Unit 8: The Total Differential

3.8.1 (L)

The main reason for this exercise, relative to the material in this unit, is to justify a technique used in the lecture. We were trying to solve a differential equation of the form

$$M(x,y)dx + N(x,y)dy = 0 \tag{1}$$

We then pointed out that if $M(x,y)dx + N(x,y)dy$ were exact, we could find a function, $w = f(x,y)$, such that

$$dw = M(x,y)dx + N(x,y)dy \tag{2}$$

We then substituted (2) into (1) to obtain

$$dw = 0 \tag{3}$$

whence we concluded that

$$w = c \tag{4}$$

and that, therefore, the solution of equation (1) was

$$f(x,y) = c \tag{5}$$

The point is that in (2), dw is a function of the two independent variables dx and dy, but in (3) dw is a function of the two dependent variables dx and dy. That is, once we equate $M(x,y)dx + N(x,y)dy$ to zero then dx and dy are no longer independent.

What we want to show in this exercise is that the use of dw in (2) is compatible with its use in (3). In essence, we wish to show that even if x and y are dependent variables the "recipe" of computing dw as $f_x dx + f_y dy$ is correct.

To show this properly we must make the distinction, once again (as we did in Part 1), between an equation (conditional equality) and an identity.

a. When we write

$$f(x,y) = 0 \tag{6}$$

3.8.1 (L) continued

we mean $\{(x,y):f(x,y)=0\}$. We do not mean that $f(x,y) = 0$ for every 2-tuple (x,y).* For example, suppose $f(x,y) = x^2 - y$. Then $f(x,y) = 0$ means $x^2 - y = 0$, or $y = x^2$. Thus in this case, $f(x,y) = 0$ means the parabola $y = x^2$. If we let C denote the parabola [i.e., $C = \{(x,y):y = x^2\}$] then equation (6) is an <u>identity provided that</u> the <u>domain of f</u> is <u>restricted to C</u>. That is, <u>on C</u> $f(x,y) = x^2 - y = x^2 - x^2 \equiv 0$, but if $(x,y) \notin C$ then $f(x,y) \neq 0$ [e.g., $f(3,2) = 9-2 = 7 \neq 0$].

b. With this as background, it should now seem clear as to what we mean when we say, "suppose $f(x,y) = 0$ determines y as a differentiable function of x". Relative to the present example $x^2 - y = 0$ implies that $y = x^2$ and x^2 is certainly a differentiable function of x. Geometrically, the statement would be that $f(x,y) = 0$ determines a smooth curve C in the plane (in our example, the parabola $y = x^2$).*

At any rate, under the present assumptions, we have that

$$f(x,y) = 0 \qquad (6)$$

is equivalent to

$y = g(x)$ where g' exists.

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* If we do mean that $f(x,y) = 0$ for <u>each</u> $(x,y)\epsilon E^2$ we then write $f(x,y) \equiv 0$, and say that $f(x,y)$ is <u>identically</u> equal to zero. In the language of functions if $f = E^2 \to R$ (where R = real numbers) we define f to be the zero function on E^2 if $f(x,y) = 0$ for each $(x,y)\epsilon E^2$. More generally if S is any set and $f = S \to R$ we define f to be the zero function if $f(s) = 0$ for each $s\epsilon S$.

* $f(x,y) = 0$ does not always guarantee that y is a differentiable function of x (although it happens in the example we chose). This is why the exercise is worded as it is.

3.8.1 (L) continued

In other words, if we let $C = \{(x,y): y = g(x)\}$ then on C

$f(x,y) \equiv f(x,g(x)) \equiv 0$ (7)

Now $f(x,g(x))$ is a function of x alone.
Hence we may write that

$h(x) = f(x,g(x))$* (8)

Therefore on C, equation (6) is equivalent to the identity $h(x) \equiv 0$.
Since $h(x) \equiv 0$ then it follows that

$h'(x) \equiv 0$ (9)

In differential form, we are saying that $w = f(x,y)$ implies, on C, that $w \equiv h(x)$; therefore

(9')

$dw \equiv h'(x)\,dx$

But $h'(x)$ can also be computed by the chain rate, using the facts that

$h(x) = f(x,y)$ and $y = g(x)$.** (10)

* In terms of part (a), $f(x,y) = x^2 - y$ and $g(x) = x^2$. Hence, $f(x,g(x)) = x^2 - x^2 \equiv 0$.

** Note that in this form, x and y are treated as being independent in the expression $f(x,y)$, with the dependency coming from $y = g(x)$.

3.8.1 (L) continued

Applying the chain rule to (10) we have

$$h'(x) = f_x \frac{dx}{dx} + f_y \frac{dy}{dx}$$

$$= f_x + f_y \frac{dy}{dx}, \qquad (11)$$

and since $y = g(x)$, $\frac{dy}{dx} = g'(x)$*, therefore

$$h'(x) = f_x + f_y\, g'(x) \qquad (11')$$

If we now replace $h'(x)$ in (9') by its value in (11'), we obtain,

$$dw = [f_x + f_y\, g'(x)]\, dx$$

$$= f_x dx + f_y\, [g'(x) dx]$$

$$= f_x dx + f_y dy \quad \text{[since on C, } y = g(x);\text{, therefore}$$

$$dy = g'(x)dx]. \qquad (12)$$

Equation (12) shows us that on C, we may still view dw as $f_x dx + f_y dy$ even though x and y are not independent on C.

c. Implicit differentiation applies to equations of the form $f(x,y) = 0$. If we assume that $f(x,y) = 0$ determines y as a differentiable function of x, say, $y = g(x)$, then if $C = \{(x,y): y = g(x)\}$,

$$f(x,y) = \underbrace{f(x,g(x)}_{h(x)} \equiv 0 \quad \text{on C.}$$

Therefore, $h'(x) \equiv 0$ and [as seen in (11)] it is also $f_x + f_y \frac{dy}{dx}$.

Comparing these two expressions for $h'(x)$, we obtain

$$f_x + f_y \frac{dy}{dx} = 0, \text{ or } \frac{dy}{dx} = -\frac{f_x}{f_y} \quad (\text{provided } f_y \neq 0).$$

* This is the first time we are making use of the assumption that g is differentiable.

3.8.1 (L) continued

More mechanically,

$w = f(x,y) = 0 \rightarrow$
$dw = f_x dx + f_y d_y = 0 \rightarrow \frac{dy}{dx} = -\frac{f_x}{f_y} \ (f_y \neq 0).$

Most likely, the mechanical way seems natural to you and in the case of most students, it would be accepted without question (although perhaps not without confusion). The key part is that the mechanical way utilizes the fact that dw is still $f_x dx + f_y dy$ even when x and y are dependent, since we are using dw in the case $f(x,y) = 0$.

d. As we did (almost trivially) in Part 1,

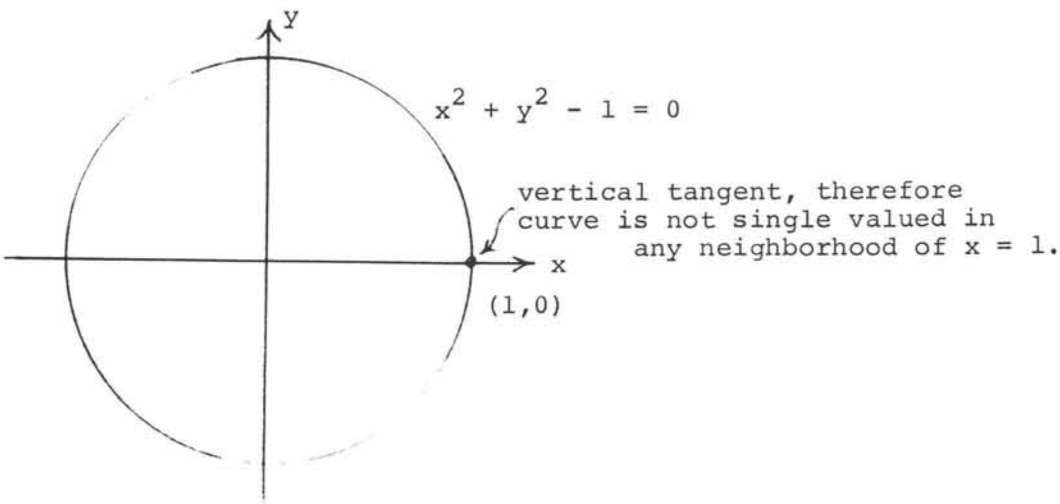
$$x^2 + y^2 - 1 = 0 \qquad (13)$$

$$2x + 2y\frac{dy}{dx} = 0, \text{ therefore } \frac{dy}{dx} = -\frac{x}{y} \qquad (14)$$

Notice that in getting from (13) to (14) we <u>assumed</u> that (13) defined y as a differentiable function of x, say g(x), such that

$$x^2 + g^2(x) - 1 \equiv 0.$$

Indeed in this case, $y = g(x) = \pm\sqrt{1 - x^2}$ and the only "trouble spots" occurred when $x = \pm 1$ since there y could not be defined as a (single-valued) function of x.



3.8.2

a. For dw to be identically zero all coefficients of dw must be zero. In this exercise $dw \equiv 0$ only on the set of 3-tuples S (usually a surface) defined by the equation $f(x,y,z) = 0$. In other words, $dw \equiv 0$ on the set $S = \{(x,y,z): f(x,y,z) = 0\}$.

b. If $f(x,y,z) = 0$ defines z as a continuously differentiable function of x and y, say, $z = g(x,y)$, then $f(x,y,g(x,y)) = h(x,y)$. Equivalently $h(x,y) = f(x,y,z)$, $z = g(x,y)$. Hence, by the chain rule,

$$h_x = f_x \frac{\partial x}{\partial x} + f_y \frac{\partial y}{\partial x} + f_z \frac{\partial z}{\partial x} \tag{1}$$

Now $\frac{\partial x}{\partial x} = 1$, $\frac{\partial y}{\partial x} = 0$ since x and y are independent variables. Thus (1) yields

$$h_x = f_x + f_z \frac{\partial z}{\partial x} = f_x + f_z g_x \tag{2}$$

Similarly,

$$h_y = f_y + f_z \frac{\partial z}{\partial y} = f_y + f_z g_y \tag{3}$$

Therefore if $S = \{(x,y,z) : f(x,y,z) = 0\}$, then <u>on S</u> $h(x,y) \equiv 0$.

$$\text{Therefore, } h_x \equiv h_y \equiv 0 \tag{4}$$

[Notice that it is $h(x,y)$ that is identically zero, <u>not</u> $f(x,y,z)$, i.e., $f(x,y,z) = 0$ only on S].

Combining (4) with (2) and (3) we obtain $f(x,y,z) = 0 \rightarrow$

$$\left.\begin{aligned} \frac{\partial z}{\partial x} &= -\frac{f_x}{f_z} \\ \frac{\partial z}{\partial y} &= -\frac{f_y}{f_z} \end{aligned}\right\} \quad \text{provided } f_z(x,y) \neq 0.$$

c. In this case $S = \{(x,y,z): z^5 + x^2y^3 = 0\}$. Now, since $w = z^5 + x^2y^3$

$$dw = 2xy^3dx + 3x^2y^2dy + 5z^4dz. \tag{5}$$

3.8.2 (continued)

For $(x,y,z) \varepsilon S$, $z^5 = -x^2y^3$, or $z = -x^{\frac{2}{5}} y^{\frac{3}{5}}$. Therefore on S,

$$z^4 = x^{\frac{8}{5}} y^{\frac{12}{5}} \tag{6}$$

and

$$dz = -\frac{2}{5}x^{-\frac{3}{5}} y^{\frac{3}{5}} dx - \frac{3}{5}x^{\frac{2}{5}} y^{-\frac{2}{5}} dy \tag{7}$$

Substituting (6) and (7) into (5), we obtain that for
$\underline{(x,y,z) \varepsilon S}$

$$\begin{aligned} dw &= 2xy^3dx + 3x^2y^2dy + 5x^{\frac{8}{5}} y^{\frac{12}{5}} \left(-\frac{2}{5}x^{-\frac{3}{5}} y^{\frac{3}{5}}dx - \frac{3}{5}x^{\frac{2}{5}} y^{-\frac{2}{5}} dy\right) \\ &= 2xy^3dx + 3x^2y^2dy - 2xy^3dx - 3x^2y^2dy \\ &= \underline{0dx + 0dy}. \end{aligned}$$

3.8.3 (L)

a. Our aim here is simply to emphasize the role of each of the two definitions of an exact differential in terms of a specific example.

We wish to find a continuously differentiable function f such that

$$df = (e^xy^3 + 2x \sin y + 4x^3y^5)dx + (3e^xy^2 + x^2 \cos y + 5x^4y^4)dy \tag{1}$$

or, alternatively without using differential notation,

$$\left.\begin{aligned} f_x &= e^xy^3 + 2x \sin y + 4x^3y^5 \\ &\text{and} \\ f_y &= 3e^xy^2 + x^2 \cos y + 5x^4y^4 \end{aligned}\right\} \tag{1'}$$

Now before we try to construct the desired f, it would be helpful to know that such a function exists. It is here that we use the

3.8.3 (L) continued

criterion of comparing M_y and N_x, where M denotes the coefficient of dx and N the coefficient of dy in Equation (1). We have

$$M = e^x y^3 + 2x \sin y + 4x^3 y^5$$

$$N = 3e^x y^2 + x^2 \cos y + 5x^4 y^4,$$

hence,

$$M_y = 3e^x y^2 + 2x \cos y + 20\, x^3 y^4$$

$$N_x = 3e^x y^2 + 2x \cos y + 20\, x^3 y^4.$$

Therefore,

$$M_y \equiv N_x \qquad (2)$$

from which we may now conclude that a function f which satisfies equation (1) exists;knowing this, we set out to construct f, and our technique mimics the proof of why $M_y = N_x$ implies exactness. Namely, we know that whatever f looks like it must satisfy

$$f_x = e^x y^3 + 2x \sin y + 4x^3 y^5.$$

If we now integrate with respect to x, treating y as a constant, we obtain,

$$f = e^x y^3 + x^2 \sin y + x^4 y^5 + g(y)* \qquad (3)$$

Notice that (3) tells us that if f exists, all we need to do is determine g(y) explicitly to determine f. What we would not know

* Recall that since x and y are independent, $\partial g/\partial x \equiv 0 \leftrightarrow$ g is a function of y alone. Our definition of a "constant" when we integrate holding y constant is any function of y. This parallels the usual definition of constant in the case of a single variable, i.e., $dc/dx \equiv 0 \leftrightarrow c$ is a constant.

3.8.3 (L) continued

without (2) is that f does exist. More specifically, if we differentiate equation (3) with respect to y, we obtain

$$f_y = 3e^x y^2 + x^2 \cos y + 5x^4 y^4 + g'(y) \tag{4}$$

but we also know from (1') that

$$f_y = 3e^x y^2 + x^2 \cos y + 5x^4 y^4.$$

Comparing this with (4), since $f_y \equiv f_y$, it follows that

$$f'(y) = 0$$

whereupon

$$g = c, \text{ c an arbitrary constant} \tag{5}$$

[The fact that $M_y = N_x$ guaranteed that when we equated the two expressions for f_y, the resulting equation would yield $g'(y)$ in terms of y alone, so that g(y) could be determined].

At any rate, substituting (5) into (3) yields

$$\underline{f[= f(x,y)] = e^x y^3 + x^2 \sin y + x^4 y^5 + c}. \tag{6}$$

As a check, we see from (6) that

$$\left.\begin{aligned} f_x &= e^x y^3 + 2x \sin y + 4x^3 y^5 \\ f_y &= 3e^x y^2 + x^2 \cos y + 5x^4 y^4 \end{aligned}\right]$$

which checks with equations (1').

Aside:

The technique by which we derived (6) works in general (provided of course, that $M_y = N_x$). This does not mean that the more astute student could not have found f by "inspection". Namely,

3.8.3 (L) continued

$$(e^x y^3 + 2x \sin y + 4x^3 y^5)dx + (3e^x y^2 + x^2 \cos y + 5x^4 y^4)dy$$
$$= (e^x y^3 dx + 3e^x y^2 dy) + (2x \sin y\, dx + x^2 \cos y\, dy)$$
$$+ (4x^3 y^5 dx + 5x^4 y^4 dy)$$
$$= d(e^x y^3\ [+c]) + d(x^2 \sin y) + d(x^4 y^5)$$
$$= d(e^x y^3 + x^2 \sin y + x^4 y^5 + c).$$

The point is that we do not have to be this astute to obtain f, but the above may help to explain why one often thinks of exact differentials as being ones that are "integrable at sight".

b. Given that $f_x = 4x^3 \sin y$ we obtain that the required f, if it exists, must have the form

$$f = x^4 \sin y + g(y). \tag{7}$$

To determine g, we compute f_y from (7) to obtain

$$f_y = x^4 \cos y + g'(y)$$

and compare this with the requirement that

$$f_y = x^4 \cos y + x.$$

This leads to the equation

$$x^4 \cos y + g'(y) = x^4 \cos y + x$$

or

$$g'(y) = x. \tag{8}$$

Equation (8) is a contradiction since it shows that x depends on y, contrary to the known fact that x and y are independent variables. This contradiction stemmed from the assumption that the required f existed, which means, therefore, that no such f exists.

3.8.3 (L) continued

In this example $M = 4x^3 \sin y$ and $N = x^4 \cos y + x$. Therefore

$M_y = 4x^3 \cos y$ and $N_x = 4x^3 \cos y + 1$

from which it follows that

$M_y \not\equiv N_x$

so that we know at once that

$4x^3 \sin y\, dx + (x^4 \cos y + x)dy$

is <u>not</u> exact.

c. One use of exact differentials is in the solution of certain types of differential equations. In this exercise we are given that

$$\frac{dy}{dx} = -\frac{(e^x y^3 + 2x \sin y + 4x^3 y^5)}{(3e^x y^2 + x^2 \cos y + 5x^4 y^4)}. \tag{9}$$

Therefore,

$$(e^x y^3 + 2x \sin y + 4x^3 y^5)dx + (3e^x y^2 + x^2 \cos y + 5x^4 y^4)dy$$

$$= 0. \tag{10}$$

Now in part (a) of this exercise we showed that the left side of (10) was df where $f = e^x y^3 + x^2 \sin y + x^4 y^5 + c$. In other words, (10) becomes $df = 0$ so that by our discussion in Exercise 3.8.1 (L), we have that

$$f(x,y) = k,\ k \text{ an arbitrary constant.} \tag{11}$$

Comparing (11) and (6), it follows that

$$e^x y^3 + x^2 \sin y + x^4 y^5 + c = k \tag{12}$$

and since c and k are both arbitrary constants they can be

3.8.3 (L) continued

"amalgamated", so that (12) becomes

$$e^x y^3 + x^2 \sin y + x^4 y^5 = c. \qquad (13)$$

Equation (13), with any value for c, satisfies the given differential equation (9), but we wish to determine a solution of (9) such that when $x = 0$, $y = 2$. To this end we let $x = 0$ and $y = 2$, and obtain

$$e^0 (2)^3 + 0 + 0 = c$$

so that

$$c = 8$$

whereupon (13) becomes

$$\left.\begin{aligned} &e^x y^3 + x^2 \sin y + x^4 y^5 = 8 \\ &\text{or} \\ &g(x,y) = 0 \text{ where } g(x,y) = e^x y^3 + x^2 \sin y + x^4 y^5 - 8. \end{aligned}\right\} \qquad (14)$$

[Geometrically, the curve (14) passes through the point (0,2) and its slope at any point (x,y) is given by equation (9)]

<u>Note:</u>

More generally, to solve the differential equation

$$Mdx + Mdy = 0 \qquad (15)$$

we first test to see whether $Mdx + Ndy$ is exact. This is done by seeing whether $M_y = N_x$. If it is exact, we use the procedure of part (a) to find explicitly a function f such that $df = Mdx + Ndy$. This function f "converts" equation (15) into

$$df = 0$$

3.8.3 (L) continued

from which it follows that

$$f(x,y) = c \qquad (16)$$

and (16) is then the solution to (15).

The reason that differential equations are not this easy to handle, in general, is that Mdx + Ndy need not be exact. The main study of first order differential equations centers around the problem of what happens in equation (15) when the left side is not exact, and this will be investigated by us in more detail in Block 7.

3.8.4

The slope of our curve is

$$\frac{dy}{dx} = -\frac{(2xe^y + e^x)}{(x^2 + 1)e^y}$$

so that

$$(2xe^y + e^x)dx + (x^2 + 1)e^y\,dy = 0. \qquad (1)$$

Letting $M = 2xe^y + e^x$ and $N = (x^2 + 1)e^y$, we see that

$$M_y = 2xe^y,\ N_x = 2xe^y.$$

Since $M_y \equiv N_x$, Mdx + Ndy is exact and we may now construct f such that df = Mdx + Ndy.

In particular, we have

$$f_x = 2xe^y + e^x.$$

Therefore,

$$f = x^2e^y + e^x + g(y) \qquad (2)$$

3.8.4 (continued)

so that

$$f_y = x^2e^y + g'(y).$$

But we also know that

$$f_y = (x^2 + 1)e^y$$

and comparing these two expressions for f_y, we see that

$$x^2e^y + g'(y) = (x^2 + 1)e^y.$$

Therefore,

$$g'(y) = e^y$$

so

$$g(y) = e^y + c. \qquad (3)$$

Substituting g(y) as determined in (3) into (2), we obtain

$$f[= f(x,y)] = x^2e^y + e^x + e^y + c. \qquad (4)$$

Thus, equation (1) becomes

$$df = 0$$

so that

$$f = k \qquad (5)$$

and combining (4) and (5),

$$x^2e^y + e^x + e^y = c. \qquad (6)$$

Since our curve passes through the origin, we must choose c so that (6) is satisfied when $x = y = 0$. This yields

$$0e^0 + e^0 + e^0 = c$$

3.8.4 (continued)

or

$$c = 2. \quad (7)$$

Substituting (7) into (6) we find that our curve is given by

$$\underline{x^2e^y + e^x + e^y = 2} \quad (8)$$

[In this particular case we may solve for y explicitly as a function of x. Namely

$$x^2e^y + e^x + e^y = 2$$

$$e^y(x^2 + 1) = 2 - e^x$$

$$e^y = \frac{2 - e^x}{x^2 + 1}$$

$$y = \ell n \left[\frac{2 - e^x}{x^2 + 1}\right] . \quad (9)$$

Since $x^2 + 1$ is always positive and we define ℓn only for positive numbers, equation (9) tells us that $2 - e^x$ must be positive, or $e^x < 2$. That is, our curve is restricted to x values such that $x < \ell n\ 2$].

3.8.5

Our main aim here is to extend the results of this unit to exact differentials in the case when there are more than two independent variables. Rather than to proceed too abstractly, we shall look at the special case n = 3.

we are given that

$$df(x,y,z) = M(x,y,z)dx + N(x,y,z)dy + P(x,y,z)dz. \quad (1)$$

This is equivalent to

$$f_x = M \quad (2)$$

3.8.5 (continued)

$$f_y = N \tag{3}$$

$$f_z = P \tag{4}$$

Now from (2) and (3) we have that

$$f_{xy} = M_y \text{ and } f_{yx} = N_x. \tag{5}$$

Under the hypotheses of the problem, $f_{xy} = f_{yx}$, so we may conclude from (5) that

$\underline{M_y = N_x}$.

Similarly, from (2) and (4), we have that

$f_{xz} = M_z$ and $f_{zx} = P_x$, and since $f_{xz} = f_{zx}$, it follows that

$\underline{M_z = P_x}$.

Finally, from (3) and (4) we have that

$f_{yz} = N_z$ and $f_{zy} = P_y$, so since $f_{yz} = f_{zy}$, we conclude that

$\underline{N_z = P_y}$.

[Hopefully, it is easy to see that this procedure generalizes to the case of n independent variables. Namely, if

$$df(x_1,\ldots,x_n) = P_1(x_1,\ldots,x_n)dx_1 + \ldots + P_n(x_1,\ldots,x_n)dx_n$$

then for $i \neq j$ ($i = 1,\ldots,n$; $j = 1,\ldots,n$),

$$\frac{\partial P_i}{\partial x_j} = \frac{\partial P_j}{\partial x_i} .$$

That is, we pick terms in pairs and we differentiate each coefficient with respect to the differential of the other term, and each of these pairs of derivatives are equal. The case n = 2 discussed in the text and in our lecture is but a special case of this.]

3.8.6

a. Unless $\frac{\partial}{\partial z}$ [xz - e^x sin y] = $\frac{\partial}{\partial y}$ [xy + z], no choice of M can make our differential exact. A quick check shows that $\frac{\partial}{\partial z}$ [xz - e^x sin y] = x = $\frac{\partial}{\partial y}$ [xy + z], so it make sense to continue.

We must have

$$\frac{\partial M}{\partial y} = \frac{\partial(xz - e^x \sin y)}{\partial x} = z - e^x \sin y \qquad (1)$$

and

$$\frac{\partial M}{\partial z} = \frac{\partial(xy + z)}{\partial x} = y\ . \qquad (2)$$

From (2),

$$M = yz + g(x,y) \qquad (3)$$

therefore,

$$\frac{\partial M}{\partial y} = z + g_y(x,y). \qquad (4)$$

Comparing (1) and (4), we have

$$z - e^x \sin y = z + g_y(x,y)$$

therefore,

$$g_y(x,y) = -\ e^x \sin y$$

or

$$g(x,y) = e^x \cos y + h(x). \qquad (5)$$

Substituting (5) into (3) yields

$$M = yz + e^x \cos y + h(x) \qquad (6)$$

where h is any function of x.

3.8.6 (continued)

b. We want to find f(x,y,z) such that

$$f_x = yz + e^x \cos y + h(x) \qquad (7)$$

$$f_y = xz - e^x \sin y \qquad (8)$$

$$f_z = xy + z. \qquad (9)$$

Integrating (7) with respect to x, we obtain

$$f = xyz + e^x \cos y + \int h(x)\,dx^* + c(y,z) \qquad (10)$$

so that

$$f_y = xz - e^x \sin y + c_y(y,z)$$

and combining this with (8) yields

$$xz - e^x \sin y = xz - e^x \sin y + c_y(y,z)$$

therefore, $c_y(y,z) = 0$

therefore, $c(y,z) = k(z)$. (11)

Substituting (11) into (10) yields

$$f(x,y,z) = xyz + e^x \cos y + \int h(x)\,dx + k(z) \qquad (12)$$

From (12)

$$f_z = xy + k'(z)$$

and comparing this with (9) leads to

$$xy + z = xy + k'(z)$$

* The fact that h(x) is continuous is sufficient to guarantee the existence of $\int h(x)\,dx$.

3.8.6 (continued)

therefore, $k'(z) = z$

therefore, $k(z) = \frac{1}{2} z^2 + c$

and putting this into (12) yields

$$f(x,y,z) = xyz + e^x \cos y + \int h(x)\,dx + \frac{1}{2} z^2 + c \tag{13}$$

c. $$f_x = e^{y + 2z} = M \tag{14}$$

$$f_y = xe^{y + 2z} = N \tag{15}$$

$$f_z = 2xe^{y + 2z} = P \tag{16}$$

Before looking for f, we make the following checks

$$M_y = \frac{\partial}{\partial y} (e^{y + 2z}) = e^{y + 2z} = \frac{\partial}{\partial x} (xe^{y + 2z}) = N_x$$

$$M_z = \frac{\partial}{\partial z} (e^{y + 2z}) = 2e^{y + 2z} = \frac{\partial}{\partial x} (2xe^{y + 2z})\ P_x$$

$$N_z = \frac{\partial}{\partial z} (xe^{y + 2z}) = 2xe^{y + 2z} = \frac{\partial}{\partial y} (2xe^{y + 2z}) = P_y.$$

Now we set out to construct f.

From (14)

$$f = \int e^{y + 2z}\,dx + g(y,z)$$

$$= xe^{y + 2z} + g(y,z) \tag{17}$$

from which it follows that

$$f_y = xe^{y + 2z} + g_y(y,z)$$

and comparing this with (15) yields

$$xe^{y + 2z} + g_y(y,z) = xe^{y + 2z}$$

3.8.6 (continued)

so that

$g_y(y,z) = 0$

and

$g = h(z)$.

Thus from (17) we have now that

$$f(x,y,z) = xe^{y + 2z} + h(z) \qquad (18)$$

and from (18) we conclude that

$f_z(x,y,z) = 2xe^{y + 2z} + h'(z)$

and comparing this with (16), yields

$h'(z) = 0$

whereupon

$h = c$.

Putting this into (18) yields the final result:

$f(x,y,z) = xe^{y + 2z} + c$.

3.8.7

$d(ST - u) =$

$$SdT + TdS - du. \qquad (1)$$

We are given that

$$TdS = du + pdv. \qquad (2)$$

Substituting (2) into (1) yields

3.8.7 (continued)

$$d(ST - u) = SdT + du + pdv - du$$

$$= SdT + pdv \qquad (3)$$

is exact (since ST - u has it as its total differential).

Therefore, $\left(\frac{\partial S}{\partial v}\right)_T = \left(\frac{\partial p}{\partial T}\right)_v$

(where the subscripts indicate that v and T are being used as the independent variables).

Quiz

1. From $u = x^3 - 3xy^2$, we obtain

$$\frac{\partial u}{\partial u} = \frac{\partial(x^3 - 3xy^2)}{\partial u},$$

or

$$1 = 3x^2 \frac{\partial x}{\partial u} - 3y^2 \frac{\partial x}{\partial u} - 6xy \frac{\partial y}{\partial u}$$

Therefore,

$$3(x^2 - y^2) \frac{\partial x}{\partial u} - 6xy \frac{\partial y}{\partial u} = 1. \tag{1}$$

Similarly, from $v = 3x^2y - y^3$, we obtain

$$\frac{\partial v}{\partial u} = \frac{\partial(3x^2y - y^3)}{\partial u},$$

or

$$0^* = 6xy \frac{\partial x}{\partial u} + 3x^2 \frac{\partial y}{\partial u} - 3y^2 \frac{\partial y}{\partial u}$$

Therefore,

$$6xy \frac{\partial x}{\partial u} + 3(x^2 - y^2) \frac{\partial y}{\partial u} = 0. \tag{2}$$

Multiplying both sides of (1) by $(x^2 - y^2)$, both sides of (2) by $2xy$, and then adding the two resulting equations, we obtain

$$\left.\begin{aligned} 3(x^2 - y^2)^2 \frac{\partial x}{\partial u} - 6xy\,(x^2 - y^2) \frac{\partial y}{\partial u} &= x^2 - y^2 \\ 12x^2y^2 \frac{\partial x}{\partial u} + 6xy\,(x^2 - y^2) \frac{\partial y}{\partial u} &= 0 \end{aligned}\right\} \rightarrow$$

*Notice that we are assuming that $x = x(u,v)$, $y = y(u,v)$ where u and v are independent variables. Hence, $\frac{\partial v}{\partial u} = 0$.

1. continued

$$3\left[(x^2 - y^2)^2 + 4x^2y^2\right] \frac{\partial x}{\partial u} = x^2 - y^2 \tag{3}$$

and since

$$(x^2 - y^2)^2 + 4x^2y^2 = x^4 - 2x^2y^2 + y^4 + 4x^2y^2$$

$$= x^4 + 2x^2y^2 + y^4 = (x^2 + y^2)^2$$

equation (3) becomes

$$3(x^2 + y^2)^2 \frac{\partial x}{\partial u} = x^2 - y^2$$

Therefore,

$$\frac{\partial x}{\partial u} = \frac{x^2 - y^2}{3(x^2 + y^2)^2}. \tag{4}$$

[Note that $u = x^3 - 3xy^2 \rightarrow \left(\frac{\partial u}{\partial x}\right)_y = 3x^2 - 3y^2$. Hence, $\left(\frac{\partial x}{\partial u}\right)_y = \frac{1}{3(x^2 - y^2)}$. This is not the same as equation (4) since in (4), $\frac{\partial x}{\partial u}$ means $\left(\frac{\partial x}{\partial u}\right)_v$.]

2. (a) $w = f(u,v)$

where

$$\begin{cases} u = 3x + 2y \\ v = 8x + 5y \end{cases}$$

Therefore

$$w = f(3x + 2y,\ 8x + 5y) = g(x,y)$$

2. continued

$$\frac{\partial w^*}{\partial x} = \frac{\partial w^*}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x}$$

$$= 3\frac{\partial w}{\partial u} + 8\frac{\partial w}{\partial v}$$

$$\frac{\partial^2 w}{\partial x^2} = 3\frac{\partial\left[\frac{\partial w}{\partial u}\right]}{\partial x} + 8\frac{\partial\left[\frac{\partial w}{\partial v}\right]}{\partial x}$$

$$= 3\left[\frac{\partial\left(\frac{\partial w}{\partial u}\right)}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial\left(\frac{\partial w}{\partial u}\right)}{\partial v}\frac{\partial v}{\partial x}\right] + 8\left[\frac{\partial\left(\frac{\partial w}{\partial v}\right)}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial\left(\frac{\partial w}{\partial v}\right)}{\partial v}\frac{\partial v}{\partial x}\right] \qquad (1)$$

$$= 9\frac{\partial^2 w}{\partial u^2} + 24\frac{\partial^2 w}{\partial v\partial u} + 24\frac{\partial^2 w}{\partial u\partial v} + 64\frac{\partial^2 w}{\partial v^2}$$

$$= 9\frac{\partial^2 w}{\partial u^2} + 48\frac{\partial^2 w}{\partial u\partial v} + 64\frac{\partial^2 w}{\partial v^2} \qquad (2)$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial y}$$

$$= 2\frac{\partial w}{\partial u} + 5\frac{\partial w}{\partial v}$$

$$\frac{\partial^2 w}{\partial y^2} = 2\left[\frac{\partial^2 w}{\partial u^2}\frac{\partial u}{\partial y} + \frac{\partial^2 w}{\partial v\partial u}\frac{\partial v}{\partial y}\right] + 5\left[\frac{\partial^2 w}{\partial u\partial v}\frac{\partial u}{\partial y} + \frac{\partial^2 w}{\partial v^2}\frac{\partial v}{\partial y}\right]. \qquad (3)$$

[Notice the similarity in structure of the bracketed expressions of equations (1) and (3). All that we have done in (3) is replaced each x in (1) by y.]

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*As usual, $\frac{\partial w}{\partial x}$ refers to the notation $w = g(x,y)$ while $\frac{\partial w}{\partial u}$ refers to the notation $w = f(u,v)$.

2. continued

$$\frac{\partial^2 w}{\partial y^2} = 4\,\frac{\partial^2 w}{\partial u^2} + 10\,\frac{\partial^2 w}{\partial v\partial u} + 10\,\frac{\partial^2 w}{\partial u\partial v} + 25\,\frac{\partial^2 w}{\partial v^2}$$

$$= 4\,\frac{\partial^2 w}{\partial u^2} + 20\,\frac{\partial^2 w}{\partial u\partial v} + 25\,\frac{\partial^2 w}{\partial v^2}. \qquad (4)$$

Adding (2) and (4) yields

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 13\,\frac{\partial^2 w}{\partial u^2} + 68\,\frac{\partial^2 w}{\partial u\partial v} + 89\,\frac{\partial^2 w}{\partial v^2}. \qquad (5)$$

(b) If

$$w = u^3 + v^2 + uv,$$

then

$$\frac{\partial^2 w}{\partial u^2} = 6u,\ \frac{\partial^2 w}{\partial u\partial v} = \frac{\partial^2 w}{\partial v\partial u} = 1, \text{ and } \frac{\partial^2 w}{\partial v^2} = 2.$$

Hence, (5) yields

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 13(6u) + 68(1) + 89(2)$$

$$= 78u + 246$$

$$= 78(3x + 2y) + 246$$

$$= 234x + 156y + 246. \qquad (6)$$

On the other hand, direct substitution yields

2. continued

$$w = (3x + 2y)^3 + (8x + 5y)^2 + (3x + 2y)(8x + 5y)$$

$$= (27x^3 + 54x^2y + 36xy^2 + 8y^3) + (64x^2 + 80xy + 25y^2)$$

$$+ (24x^2 + 31xy + 10y^2)$$

Therefore,

$$w = 27x^3 + 54x^2y + 36xy^2 + 8y^3 + 88x^2 + 111xy + 35y^2$$

Therefore

$$\frac{\partial w}{\partial x} = 81x^2 + 108xy + 36y^2 + 176x + 111y$$

$$\frac{\partial^2 w}{\partial x^2} = 162x + 108y + 176 \tag{7}$$

and

$$\frac{\partial w}{\partial y} = 54x^2 + 72xy + 24y^2 + 111x + 70y$$

$$\frac{\partial^2 w}{\partial y^2} = 72x + 48y + 70. \tag{8}$$

Adding (7) and (8) yields

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 234x + 156y + 246,$$

which checks with our result in (6).

3. We first find the equation of M. Since M is an equipotential surface of $w = z^5 + 6xyz + x^4y^5$, we know that $\vec{\nabla}w\Big|_{(1,1,1)}$ is normal to M.

Now,

$$\vec{\nabla}w = (6yz + 4x^3y^5,\ 6xz + 5x^4y^4,\ 5z^4 + 6xy).$$

3. continued

Therefore,

$$\vec{\nabla}w \Big|_{(1,1,1)} = (10,11,11).$$

Therefore,

$$10\vec{i} + 11\vec{j} + 11\vec{k}$$

is normal to M at (1,1,1). Therefore, the equation of M is

$$10(x - 1) + 11(y - 1) + 11(z - 1) = 0,$$

or

$$10x + 11y + 11z = 32. \qquad (1)$$

To find where M meets $z = 2y = 4x$ (i.e. $z = 4x$ and $y = 2x$), we replace z by 4x and y by 2x in (1) to obtain

$$10x + 22x + 44x = 32,$$

or,

$$x = \frac{32}{76} = \frac{8}{19}.$$

Therefore,

$$y = 2x \rightarrow y = \frac{16}{19} \quad \text{and} \quad z = 4x \rightarrow z = \frac{32}{19}.$$

Hence, the point of intersection is $(\frac{8}{19}, \frac{16}{19}, \frac{32}{19})$.

4. We want the gradient, $\vec{\nabla}f$, at (2,3). This is given by

$$\vec{\nabla}f \Big|_{(2,3)} = f_x(2,3)\vec{i} + f_y(2,3)\vec{j}. \qquad (1)$$

4. continued

We are told that in the direction, s_1, from (2,3) to (5,7) $\frac{dw}{ds_1} = 4$. But, $\frac{dw}{ds_1} = \vec{\nabla} f_{(2,3)} \cdot \vec{u}_{s_1}$.

The vector from (2,3) to (5,7) is $3\vec{i} + 4\vec{j}$. Therefore,

$$\vec{u}_{s_1} = \frac{3}{5}\vec{i} + \frac{4}{5}\vec{j}.$$

Therefore, from (1)

$$4 = \vec{\nabla} f \Big|_{(2,3)} \cdot \vec{u}_{s_1} = [f_x(2,3)\vec{i} + f_y(2,3)\vec{j}] \cdot [\tfrac{3}{5}\vec{i} + \tfrac{4}{5}\vec{j}]$$

Therefore,

$$4 = \frac{3}{5} f_x(2,3) + \frac{4}{5} f_y(2,3)$$

or

$$3\, f_x(2,3) + 4\, f_y(2,3) = 20. \qquad (2)$$

Similarly, $\vec{u}_{s_2} = \frac{4}{5}\vec{i} + \frac{3}{5}\vec{j}$ if s_2 is the direction from (2,3) to (6,6). Therefore,

$$10 = \vec{\nabla} f \Big|_{(2,3)} \cdot \vec{u}_{s_2} = \frac{4}{5} f_x(2,3) + \frac{3}{5} f_y(2,3)$$

Therefore,

$$4\, f_x(2,3) + 3\, f_y(2,3) = 50. \qquad (3)$$

Solving equations (2) and (3) simultaneously yields

$f_x(2,3) = 20$ and $f_y(2,3) = -10$.

Hence,

$$\vec{\nabla} f \Big|_{(2,3)} = 20\vec{i} - 10\vec{j}.$$

4. continued

Therefore, $\frac{dw}{ds}$ is maximum at (2,3) in the direction $20\vec{i} - 10\vec{j}$ [i.e., in the direction from (2,3) to (4,2)] and this maximum value is

$$\sqrt{(20)^2 + (-10)^2} = 10\sqrt{5} \ (\approx 22.4).$$

5. $f(x) = \int_a^x (x - y)h(y)dy \rightarrow$

$$f'(x) = \int_a^x \frac{\partial}{\partial x}[(x - y)h(y)]dy + \underbrace{(x - x)}_{0}h(x)\frac{dx}{dx} - (x - a)h(a)\underbrace{\frac{da}{dx}}_{0}$$

Therefore,

$$f'(x) = \int_a^x h(y)dy.$$

Hence, the resulting differential equation is

$$f''(x) = h(x).$$

6. (a) Letting $M = 3x^2y + e^x \cos y$ and $N = x^3 - e^x \sin y$, we see that

$$\frac{\partial M}{\partial y} = 3x^2 - e^x \sin y = \frac{\partial N}{\partial x}$$

Therefore, Mdx + Ndy is exact. Therefore, there exists f(x,y) such that

$$df = (3x^2y + e^x \cos y)dx + (x^3 - e^x \sin y)dy$$

$$= f_x dx + f_y dy.$$

Therefore,

6. continued

$f_x = 3x^2y + e^x \cos y$ and $f_y = x^3 - e^x \sin y$

$f = x^3y + e^x \cos y + g(y)$

$f_y = x^3 - e^x \sin y + g'(y)$

$g'(y) = 0,\ g(y) = C$

or:

$f = x^3y + e^x \cos y + C$

(b) $\dfrac{dy}{dx} = \dfrac{3x^2y + e^x \cos y}{e^x \sin y - x^3} \rightarrow$

$(e^x \sin y - x^3)dy = (3x^2y + e^x \cos y)dx \rightarrow$

$(3x^2y + e^x \cos y)dx + (x^3 - e^x \sin y)dy = 0 \rightarrow$

$d(x^3y + e^x \cos y + C) = 0 \rightarrow$

$x^3y + e^x \cos y + C = \text{constant} \rightarrow$

$x^3y + e^x \cos y = \text{constant}.$