Unit 7: More on Derivatives of Integrals

1. (Optional) Read: Thomas Section 15.14.

[For the purpose of this unit we need the prerequisite result obtained in Part 1 of our course, that

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)\,dt = f[v(x)]\,\frac{dv}{dx} - f[u(x)]\,\frac{du}{dx} \quad , \tag{1}$$

where f is cont. on $a \leq t \leq b$, and u and v are differentiable functions of x; and $a \leq u(x) \leq b$, $a \leq v(x) \leq b$.

If you feel that you are familiar with equation (1), you may proceed at once to Lecture 3.050.

In Lecture 3.050 we shall discuss a generalization of Equation (1) in which x appears in the integrand as well as in the limits of integration. That is, we shall investigate

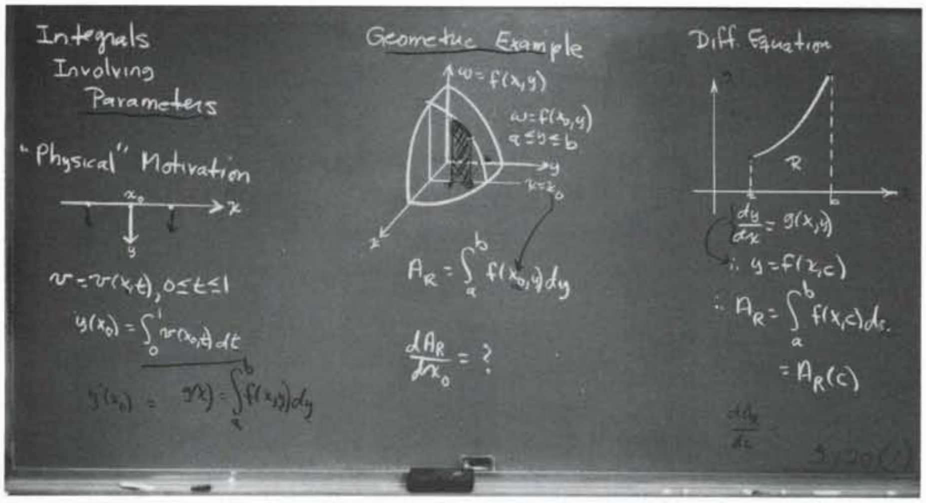
$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(x,t)\,dt \tag{2}$$

$\int_{u(x)}^{v(x)} f(x,t)\,dt$ is more complicated than $\int_{u(x)}^{v(x)} f(t)\,dt$

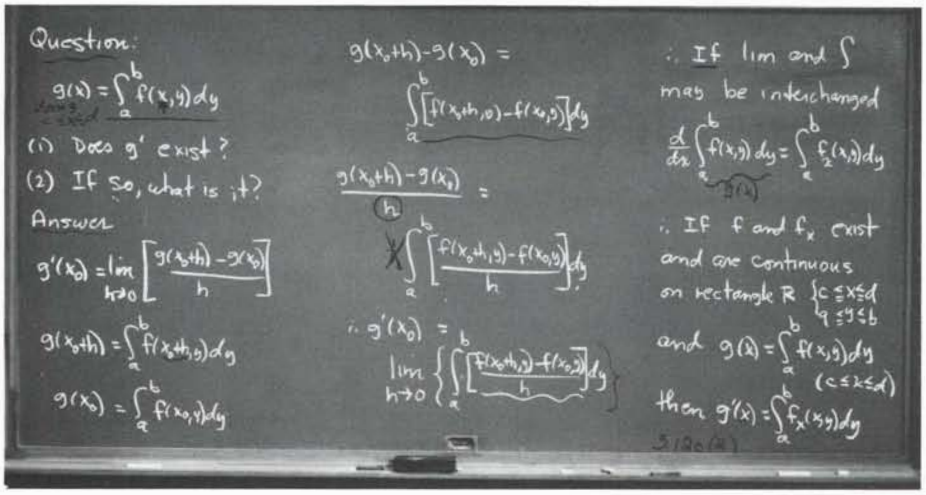
since the integrand also changes as x changes.

The main point is that equation (1) is important in evaluating (2)].

2. Lecture 3.050



a.



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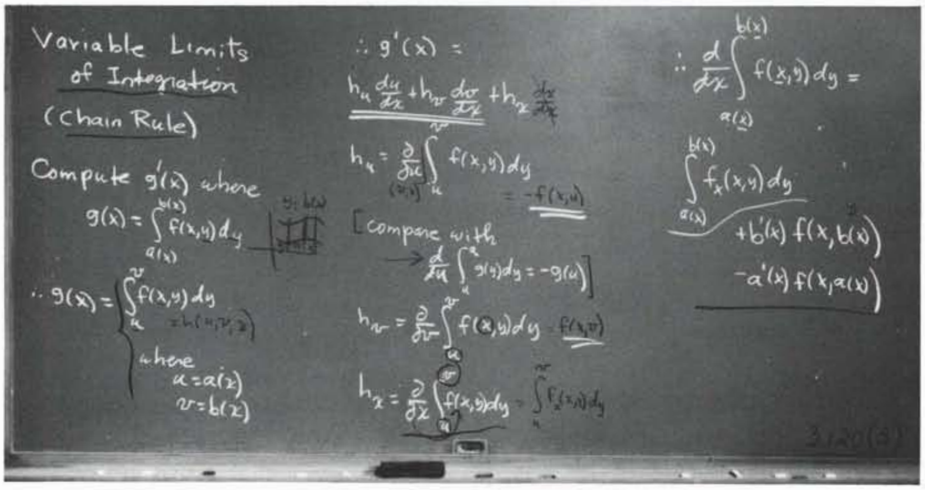


c.

3. Exercises: [If you understand and/or are willing to accept the results derived in the lecture you may omit the first three exercises. You will notice that these three exercises repeat, in more detail, what was done in the lecture. Our feeling is that since these derivations are not in the text they should be included in the exercises for the interested student. With the exception of the "precision," all the results of these three exercises are contained in the photographs of the lecture].

3.7.1 (Optional)

Assuming that $f(x,y) \geqslant 0$ for all (x,y) and that $w = f(x,y)$, illustrate geometrically the meaning of

$$\int_a^b f(x_0,y)\,dy \quad .$$

3.7.2 (Optional)

Let R be the rectangle defined by $c \leqslant x \leqslant d$, $a \leqslant y \leqslant b$. Prove that if f and f_x exist and are continuous on R then

$$\frac{d}{dx} \int_a^b f(x,y)\,dy = \int_a^b f_x(x,y)\,dy$$

3.7.3 (Optional)

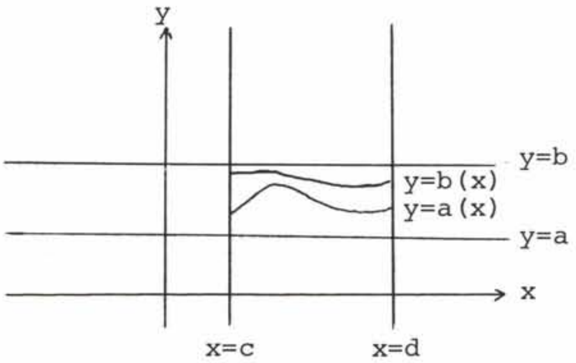
Let R be as in Exercise 3.7.2 and suppose again that f and f_x are continuous on R. Now let $a(x)$ and $b(x)$ be continuously differentiable functions with range in the interval $a \leqslant y \leqslant b$ (see diagram). Then if

$$F(x) = \int_{a(x)}^{b(x)} f(x,y)\,dy,$$

$$F'(x) = \int_{a(x)}^{b(x)} f_x(x,y)\,dy + f(x,b(x))b'(x) - f(x,a(x))a'(x)$$

(continued on next page)

3.7.3 continued



(1) The "b" [or "a"] in b(x) [or a(x)] is not the same as b [or a]

(2) The curves y=b(x) and y=a(x) may cross. They need only be in R)

3.7.4(L)

a. Verify the correctness of

$$\frac{d}{dx}\int_a^b f(x,y)\,dy = \int_a^b f_x(x,y)\,dy$$

in the case a=0, b=1, and $f(x,y) = x^2y+x^5y^3+3$.

b. Compute $g'(x)$ if $g(x) = \int_1^2 \sin(xe^y)\,dy$.

3.7.5(L)

Find the values of c, $0\leqslant c\leqslant \pi/2$, for which

$$\int_1^2 \frac{\sin cy\, dy}{y}$$

has maximum or minimum values.

3.7.6

a. Given that

$$g(y) = \int_0^1 \frac{x^y - x^b}{\ln x}\,dx, \text{ where } y>b>-1$$

show that $g'(y) = \frac{1}{y+1}$

b. Noticing that $\frac{d[\ln(y+1)]}{dy} = \frac{1}{y+1}$, use part (a) to show that

$$\int_0^1 \frac{x^y - x^b}{\ln x}\,dx = \ln \frac{y+1}{b+1} \quad \text{if } y>b>-1 \quad .$$

c. Use part (b) to evaluate

$$\int_0^1 \frac{x^3 - x^2}{\ln x}\,dx \quad .$$

3.7.7

a. Verify the result stated in Exercise 3.7.3 if $a(x)=x$, $b(x)=x^2$, and $f(x,y)=xy$.

b. Compute $g'(x)$ if

$$g(x) = \int_{-x}^{x} \frac{1-e^{-xy}}{y}\,dy; \; x>0, \; y\neq 0 \; .$$

3.7.8(L)

a. We are given that y is a function of x, defined by

$$y(x) = \int_a^x h(t)\,\sin(x-t)\,dt$$

3.7.8(L) continued

Use the result of Exercise 3.7.3 to show that $y(x)$ is determined by the differential equation

$$y''(x) + y(x) = h(x), \text{ where } y(a) = y'(a) = 0.$$

b. If

$$y(x) = \int_0^x 2e^t \sin(x-t)\,dt$$

show that

$$y''(x)+y(x) = 2e^x \text{ and } y(0) = y'(0) = 0.$$

3.7.9

a. If

$$y(x) = \int_0^x (t-x)\,[y(t)-e^t]\,dt^*$$

show that

$$y''(x) + y(x) = e^x \text{ and } y(0) = y'(0) = 0$$

b. Use the result of part (a) to show that $y(x) = \frac{1}{2}(e^x-\sin x-\cos x)$ is a solution of the integral equation

$$y(x) = \int_0^x (t-x)[y(t)-e^t]dt \quad .$$

*An equation in which we seek a function, say, $y(x)$ is called an <u>integral equation</u> if y appears as part of a definite integral whose limits depend on x. Thus

$$y(x) = \int_0^x (t-x)[y(t)-e^t]dt$$

is an integral equation.

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Resource: Calculus Revisited: Multivariable Calculus
Prof. Herbert Gross

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