle than our study of arc length. Namely, there were three important axioms that were valid in the treatment of area and volume, but only two of these three were valid for arc length. In this sense (even though the degree of sophistication is a dimension higher) we are faced with the same problem in extending the notions of volume as a double (or triple) integral to the notion of surface area as a double integral. [This topic is covered in detail in Section 16.9 of the text <u>in the special case</u> that the surface has the explicit form $z = f(x,y)$. While we shall try to make our discussion more general than that in the text, the main points will be the same.] Namely, we are without the key axiom that if the one region is contained within another the surface area of the containing region is at least as great as that of the contained region. In other words, when we deal with surface area, we cannot apply that nice device of "squeezing" the desired region between two other regions. Consequently, we must be extra careful (just as we had to be when we studied arch length in calculus of a single variable) in how we define surface area rigorously. (The author concludes Section 16.9 with a rather paradoxical conclusion based on a seemingly harmless way of computing a surface area.)

From an intuitive point of view, it seems that we should be able to partition a surface into a grid in much the same way that we partitioned regions in the xy-plane into rectangular grids. The problem, of course, is that it is not easy to define a basic element of area on an arbitrary surface. For example, in planes straight lines are very natural for measuring distances; on a sphere arcs of great circles (i.e., circles on the sphere whose center is also the center of the sphere) are very natural for measuring distances; but what is natural surely depends on the shape of the particular surface. (As a point of information geodesic lines on a surface are those curves on the surface which yield the shortest path between two points on the surface; so that for planes, the geodesics are straight lines while for a sphere the geodesics are arcs of great circles.)

The typical ploy is to reduce the study of surface area to the study of plane area. One way of doing this is to project the given surface onto the xy-plane (we need not restrict our usage to Cartesian coordinates except that it seems convenient. The key idea is that we project the surface onto a plane). If the surface is single valued (meaning that a line perpendicular to the xy-plane meets the surface in no more than one point) then there is a natural correspondence between the points on the surface S and the points on its projection R; where by "natural" we mean that the correspondence is given by the "matchup" of points on lines perpendicular to the xy-plane (see the pictorial summary in Figures 1, 2, and 3). If the surface is not single valued, we can, as usual, divide it into single-valued branches.

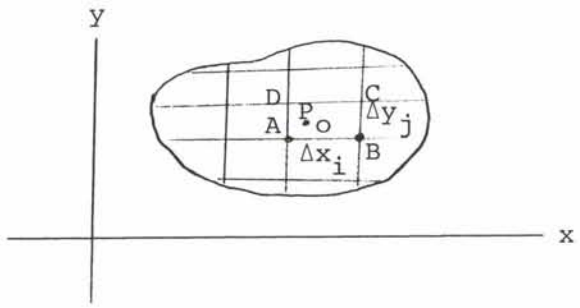
The point is that since R is a plane region we can partition it in the usual way into a rectangular grid. If we then pick a point (P_o) in the rectangle of dimension Δx_i by Δy_j, we can project this rectangle onto the plane tangent to S at the point P_o at which the line through (P_o) perpendicular to the xy-plane meets S. Note especially that this requires the surface S to be smooth; otherwise there might not be a tangent plane at each point on the surface.

The gist of the argument is to assume that if the rectangles into which R is divided are sufficiently small then the area of the region obtained when the rectangles are projected onto the tangent planes is essentially the same as the area of the region obtained when the rectangle is projected onto the surface S itself. Then since there is a 1-1 correspondence between the rectangles in R and the projections of them onto the surface S, and since the projections onto S yield precisely S itself, we can find the area of S by appropriately finding the area of the individual rectangles in R. Pictorially,



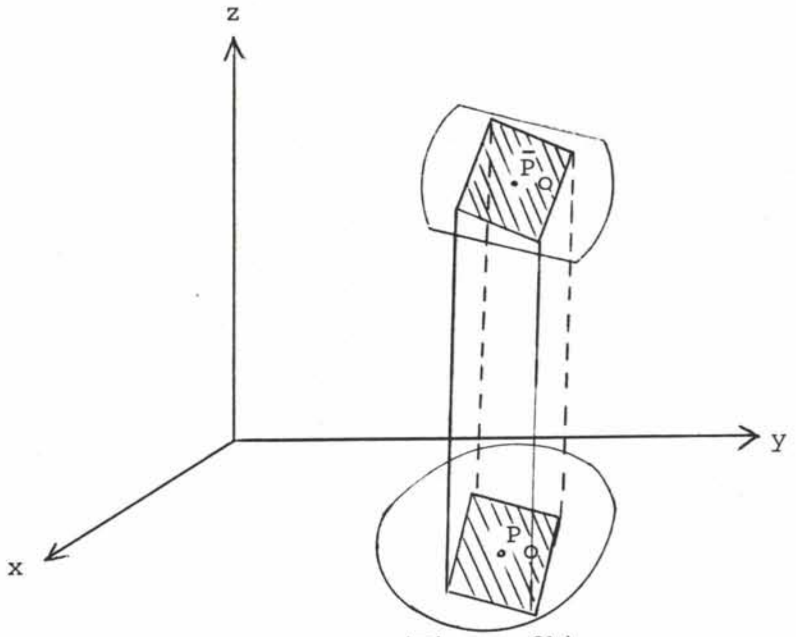
(Figure 1)

Since S is single valued there is a 1-1 correspondence between points on S and points on R. If S is given in the form $z = f(x,y)$ then the correspondence matches (x_o,y_o) in R with (x_o,y_o,z_o) in S where $z_o = f(x_o,y_o)$. That is the point on S(R) is directly above (below) the corresponding point on R(S).



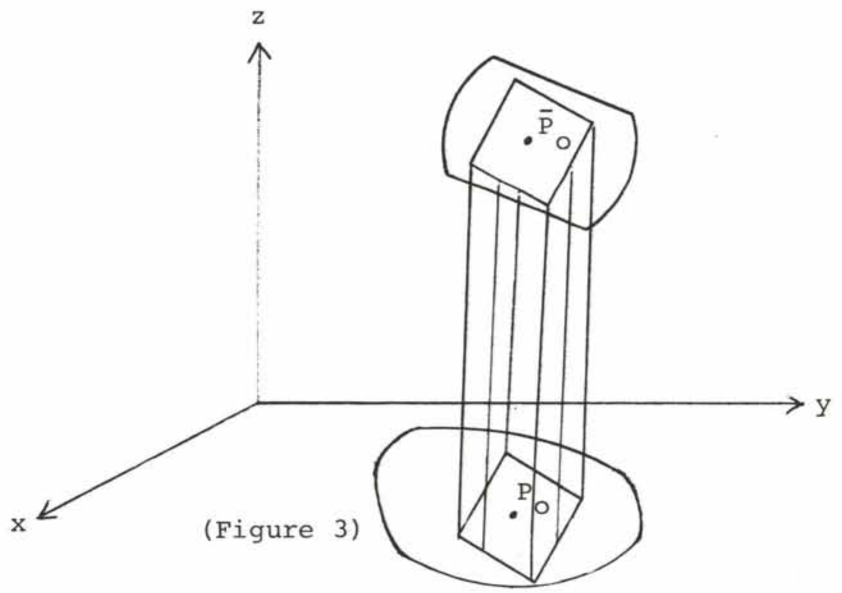
(Figure 2a)

We may then partition R in the "usual way" (ABCD is henceforth referred to as ΔA_{ij}).



(Figure 2b)

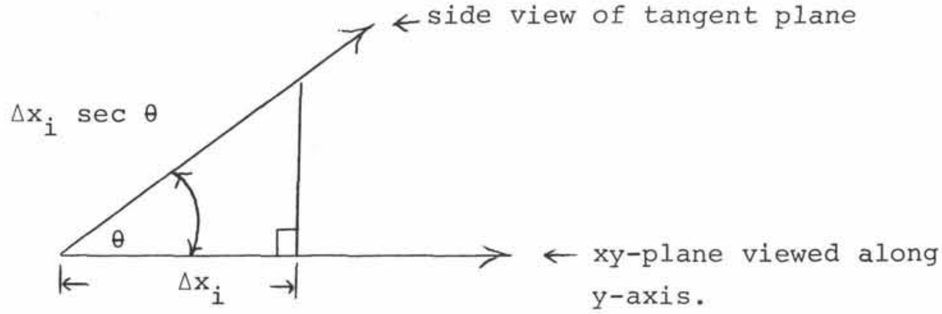
We then project the rectangle A_{ij} onto the plane tangent to S at the point $\bar{P}_o$ on S directly above P_o.



(Figure 3)

For sufficiently small Δx_i, Δy_j, the area of projection of ΔA_{ij} onto the plane tangent to S at $\bar{P}_o$ (Figure 2b) is essentially the same as the area of the projection of ΔA_{ij} onto S itself.

Now, if the plane tangent to S at $\overline{P}_o$ makes an angle θ with the xy-plane, then the area of the projection of ΔA_{ij} onto this plane is $(\Delta x_i \Delta y_j) \sec\theta$. Again, pictorially



Now, if u_n denotes a unit normal vector to the plane then $\vec{u}_n \cdot \vec{k} = \cos\theta$. Hence (sparing the details of taking limits) an element of area on the surface is given by

$$\frac{dx\ dy}{\vec{u}_n \cdot \vec{k}}$$

where u_n is the unit normal* to the surface S at the point $\overline{P}_o$.

In this way we obtain the result, by definition,

$$A_S = \iint_R \frac{dA_R}{|\vec{u}_n \cdot \vec{k}\,|} \, . \tag{1}$$

The only remaining difficulty is in expressing $\vec{u}_n$. In the approach used in the text S is given in the form $z = f(x,y)$. In this case, the normal vector is $f_x\vec{i} + f_y\vec{j} - \vec{k}$, so that a unit normal u_n is given by

*Actually there are two such normals depending on the sense we choose. By convention one picks the outward normal, i.e., the normal point out from the surface which means in the direction from the origin to the surface, but since area is non-negative, it is reasonably safer to write

$\dfrac{dx\ dy}{|\vec{u}_n \cdot \vec{k}|}$ and not worry about the sense of u_n.

$$\pm\left[\frac{f_x\vec{i} + f_y\vec{j} - \vec{k}}{\sqrt{f_x^{\,2} + f_y^{\,2} + 1}}\right]$$

Therefore,

$$\vec{u}_n \cdot \vec{k} = \frac{\pm 1}{\sqrt{f_x^{\,2} + f_y^{\,2} + 1}}$$

and formula (1) becomes

$$A_S = \iint_R \sqrt{f_x^{\,2} + f_y^{\,2} + 1}\; dA_R. \qquad (2)$$

Formula (2) is developed quite rigorously in the reading assignment in the text. Its only deficiency from our point of view is that it requires that the surface be expressed in the special form $z = f(x,y)$. Yet there are times when the surface may be defined implicitly in the form $g(x,y,z) = 0$, or parametrically in the form

$$\begin{aligned} x &= x(u,v) \\ y &= y(u,v) \\ z &= z(u,v). \end{aligned} \qquad (3)$$

(In fact formula (3) is perhaps the best since the key structural property of a surface in that we have two degrees of freedom at our disposal. That is, any definition of a surface is a form of (3). In particular the form $z = f(x,y)$ is the simple special case in which $u = x$ and $y = v$.)

In the optional exercises in this unit we shall develop recipes for finding surface area when the surface is given either implicitly or parametrically.

2. Read: Thomas; Section 16.9

3. Exercises

5.6.1(L)

Find the surface area of the region S if S is the portion of the paraboloid $z = 1 - x^2 - y^2$ for which $z \geq 0$.

5.6.2

Find the area of the portion of the surface $2z = x^2 + y^2$ cut off by the plane $z = 1$.

5.6.3

Find the area cut from the plane $x + y + z = 1$ by the cylinder $x^2 + y^2 = 1$.

5.6.4

Find the surface area of that portion of the sphere $x^2 + y^2 + z^2 = a^2$ cut off by the cylinder $x^2 - ax + y^2 = 0$.

5.6.5(L)

Find the mass of the hemispherical shell S defined by $x^2 + y^2 + z^2 = 1$ and $z \geq 0$, if the density of S at any point is proportional to the distance of the point from the xy-plane.

Optional Exercises

[The aim of 5.6.6 is to show how we develop the formula for the surface area of S if S is given in the impllicit form $g(x,y,z) = 0$.

In 5.6.7 we show how to compute the surface area of S if S is given in the parametric form

$$\begin{cases} x = f(u,v) \\ y = g(u,r) \\ z = h(u,r). \end{cases}$$

In 5.6.8 we apply the results of 5.6.7 to the special case of polar coordinates (i.e., $x = u \cos v$, $y = u \sin v$) and show

how surface area is computed when the surface is given in the polar form $z = f(r,\theta)$.]

5.6.6

Determine the surface area found in Exercise 5.6.4, but without solving for z explicitly in terms of x and y.

5.6.7

If the surface S is defined by

$$\left.\begin{array}{l} x = f(u,v) \\ y = g(u,v) \\ z = h(u,v) \end{array}\right\} \quad (u,v)\in R$$

let $\vec{F}(u,v) = (f(u,v), g(u,v), h(u,v)$.

a. Show that the surface area of S is given by

$$\iint_R |\vec{F}_u \times \vec{F}_v| dA_R.$$

b. Check the above result in the special case $u = x$, $v = y$.

5.6.8

a. Apply the result of Exercise 5.6.7(a) to the case in which our surface is given by

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = f(r,\theta).$$

b. Check the result obtained in (a) by seeing what happens in the special case $z = 0$.

c. What happens in the special case of $z = f(r,\theta)$ where $f_\theta \equiv 0$?

d. Compute the surface area of S if S is the portion of the sphere $x^2 + y^2 + z^2 = 4$ cut off by the cylinder $r = \frac{\theta}{2}$ where $0 \leq \theta \leq \pi$.