Unit 6: Vectors in Terms of Polar Coordinates

2.6.1(L)

In this exercise, we simply want to reinforce the idea that our study of polar coordinates was motivated by problems involving particles in motion under the influence of various forces. To keep ourselves free of any coordinate system for the moment, let us assume that a particle moves along the curve $\vec{R}(t)$. Notice that by writing $\vec{R}(t)$ we are indicating that the position of the particle is a function of time, and, as yet, we have not specified what coordinate system is to be used. In this exercise, we want to investigate how the position, velocity, and acceleration of the particle will be represented _if_ we elect to use polar coordinates rather than Cartesian coordinates or tangential and normal coordinates. Obviously, the quantities themselves do not depend on the coordinate system, but their representation in terms of components does.

Our first major problem concerns the subtle pitfall that occurs when we try to define a pair of orthogonal unit polar vectors in the vein of $\vec{i}$ and $\vec{j}$ or $\vec{T}$ and $\vec{N}$.

The problem stems from the fact that when a curve C is written in the polar form $r = f(\theta)$, r may be negative for some values of θ (which is why we stressed this point so much in the previous units). Namely, suppose we pick a value of θ, say $\theta = \theta_o$, and we locate the point, $P_o(r_o,\theta_o)$, on the curve C where $r_o = f(\theta_o)$. It is then natural to think of the radius vector, $\vec{R}$, which joins the origin to P_o. Pictorially,

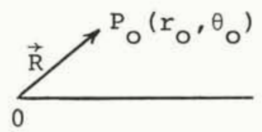
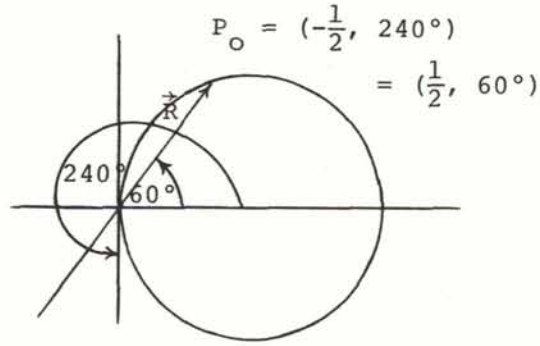


Figure 1

2.6.1(L) continued

Now it seems that a very obvious candidate for a unit vector when we deal with polar coordinates is the vector obtained when we divide $\vec{R}$ by its magnitude. From Figure 1, it would appear that this vector is $\vec{R}/r_o$, since the magnitude of $\vec{R}$ appears to be r_o. The trouble is that P_o may be located in the first quadrant for a third quadrant value of θ_o and a negative value of r_o, since (r_o, θ_o) and $(-r_o, \underline{\theta} + 180°)$ name the same point, even though one of the "names" may not satisfy the polar equation of C. By way of an illustration, consider the curve C whose polar equation is $r = \cos\theta$. If we let $\theta_o = 240°$, $r_o = \cos 240° = -\frac{1}{2}$. Thus, the point $(-\frac{1}{2}, 240°)$ belongs to C but it appears in the first quadrant, not the third. Again pictorially,



In this case,
$|\vec{R}| = \frac{1}{2} = -r_o$
since $r_o = -\frac{1}{2}$

Figure 2

Notice that in looking at Figure 2 without reference to the equation of C, we would be tempted to say that r_o is $\frac{1}{2}$ and that $\theta_o = 60°$, but we know that $\theta_o = 240°$ and that $r = -\frac{1}{2}$.

We could avoid this dilemma by insisting that in dividing $\vec{R}$ by its magnitude we always think of r_o as being non-negative. In fact, had we imposed this stringent condition when we first introduced polar coordinates (and this condition is imposed, as we shall see in Block 6, when one views complex numbers geometrically) the present problem would not have occurred. On the other hand, this condition, as we have tried to show in some of the exercises of the previous units, would have introduced other unpleasantries.

2.6.1(L) continued

From our point of view, the key idea is that we are already committed to an interpretation that allows r to be negative. If we now introduce another definition of r that does not allow r to be negative, we would then have two different concepts both named by the same letter, and so similar in nature that misinterpretation is almost assured of taking place.

To avoid this dilemma, we agree not to change the definition of r and we also agree that we like the notion of defining a unit vector, $\vec{u}_r$, as being $\vec{R}$ divided by r_o. For if we let $\vec{u}_r = \vec{R}/r_o$, we obtain the result that

$$\vec{R} = r_o\vec{u}_r \tag{1}$$

which agrees with the first quadrant interpretation expressed in Figure 1. The major observation to be made, however, is that since r_o may be negative, the most we can imply from equation (1) is that $\vec{R}$ and $\vec{u}_r$ have the same direction - but they may (and, in fact will, if r_o is negative) have opposite sense.

Defining $\vec{u}_r$ in this way is in keeping with our intuitive beliefs. That is, when we study a curve C, we are interested in the curve, not in the equation which expresses this curve. Thus, while the equations are different $r = \cos\theta$ and $r = -\cos(\theta + \pi)$ name the same curve. Therefore, given a point on this curve, we would like $\vec{u}_r$ to be defined independently of which equation represents the curve. One way of doing this is to have $\vec{u}_r$ determined by the ray $\theta = \theta_o$, without reference to the equation $r = f(\theta)$ which represents C.

This is done in the textbook's definition in which $\vec{u}_r$ is defined by

$$\vec{u}_r = (\cos\theta_o)\vec{i} + (\sin\theta_o)\vec{j}. \tag{2}$$

Namely, equation (2) defines $\vec{u}_r$ solely in terms of $\theta = \theta_o$. Our only objection to definition (2) is that it is cloaked in the language of Cartesian coordinates, rather than in a language which does not depend on the coordinate system. Definition (1), on the other hand, is independent of any coordinate system.

2.6.1(L) continued

As a final illustration of our remarks, consider the point on the curve $r = \cos\theta$ which corresponds to the value of $\theta = 120°$ (where we have changed the previous example slightly to get away from the first and third-quadrant orientation). The point in question is then $(-\frac{1}{2}, 120°)$ and this point is in the fourth quadrant, even though the ray $\theta = 120°$ is in the second quadrant. Pictorially,

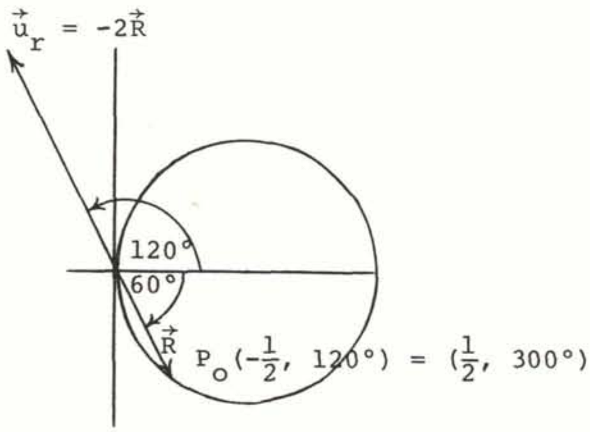


Figure 3

From Figure 3, we see that $\vec{R}$ is $\vec{OP_o}$, and if we elect to use Cartesian coordinates, R is given by

$$\vec{R} = (\tfrac{1}{2}\cos 300°)\vec{i} + (\tfrac{1}{2}\sin 300°)\vec{j} + \tfrac{1}{4}\vec{i} - \tfrac{\sqrt{3}}{4}\vec{j}$$

and since $r_o = -\frac{1}{2}$, we obtain from definition (1) that

$$\vec{u}_r = \vec{R}/r_o = -2\vec{R} = -\tfrac{1}{2}\vec{i} + \tfrac{\sqrt{3}}{2}\vec{j}. \qquad (3)$$

It is now easy to verify that equation (3) agrees with definition (2), namely

$$\vec{u}_r = \cos 120°\ \vec{i} + \sin 120°\ \vec{j} = -\tfrac{1}{2}\vec{i} + \tfrac{\sqrt{3}}{2}\vec{j}.$$

At any rate, our results may be briefly summarized as follows: Given the point (r_o,θ_o) on the curve C whose polar equation is $r = f(\theta)$, we draw the radius vector $\vec{R}$ from the origin to (r_o,θ_o).

2.6.1(L) continued

If r_o is positive, then $\vec{u}_r$ is simply $\vec{R}/|\vec{R}|$ (or the vector of unit length having the same direction and sense as $\vec{R}$). If r_o is negative, we define $\vec{u}_r$ to be $-\vec{R}/|\vec{R}|$ (or the unit vector having the same direction as $\vec{R}$ but the opposite sense). In either case, $\vec{u}_r = \vec{R}/r_o$, and in Cartesian coordinates, this is equivalent to

$$\vec{u}_r = (\cos\theta_o)\vec{i} + (\sin\theta_o)\vec{j}.$$

We now need an orthogonal companion for $\vec{u}_r$. The quickest way to get one is from our earlier result that a variable vector of <u>constant magnitude</u> is <u>orthogonal</u> to its <u>derivative</u>. Thus, if we assume that $\vec{u}_r$ is a differentiable function of θ (which means that $\vec{u}_r$ varies "smoothly" with respect to θ), then $\frac{d\,\vec{u}_r}{d\theta}$ is orthogonal to $\vec{u}_r$. Now, because we are free to choose what sense a vector can have, there are two unit vectors in the same direction as $\frac{d\,\vec{u}_r}{d\theta}$. Since we want our system to have the same structure as $\vec{i}$ and $\vec{j}$, we choose the unit vector which is obtained from $\vec{u}_r$ when $\vec{u}_r$ is rotated 90° in the <u>positive</u> (counterclockwise) direction. The unit vector in the direction of $\frac{d\,\vec{u}_r}{d\theta}$ thus obtained is denoted by $\vec{u}_\theta$.

Once these ideas are clear in your mind, the computational notions that (1) $\vec{R} = r\,\vec{u}_r$, (2) $\vec{u}_\theta = \frac{d\,\vec{u}_r}{d\theta}$, and (3) $-\vec{u}_r = \frac{d\,\vec{u}_\theta}{d\theta}$ are quite routine and they have been derived both in the text and in our lecture.

From this point on, the exercise proceeds as an example of our structure of vector calculus, with every step being an accepted rule of the game. Thus,

$$\vec{R} = r\,\vec{u}_r \qquad (1)$$

Therefore,

$$\vec{v} = \frac{d\vec{R}}{dt} = r\,\frac{d\,\vec{u}_r}{dt} + \frac{dr}{dt}\,\vec{u}_r. \qquad (2)$$

2.6.1(L) continued

While (2) is a correct statement, the idea is that we would like $\vec{v}$ expressed in terms of $\vec{u}_r$ and $\vec{u}_\theta$ coordinates (not $\vec{u}_r$ and $\frac{d\,\vec{u}_r}{dt}$ coordinates).

Knowing that $\frac{d\,\vec{u}_r}{d\theta} = u_\theta$ and (by the chain rule) that $\frac{d\,\vec{u}_r}{dt} = \frac{d\,\vec{u}_r}{d\theta}\frac{d\theta}{dt}$, we may replace (2) by

$$\vec{v} = r\,\frac{d\,\vec{u}_r}{d\theta}\frac{d\theta}{dt} + \frac{dr}{dt}\,\vec{u}_r$$

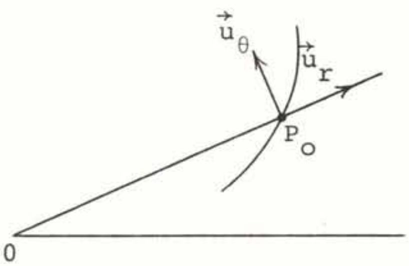
$$= r\,\vec{u}_\theta\,\frac{d\theta}{dt} + \frac{dr}{dt}\,\vec{u}_r$$

or

$$\vec{v} = \frac{dr}{dt}\,\vec{u}_r + r\,\frac{d\theta}{dt}\,\vec{u}_\theta. \qquad (3)$$

Notice that (3) lends itself to a nice physical interpretation (as shown below) but that the validity of (3) requires no physical insight. In other words, one beauty of our derivation, in terms of the "game" idea, is that once a few "simple" properties are assumed, more difficult properties follow as inescapable consequences of the simpler ones - without the necessity of referring to external models.

① At the instant the particle is at P_o, its velocity component in the direction of $\vec{u}_r$ is $\frac{dr}{dt}$ which is the rate of change of the position vector. i.e., $|v_r|$ measures the instantaneous rate of change of $|\vec{OP}_o|$.



② If $\Delta\theta$ is taken to be of infinitesimal length, $rd\theta$ may be viewed as circular arc length. Thus, $r\,\frac{d\theta}{dt}$ may be viewed as the instantaneous speed of the particle at P_o along the circle centered at 0.

2.6.1(L) continued

Now the definition of $\vec{a}$ is still $\frac{d\vec{v}}{dt}$. Thus, from (3) we obtain

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\frac{dr}{dt}\,\vec{u}_r + r\,\frac{d\theta}{dt}\,\vec{u}_\theta\right). \tag{4}$$

Since the derivative recipes are valid here, we may expand (4) to obtain

$$\vec{a} = \frac{d^2r}{dt^2}\,\vec{u}_r + \frac{dr}{dt}\,\frac{d\,\vec{u}_r}{dt} + \frac{dr}{dt}\,\frac{d\theta}{dt}\,\vec{u}_\theta + r\,\frac{d^2\theta}{dt^2}\,\vec{u}_\theta + r\,\frac{d\theta}{dt}\,\frac{d\,\vec{u}_\theta}{dt} \tag{5}$$

where the last three terms in (5) come from differentiating $r\,\frac{d\theta}{dt}\,\vec{u}_\theta$ in (4), which is a product of three functions of t.

To convert (5) into terms involving only $\vec{u}_r$ and $\vec{u}_\theta$, we write $\frac{d\,\vec{u}_r}{dt} = \frac{d\,\vec{u}_r}{d\theta}\,\frac{d\theta}{dt} = \frac{d\theta}{dt}\,\vec{u}_\theta$ (since $\vec{u}_\theta = \frac{d\,\vec{u}_r}{d\theta}$), and $\frac{d\,\vec{u}_\theta}{dt} = \frac{d\,\vec{u}_\theta}{d\theta}\,\frac{d\theta}{dt} = -\frac{d\theta}{dt}\,\vec{u}_r$ (since $\frac{d\,u_\theta}{d\theta} = -u_r$). With these substitutions (5) becomes

$$\vec{a} = \frac{d^2r}{dt^2}\,\vec{u}_r + \frac{dr}{dt}\,\frac{d\theta}{dt}\,\vec{u}_\theta + \frac{dr}{dt}\,\frac{d\theta}{dt}\,\vec{u}_\theta + r\,\frac{d^2\theta}{dt^2}\,\vec{u}_\theta + r\,\frac{d\theta}{dt}\left(-\frac{d\theta}{dt}\,\vec{u}_r\right)$$

or

$$\vec{a} = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\vec{u}_r + \left(r\,\frac{d^2\theta}{dt^2} + 2\,\frac{dr}{dt}\,\frac{d\theta}{dt}\right)\vec{u}_\theta. \tag{6}$$

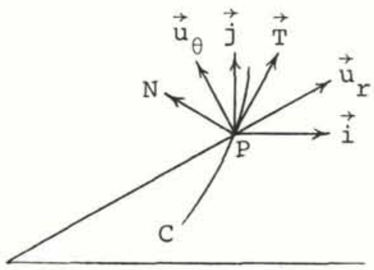
You may feel that the components of $\vec{u}_r$ and $\vec{u}_\theta$ don't seem "intuitive" in (6). Should this be the case, notice, again, the power of "pure" mathematics. That is, (6) is an escapable consequence of our structure - intuitive or not.*

*This is not really a new idea to us. In plane geometry, the axioms were reasonably "self-evident" to us. Yet something like the Pythagorean Theorem which followed inescapably from these axioms was not self-evident. In fact, it seems amazing that the statement of the Pythagorean Theorem is even true!

2.6.1(L) continued

Obviously, one can use equations (3) and (6) without understanding their derivation, but somehow the ability to work with polar equations in a meaningful way is seriously impaired if the derivations are omitted. Even more to the point, the derivations should be understood if we wish to use vector calculus in polar coordinates effectively.

As a final remark, let us summarize by a diagram that $(\vec{u}_r, \vec{u}_\theta)$, $(\vec{T}, \vec{N})$, and $(\vec{i}, \vec{j})$ are entirely different coordinate systems.

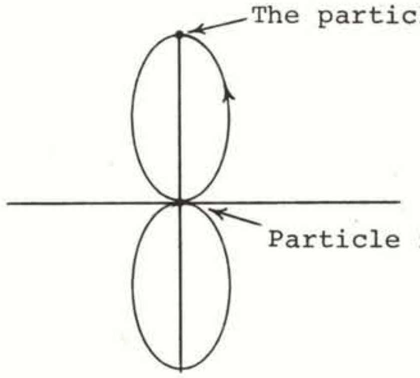


Given an acceleration vector, $\vec{a}$, at $\vec{P}$, the vector is fixed, independent of any coordinate system, but its components depend on whether we express it in terms, say, of $\vec{u}_r$ and $\vec{u}_\theta$ rather than $\vec{T}$ and $\vec{N}$.

2.6.2

a. Just as we saw in our treatment of Cartesian coordinates, there is a difference between the path traced out by a moving particle and how the particle traverses the path. By way of a trivial analogy, a winding road remains the same for each vehicle, but different vehicles traverse it in different ways. In this context, a particle which moves according to the polar equation $r = \sin^2\theta$, where θ is a function of time, traverses the same path regardless of how θ varies with time. That is, it traces the curve $r = \sin^2\theta$. The particular way in which θ depends on t affects the position, velocity, and acceleration of the particle at any time as it moves along the curve. Thus, in this exercise,

2.6.2 continued



b. In the last exercise, we established that

$$\vec{a} = a_r\vec{u}_r + a_\theta\vec{u}_\theta,$$

where

$$a_r = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2,$$

and

$$a_\theta = r\,\frac{d^2\theta}{dt^2} + 2\,\frac{dr}{dt}\,\frac{d\theta}{dt}\,.$$

In this example, $\theta = \frac{1}{2}t$; hence, $\frac{d\theta}{dt} = \frac{1}{2}$ and $\frac{d^2\theta}{dt^2} = 0$. Moreover, since $r = \sin^2\theta$, $\frac{dr}{d\theta} = 2\sin\theta\cos\theta = \sin 2\theta$. Therefore, $\frac{dr}{dt} = \frac{dr}{d\theta}\,\frac{d\theta}{dt} = \frac{1}{2}\sin 2\theta$, whereupon

$$\frac{d^2r}{dt^2} = \frac{d\left(\frac{dr}{dt}\right)}{d\theta}\,\frac{d\theta}{dt} = (\cos 2\theta)\left(\frac{1}{2}\right) = \frac{1}{2}\cos 2\theta.$$

Therefore,

2.6.2 continued

$$a_r = \frac{1}{2}\cos 2\theta \; - \sin^2\theta \; (\tfrac{1}{2})^2 = \frac{1}{2}\cos 2\theta - \frac{1}{4}\sin^2\theta \qquad (1)$$

$$a_\theta = (\sin^2\theta)\;(0) + 2\;(\tfrac{1}{2}\sin 2\theta)\;(\tfrac{1}{2}) = \frac{1}{2}\sin 2\theta \qquad (2)$$

or, since $\theta = \frac{1}{2}t$,

$$a_r = \frac{1}{2}\cos t - \frac{1}{4}\sin^2\frac{t}{2},\; a_\theta = \frac{1}{2}\sin t. \qquad (2')$$

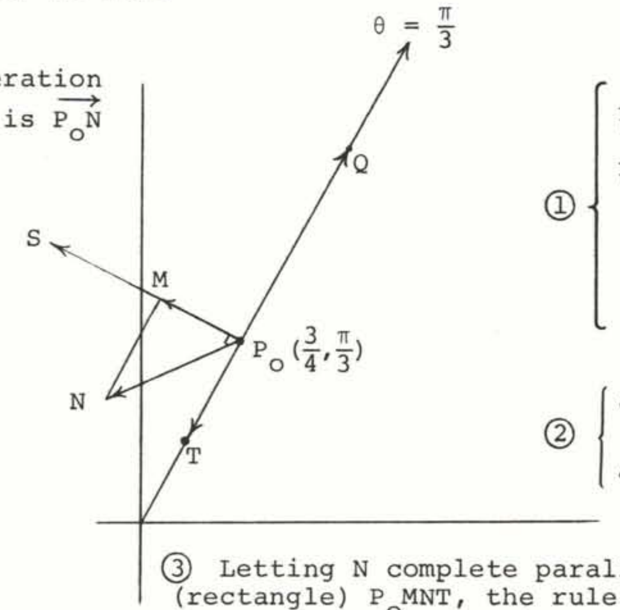
c. When $t = \frac{2\pi}{3}$, $\theta = \frac{\pi}{3}$, so the particle is at $P_o(\sin^2\frac{\pi}{3},\frac{\pi}{3})$, or, $(\frac{3}{4},\frac{\pi}{3})$. Its components of acceleration at this point are obtained from (1) and (2) with $\theta = \frac{\pi}{3}$ (or from (2') with $t = \frac{2\pi}{3}$). Hence,

$$a_r = \frac{1}{2}\cos\frac{2\pi}{3} - \frac{1}{4}\sin^2\frac{\pi}{3} = \frac{1}{2}\;(-\tfrac{1}{2}) - \frac{1}{4}\;(\tfrac{3}{4}) = -\frac{7}{16} \qquad (3)$$

and

$$a_\theta = \frac{1}{2}\sin\frac{2\pi}{3} = \frac{1}{2}\;(\tfrac{1}{2}\sqrt{3}) = \frac{1}{4}\sqrt{3}. \qquad (4)$$

d. Drawn to scale, we have



④ The acceleration vector at P_o is $\overrightarrow{P_oN}$

① $\vec{u}_r = \overrightarrow{P_oQ}$, $\vec{u}_\theta = \overrightarrow{P_oS}$, $|\overrightarrow{P_oT}| = \frac{7}{16}$, $|\overrightarrow{P_oM}| = \frac{1}{4}\sqrt{3}$

② $a_r\vec{u}_r = -\frac{7}{16}\vec{u}_r = \overrightarrow{P_oT}$, $a_\theta\vec{u}_\theta = \frac{1}{4}\sqrt{3}\;\vec{u}_\theta = \overrightarrow{P_oM}$

③ Letting N complete parallelogram (rectangle) P_oMNT, the rule for adding vectors yields $\vec{a} = \overrightarrow{P_oT} + \overrightarrow{P_oM} = \overrightarrow{P_oT} + \overrightarrow{TN}$

2.6.2 continued

e. By the Pythagorean Theorem

$$|\vec{a}|^2 = a_r{}^2 + a_\theta{}^2$$

$$= \left(-\frac{7}{16}\right)^2 + \left(\frac{1}{4}\sqrt{3}\right)^2$$

$$= \frac{49}{256} + \frac{3}{16} = \frac{97}{256}$$

Therefore,

$$|\vec{a}| = \frac{\sqrt{97}}{16} = \frac{9.88}{16}.$$

2.6.3

a. Again,

$$a_r = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 \qquad (1)$$

$$a_\theta = r\,\frac{d^2\theta}{dt^2} + 2\,\frac{dr}{dt}\,\frac{d\theta}{dt}\,. \qquad (2)$$

Now, since $\theta = e^t$, both $\frac{d\theta}{dt}$ and $\frac{d^2\theta}{dt^2} = e^t = \theta$ and $\left(\frac{d\theta}{dt}\right)^2 = (e^t)^2 = \theta^2$. Then, since $r = 1 + \cos\theta$, we have

$$\frac{dr}{d\theta} = -\sin\theta.$$

Therefore,

$$\frac{dr}{dt} = \frac{dr}{d\theta}\,\frac{d\theta}{dt} = -e^t \sin\theta = -\theta\sin\theta$$

2.6.3 continued

$$\frac{d^2r}{dt^2} = \frac{d(-\theta \sin \theta)}{d\theta} \frac{d\theta}{dt}$$

$$= (-\theta \cos \theta - \sin \theta)e^t$$

$$= (-\theta \cos \theta - \sin \theta)\theta$$

$$= -(\theta^2 \cos \theta + \theta \sin \theta).$$

Thus, (1) becomes

$$a_r = -(\theta^2 \cos \theta + \theta \sin \theta) - (1 + \cos \theta)\theta^2$$

$$= -2\,\theta^2 \cos \theta - \theta \sin \theta - \theta^2 \qquad (3)$$

$$a_\theta = (1 + \cos \theta)\theta + 2(-\theta \sin \theta)(\theta)$$

$$= \theta + \theta \cos \theta - 2\,\theta^2 \sin \theta. \qquad (4)$$

b. When $t = \ln \frac{\pi}{2}$, $\theta = e^{\ln\frac{\pi}{2}} = \frac{\pi}{2}$, and since the equation of the curve is $r = 1 + \cos \theta$, $r = 1$ when $\theta = \frac{\pi}{2}$.

Hence, the particle is at $(1,\frac{\pi}{2})$ when $t = \ln \frac{\pi}{2}$. To find its acceleration in $\vec{u}_r$ and $\vec{u}_\theta$ coordinates, we use (3) and (4) with $\theta = \frac{\pi}{2}$. Since $\sin \frac{\pi}{2} = 1$ and $\cos \frac{\pi}{2} = 0$, we obtain

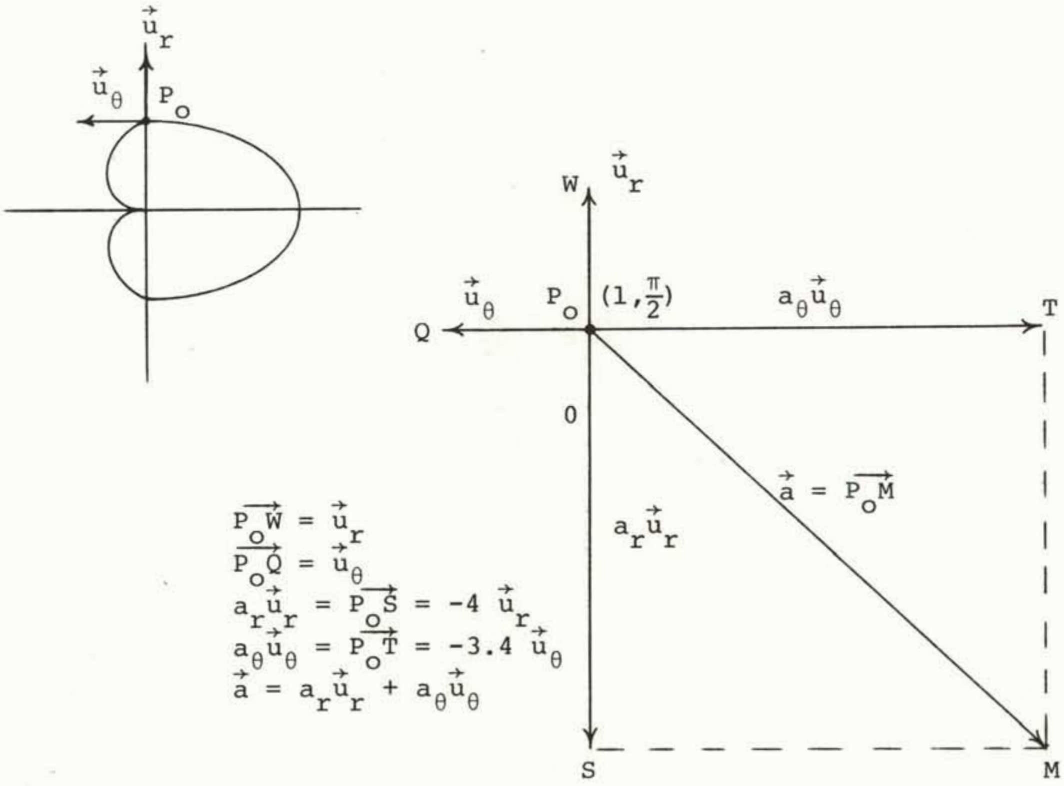
$$a_r = -\frac{\pi}{2} - \frac{\pi^2}{4} = -\frac{\pi}{4}(2 + \pi) \approx -4 \qquad (5)$$

and

$$a_\theta = \frac{\pi}{2} - 2\left(\frac{\pi^2}{4}\right) = \frac{\pi}{2}(1 - \pi) \approx -(3.4). \qquad (6)$$

2.6.3 continued

c. We have:



$\overrightarrow{P_oW} = \vec{u}_r$
$\overrightarrow{P_oQ} = \vec{u}_\theta$
$a_r\vec{u}_r = \overrightarrow{P_oS} = -4\ \vec{u}_r$
$a_\theta\vec{u}_\theta = \overrightarrow{P_oT} = -3.4\ \vec{u}_\theta$
$\vec{a} = a_r\vec{u}_r + a_\theta\vec{u}_\theta$

($\vec{u}_r$ may be thought of as either $\overrightarrow{OP_o}$ or $\overrightarrow{P_oW}$, but $\vec{a}_r$ must = $\overrightarrow{P_oS}$, since $\vec{a}_r$ by definition originates at the point in question on the curre.)

2.6.4(L)

Our aim here is to show how polar coordinates can be used for theoretical gain as well as for the solution of specific motion problems.

In a central force field, by definition, the force is in the direction of $\vec{R}$, assuming, of course, that our central force is at the origin. Now, since $\vec{F} = m\vec{a}$, we have that $\vec{F}$ is a (positive) scalar multiple of $\vec{a}$. Hence, $\vec{F}$ and $\vec{a}$ have the same direction. Therefore, since $\vec{F}$ is in the direction of $\vec{R}$, $\vec{a}$ is also in the direction of $\vec{R}$. (Note that without Newton's Second Law, we cannot conclude that $\vec{F}$ and $\vec{a}$ have the same direction.)

2.6.4(L) continued

This, in turn, makes the component of acceleration in the direction at right angles to $\vec{R}$ equal to 0. That is,

$$a_\theta = r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt} = 0. \tag{1}$$

Since θ never appears other than in a differentiated form, we may simplify the differential equation in (1) by letting, say, $u = \frac{d\theta}{dt}$. In this case, $\frac{du}{dt} = \frac{d}{dt}\left(\frac{d\theta}{dt}\right) = \frac{d^2\theta}{dt^2}$, and (1) becomes

$$r\frac{du}{dt} = -2\frac{dr}{dt}u. \tag{2}$$

Rewriting (2) in differential form, we have

$$r\,du = -2u\,dr$$

and upon separating variables, we obtain

$$\frac{du}{2u} = -\frac{dr}{r} \tag{3}$$

whereupon

$$\frac{1}{2}\ln|u| + c_1 = \ln\frac{1}{|r|}$$

or

$$\ln\left|\frac{1}{r}\right| = \frac{1}{2}\ln|u| + \ln c_2 \text{ (where } \ln c_2 = c_1)$$

$$= \ln|u|^{\frac{1}{2}} + \ln c_2$$

$$= \ln c_2|u|^{\frac{1}{2}}$$

2.6.4(L) continued

Therefore,

$$\left|\frac{1}{r}\right| = c_2|u|^{\frac{1}{2}}. \tag{4}$$

Squaring both sides of (4) yields

$$\frac{1}{r^2} = cu \quad (\text{where } c = c_2^{\,2}). \tag{5}$$

Recalling that $u = \frac{d\theta}{dt}$, (5) yields

$$\frac{1}{c} = r^2\,\frac{d\theta}{dt}. \tag{6}$$

Since c is a constant, (6) implies that $r^2\,\frac{d\theta}{dt}$ is constant. Therefore, $\frac{1}{2}\,r^2\,\frac{d\theta}{dt}$ is constant. Notice, however, that $\frac{1}{2}\,r^2\,\frac{d\theta}{dt} = \frac{dA}{dt}$, so that

$$\frac{dA}{dt} = \text{constant}.$$

In other words, the particle moves so that the rate of change of area it sweeps out is constant.

2.6.5(L)

The main aim of this exercise aside from the computational techniques involved or from the physical importance of the result is to emphasize that when we deal with central force fields the chances are that polar coordinates are going to be the most advantageous. For this reason, the ellipse is written in the polar form

$$r = \frac{c}{1 - e\cos\theta} \quad (0 \leqslant e < 1) \tag{1}$$

rather than in the more familiar Cartesian form. The interested reader can test by direct substitution of Cartesian coordinates that (1) is indeed the equation of an ellipse.

2.6.5(L) continued

The other learning experience of this exercise is to help you see how the knowledge that we are in a central force field allows us to side-step some rather tedious computations. In particular, based on the results of the previous exercise, in a central force field, we can always replace $r^2 \frac{d\theta}{dt}$ by a constant.

In any event, we have (since the force is directed toward the origin) that the acceleration is also directed toward the origin (since $\vec{F} = m\vec{a}$). Therefore, $a_\theta = 0$ and we have

$$\vec{a} = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\vec{u}_r,$$

so that

$$|\vec{a}| = \left|\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right|. \qquad (2)$$

From (1)

$$\frac{dr}{dt} = \frac{-c(e \sin \theta)}{(1 - e \cos \theta)^2}\frac{d\theta}{dt}. \qquad (3)$$

Since $r = \frac{c}{1 - e \cos \theta}$ or $r^2 = \frac{c^2}{(1 - e \cos \theta)^2}$, we rewrite (3) to take advantage of this substitution. Therefore,

$$\frac{dr}{dt} = -\frac{1}{c}\left[\frac{c^2}{(1 - e \cos \theta)^2}\right] e \sin \theta \frac{d\theta}{dt}$$

$$= -\frac{1}{c} r^2 e \sin \theta \frac{d\theta}{dt}$$

$$= -\frac{e}{c}\left(r^2 \frac{d\theta}{dt}\right) \sin \theta. \qquad (4)$$

In the last exercise, we saw that $r^2 \frac{d\theta}{dt} = k$ in any central force field.

Thus, (4) takes on the simpler form

2.6.5(L) continued

$$\frac{dr}{dt} = -\frac{ek}{c} \sin \theta. \tag{5}$$

From (5),

$$\frac{d^2r}{dt^2} = -\frac{ek}{c} \cos \theta \frac{d\theta}{dt}$$

whence (2) becomes

$$|\vec{a}| = \left| -\frac{ek}{c} \cos \theta \frac{d\theta}{dt} - r\left(\frac{d\theta}{dt}\right)^2 \right|. \tag{6}$$

Since $r^2 \frac{d\theta}{dt} = k$, we may replace $\frac{d\theta}{dt}$ in (6) by $\frac{k}{r^2}$ to obtain

$$|\vec{a}| = \left| -\frac{ek}{c} \cos \theta \frac{k}{r^2} - \frac{r\,k^2}{r^4} \right|$$

$$= \left| -\frac{ek^2}{cr^2} \cos \theta - \frac{k^2}{r^3} \right|.$$

Factoring out $\frac{1}{r^2}$ (there's no harm in keeping one eye on the desired answer) we obtain

$$|\vec{a}| = \frac{1}{r^2} \left| -\frac{ek^2}{c} \cos \theta - \frac{k^2}{r} \right|$$

$$= \frac{k^2}{r^2} \left| -\frac{e}{c} \cos \theta - \frac{1}{r} \right|$$

and since $|n| = |-n|$, we obtain

$$|\vec{a}| = \frac{k^2}{r^2} \left| \frac{e}{c} \cos \theta + \frac{1}{r} \right|. \tag{7}$$

2.6.5(L) continued

From (1),

$$\frac{1}{r} = \frac{1 - e\cos\theta}{c} = \frac{1}{c} - \frac{e}{c}\cos\theta,$$

so that (7) becomes

$$|\vec{a}| = \frac{k^2}{r^2}\left|\frac{1}{c}\right| = \left|\frac{k^2}{c}\right|\frac{1}{r^2}$$

and since $\left|\frac{k^2}{c}\right|$ is a constant, the desired result holds.

2.6.6(L)

In all the exercises of this section, we have assumed that our particle was moving in the xy-plane, even though we used polar coordinates. A more general problem is that of a particle moving in <u>space</u> in a central force field. The interesting thing is that in this case, too, the path of the particle is a <u>plane</u> curve.

This fact can be established mathematically, assuming no physical knowledge other than $\vec{F} = m\vec{a} = m\frac{d^2\vec{R}}{dt^2}$. This is precisely the aim of this exercise.

We look at

$$\vec{R} \times \frac{d\vec{R}}{dt}$$

and differentiate this with respect to t to obtain

$$\frac{d\left(\vec{R} \times \frac{d\vec{R}}{dt}\right)}{dt} = \left(\vec{R} \times \frac{d^2\vec{R}}{dt^2}\right) + \left(\frac{d\vec{R}}{dt} \times \frac{d\vec{R}}{dt}\right). \qquad (1)$$

Now in any event $\frac{d\vec{R}}{dt} \times \frac{d\vec{R}}{dt} = \vec{0}$ since any vector crossed with itself has this property. Hence, (1) may be rewritten as

2.6.6(L) continued

$$\frac{d\left(\vec{R} \times \frac{d\vec{R}}{dt}\right)}{dt} = \vec{R} \times \frac{d^2\vec{R}}{dt^2} \tag{2}$$

Equation (2) is valid for any twice-differentiable vector function of t.

The point is that since we are in a central force field, by definition

$$\vec{F} = k\vec{R} \tag{3}$$

and by Newton's Law

$$\vec{F} = m\frac{d^2\vec{R}}{dt^2} \tag{4}$$

Comparing (3) and (4) we see that $\frac{d^2\vec{R}}{dt^2}$ and $\vec{R}$ have the same direction (i.e., $\frac{d^2\vec{R}}{dt^2} = \frac{k}{m}\vec{R}$). Since vectors which have the same direction yield $\vec{0}$ as their cross product, we have that in a central force field $\vec{R} \times \frac{d^2\vec{R}}{dt^2} = 0$.

Putting this result into (2) yields

$$\frac{d\left(\vec{R} \times \frac{d\vec{R}}{dt}\right)}{dt} = \vec{0}. \tag{5}$$

Thus, from (5), we have that $\vec{R} \times \frac{d\vec{R}}{dt} = \vec{c}$ where $\vec{c}$ is a constant vector.

But what is the physical meaning of $\vec{R} \times \frac{d\vec{R}}{dt}$? $\vec{R}$ is the position vector to the particle while $\frac{d\vec{R}}{dt}$ is its velocity. Hence, $\vec{c}$ is perpendicular to the plane determined by $\vec{R}$ and $\vec{v}$. Since $\vec{c}$ is a constant, this plane can never change direction. Hence, $\vec{R}$ and $\vec{v}$ are always in the same plane when we have a central force field. We may then let this plane be denoted as the xy-plane without loss of generality.

2.6.6(L) continued

There is, of course, more information that can be deduced from $\vec{R} \times \vec{v} = \vec{c}$, but we leave such deductions to the interested reader. Our purpose was more to continue our emphasis on the value of vector calculus in the study of motion.

At least, by now, everyone should be convinced that vectors supply us with a powerful analytic tool - too powerful to be relegated to the role of merely being broken up into x, y, and z components!

Quiz

1. $\vec{F}'(t) = 5t^4\,\vec{i} - 2\sin t\,\vec{j} \rightarrow$

$\vec{F}(t) = t^5\,\vec{i} + 2\cos t\,\vec{j} + \vec{c}$ (1)

[equivalently, $\vec{F}(t) = (t^5+c_1)\vec{i} + (2\cos t + c_2)\vec{j}$, and $\vec{c} = c_1\vec{i} + c_2\vec{j}$]*

Therefore,

$\vec{F}(0) = 0\vec{i} + 2\cos 0\vec{j} + \vec{c} = 2\vec{j} + \vec{c}$ (2)

But we are also given that

$\vec{F}(0) = 2\vec{i} + 3\vec{j}$ (3)

Comparing (2) and (3) yields

$2\vec{j} + \vec{c} = 2\vec{i} + 3\vec{j}$

or

$\vec{c} = 2\vec{i} + \vec{j}$ (4)

Substituting (4) into (1), we obtain

$\vec{F}(t) = t^5\,\vec{i} + 2\cos t\,\vec{j} + 2\vec{i} + \vec{j}$

$= (t^5+2)\vec{i} + (2\cos t + 1)\vec{j}$

2. (a) When the equation of the curve has the form $y = f(x)$, the curvature, κ, is given most conveniently by

*The key point is that if $\vec{F}(t) = f_1(t)\vec{u}_1 + f_2(t)\vec{u}_2$ and $\vec{u}_1$ and $\vec{u}_2$ are <u>any constant</u> vectors (not just constant in magnitude) then $\vec{F}'(t) = f_1'(t)\vec{u}_1 + f_2'(t)\vec{u}_2$.

2. continued

$$\kappa = \left| \frac{\frac{d^2y}{dx^2}}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}} \right| \quad (1)$$

In this exercise $y = e^x$. Therefore $\frac{dy}{dx} = \frac{d^2y}{dx^2} = e^x$ and $\left(\frac{dy}{dx}\right)^2 = e^{2x}$.

Putting these results into (1) yields

$$\kappa = \frac{e^x}{[1+e^{2x}]^{3/2}} \quad (2)$$

(b) If $x = \ln\sqrt{3}$, $2x = 2\ \ln\sqrt{3} = \ln(\sqrt{3})^2 = \ln 3$.

Hence, $e^x = e^{\ln\sqrt{3}} = \sqrt{3}$, and $e^{2x} = e^{\ln 3} = 3$.
Therefore equation (2) becomes

$$\kappa = \frac{\sqrt{3}}{(1+3)^{3/2}}$$

or

$$\kappa = \frac{\sqrt{3}}{8} \quad (3)$$

(c) To find the maximum (or minimum) value of κ we compute $\frac{d\kappa}{dx}$ from equation (2) and then find the values of x for which $\frac{d\kappa}{dx} = 0$. We obtain

2. continued

$$\frac{d\kappa}{dx} = \frac{(1+e^{2x})^{3/2}\ e^x - e^x\left[\frac{3}{2}(1+e^{2x})^{1/2}\,2e^{2x}\right]}{(1+e^{2x})^3}$$

$$= \frac{(1+e^{2x})^{1/2}\ e^x\left[(1+e^{2x}) - 3e^{2x}\right]}{(1+e^{2x})^3}$$

$$= \frac{e^x}{(1+e^{2x})^{5/2}}\ [1-2e^{2x}] \qquad (4)$$

Since neither e^x nor $(1+e^{2x})^{5/2}$ can be zero (i.e., the exponential is positive), we see from equation (4) that

$$\frac{d\kappa}{dx} = 0 \leftrightarrow 1-2e^{2x} = 0$$

$$\leftrightarrow e^{2x} = \frac{1}{2}$$

$$\leftrightarrow 2x = \ln\frac{1}{2} \text{ or } x = \frac{1}{2}\ \ln\frac{1}{2}$$

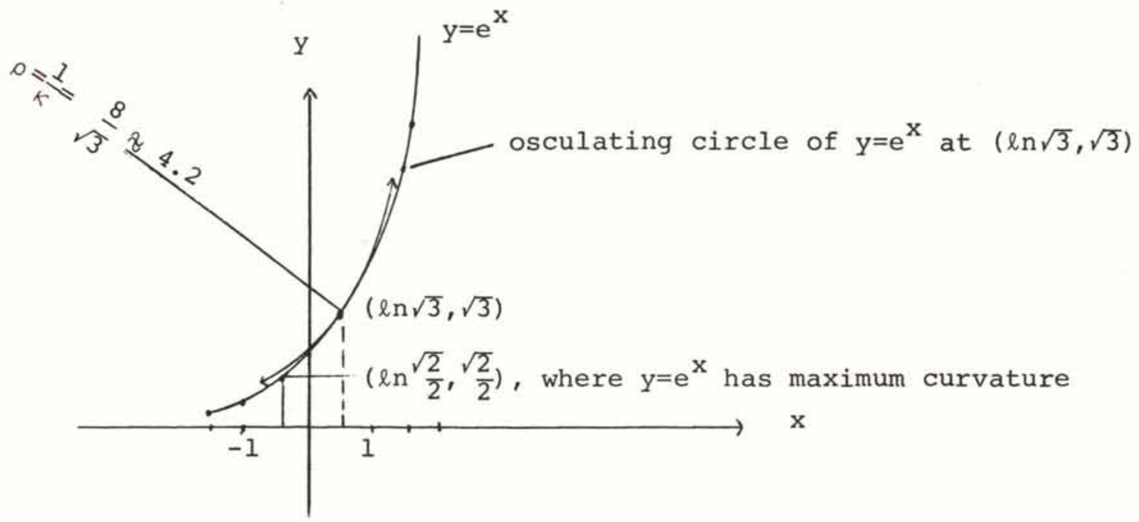
$$\leftrightarrow x = \ln\left(\frac{1}{2}\right)^{1/2} = \ln\sqrt{\frac{1}{2}} = \ln\frac{1}{2}\sqrt{2}$$

Hence the maximum* curvature of $y = e^x$ occurs at the point $\left(\ln\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

---

*Of course, one could compute $\frac{d^2\kappa}{dx^2}$ to make sure that we have found a maximum rather than a minimum curvature. However, a glance at the curve $y = e^x$ shows that the curve "flattens out" very rapidly as we move away from $\left(\ln\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ in either direction.

2. continued



(Pictorial Summary)

3. (a) $\vec{R} = \ln(t^2+1)\vec{i} + (t-2\tan^{-1}t)\vec{j}$

Therefore

$$\vec{v} = \frac{d\vec{R}}{dt} = \frac{2t}{t^2+1}\vec{i} + \left(1 - \frac{2}{1+t^2}\right)\vec{j}$$

$$= \frac{2t}{t^2+1}\vec{i} + \left(\frac{t^2-1}{t^2+1}\right)\vec{j} \qquad (1)$$

Therefore

3. continued

$$|\vec{v}| = \sqrt{\left(\frac{2t}{t^2+1}\right)^2 + \left(\frac{t^2-1}{t^2+1}\right)^2}$$

$$= \sqrt{\frac{4t^2+(t^4-2t^2+1)}{(t^2+1)^2}}$$

$$= \sqrt{\frac{(t^2+1)^2}{(t^2+1)^2}}$$

$$= 1 \text{ ft/sec}$$

Therefore the particle moves at the constant speed of 1 ft/sec.

b.

$$\frac{d}{dt}\left(\frac{2t}{t^2+1}\right) = \frac{(t^2+1)2-2t(2t)}{(t^2+1)^2} = \frac{2-2t^2}{(t^2+1)^2}$$

$$\frac{d}{dt}\left(\frac{t^2-1}{t^2+1}\right) = \frac{(t^2+1)2t-(t^2-1)2t}{(t^2+1)^2} = \frac{4t}{(t^2+1)^2}$$

Hence from equation (1) we obtain

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{2-2t^2}{(t^2+1)^2}\,\vec{i} + \frac{4t}{(t^2+1)^2}\,\vec{j} \qquad (2)$$

3. continued

c. $$|\vec{a}| = \sqrt{\left[\frac{2-2t^2}{(t^2+1)^2}\right]^2 + \left[\frac{4t}{(t^2+1)^2}\right]^2}$$

$$= \sqrt{\frac{4-8t^2+4t^4+16t^2}{(t^2+1)^4}}$$

$$= \sqrt{\frac{(2+2t^2)^2}{(t^2+1)^4}}$$

$$= \frac{2+2t^2}{(t^2+1)^2} = \frac{2(t^2+1)}{(t^2+1)^2}$$

$$= \frac{2}{(t^2+1)}$$ [and this is maximum when (t^2+1) is minimum.] (3)

$(t^2+1) \geqslant 1$ and equals $1 \leftrightarrow t=0$

Therefore $|\vec{a}|$ is maximum when $t=0$, and this maximum value is 2 ft/sec^2.

d. We know that with respect to tangential and normal components

$$\vec{a} = \frac{d^2s}{dt^2}\,\vec{T} + \kappa\left(\frac{ds}{dt}\right)^2\vec{N} \qquad (4)$$

Since $|\vec{v}| = \frac{ds}{dt}$, we have from part (a) that $\frac{ds}{dt} = 1$.

3. continued

Hence $\frac{d^2s}{dt^2} = 0$ and $\left(\frac{ds}{dt}\right)^2 = 1$*.

Putting these results into (4) yields

$$\vec{a} = \kappa\,\vec{N} \tag{5}$$

Therefore, if $\vec{a} = a_T\vec{T} + a_N\vec{N}$, $a_T = 0$ and $a_N = \kappa$.

e. Since $|\vec{v}(t)| = 1$ = constant, we have that $\vec{v} \cdot \frac{d\vec{v}}{dt} = 0$.

Therefore $\vec{a}$ $(= \frac{d\vec{v}}{dt})$ is either $\vec{0}$ or else it is perpendicular to $\vec{v}$. From equation (3), $\vec{a} \neq \vec{0}$ for all t; hence

$$\vec{a} \perp \vec{v} \quad .$$

Since $\vec{v}$ is in the direction of $\vec{T}$, $\vec{a} \perp \vec{T}$; and (since $\vec{a}$ is in the xy-plane) therefore $\vec{a}$ is parallel to $\vec{N}$. In other words, $\vec{a}$ is a scalar multiple of $\vec{N}$, which is consistent with equation (5).

f. The acceleration, $|\vec{a}|$, of the particle at any time t is independent of the coordinate system being used (although, of course, the various components of $\vec{a}$ depend on the coordinate system).

From equation (5) we have that $|\vec{a}| = |\kappa\,\vec{N}| = |\kappa|$, and since by convention κ is always non-negative, it follows that

$$\kappa = |\vec{a}| \tag{6}$$

On the other hand, we know from equation (3) that

$$|\vec{a}| = \frac{2}{(t^2+1)} \quad .$$

---

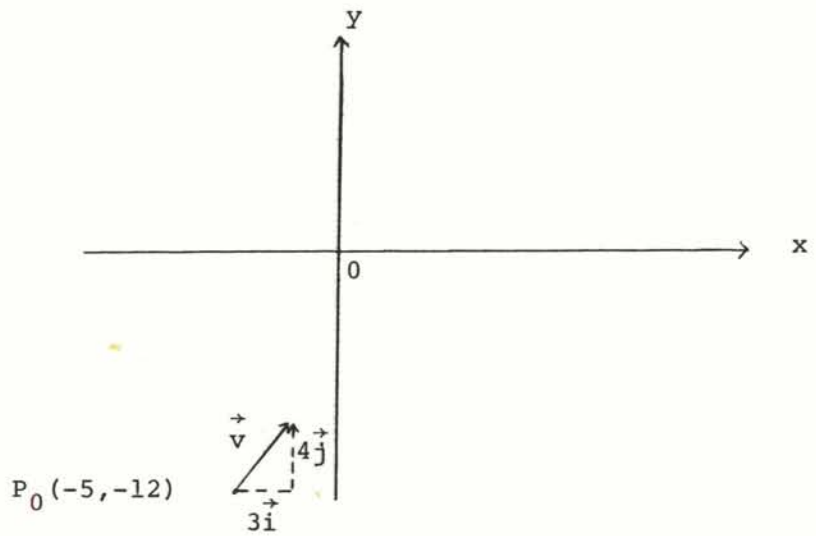
* $|\vec{v}| = \frac{ds}{dt}$ implies that the sense of the path has s increasing with t. In this problem, even if this convention weren't obeyed, the same results would follow since then $\frac{ds}{dt} = -1$ whereupon $\frac{d^2s}{dt^2} = 0$ and $\left(\frac{ds}{dt}\right)^2 = 1$.

3. continued

Comparing this with equation (6) it follows that

$$\kappa = \frac{2}{(t^2+1)}$$

4. a. Pictorially, what we have is



Now since $\vec{R} = \vec{0P_0} = -5\vec{i} - 12\vec{j}$, $|\vec{R}| = \sqrt{(-5^2)+(-12)^2} = 13$

Therefore,

$$\frac{\vec{R}}{|\vec{R}|} = -\frac{5}{13}\vec{i} - \frac{12}{13}\vec{j} \qquad (1)$$

4. continued

The key point is that since $\vec{u}_r$ is defined by $\vec{R} = r\,\vec{u}_r$ we see that $\vec{u}_r$ has the opposite sense of $\vec{R}$ (hence, also of $\frac{\vec{R}}{|\vec{R}|}$) when r is negative, which is the case in this problem.

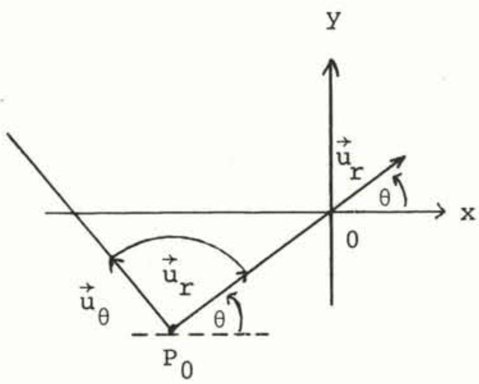
In other words,

$$\vec{u}_r = -\frac{\vec{R}}{|\vec{R}|} \quad ,$$

so that by equation (1),

$$\vec{u}_r = \frac{5}{13}\vec{i} + \frac{12}{13}\vec{j} \qquad (2)$$

$\vec{u}_\theta$ is obtained by rotating u_r, 90° in the counterclockwise direction (i.e., +90°). Again pictorially,



($\vec{u}_r$ may be viewed as being in either of the two positions shown)

There are several ways open to us for computing $\vec{u}_\theta$. One way is to recall that $\vec{u}_r = \cos\theta\vec{i} + \sin\theta\vec{j}$ and $\vec{u}_\theta = -\sin\theta\vec{i} + \cos\theta\vec{j}$ [i.e., $\vec{u}_\theta = \frac{d\vec{u}_r}{d\theta}$ or $\vec{u}_\theta = \cos(\theta+90°)\vec{i} + \sin(\theta+90°)\vec{j}$].

Since we also know that $\vec{u}_r = \frac{5}{13}\vec{i} + \frac{12}{13}\vec{j}$, the fact that $\vec{u}_r = \cos\theta\vec{i} + \sin\theta\vec{j}$ implies that

$$\cos\theta = \frac{5}{13} \text{ and } \sin\theta = \frac{12}{13} \quad .$$

4. continued

Hence, $-\sin\theta = -\frac{12}{13}$, whereupon $\vec{u}_\theta = -\sin\theta\vec{i} + \cos\theta\vec{j}$ implies

$$\vec{u}_\theta = -\frac{12}{13}\vec{i} + \frac{5}{13}\vec{j} \tag{3}$$

(Another way to find $\vec{u}_\theta$ would be to utilize the fact that $\vec{u}_r\cdot\vec{u}_\theta=0$. Then let $\vec{u}_\theta = x\vec{i} + y\vec{j}$, $\vec{u}_r = \frac{5}{13}\vec{i} + \frac{12}{13}\vec{j}$. Therefore $\vec{u}_r\cdot\vec{u}_\theta=0 \rightarrow \frac{5x}{13} + \frac{12y}{13} = 0$, or $y = -\frac{5x}{12}$. Hence $\vec{u}_\theta$ has the form

$$\vec{u}_\theta = x\vec{i} - \frac{5x}{12}\vec{j} = \frac{x}{12}(12\vec{i} - 5\vec{j})$$

Therefore, $|\vec{u}_\theta| = \frac{x}{12}|12\vec{i} - 5\vec{j}| = \frac{13x}{12}$, and since $|\vec{u}_\theta| = 1$, $x = \pm\frac{12}{13}$. The value $x = +\frac{12}{13}$ makes $\vec{u}_\theta$ a <u>clockwise</u> rotation of $\vec{u}_r$; hence, $x = -\frac{12}{13}$, whereupon $y = -\frac{5}{12}(-\frac{12}{13}) = \frac{5}{13}$, and as a result

$$\vec{u}_\theta = x\vec{i} + y\vec{j} = -\frac{12}{13}\vec{i} + \frac{5}{13}\vec{j}).$$

b. The most straightforward approach here is to express $\vec{i}$ and $\vec{j}$ in terms of $\vec{u}_r$ and $\vec{u}_\theta$ since $\vec{v}$ is already known in terms of $\vec{i}$ and $\vec{j}$. To this end we have

$$\left.\begin{aligned}\vec{u}_r &= \frac{5}{13}\vec{i} + \frac{12}{13}\vec{j}\\ \vec{u}_\theta &= -\frac{12}{13}\vec{i} + \frac{5}{13}\vec{j}\end{aligned}\right\} \text{ or } \quad \begin{aligned}13\vec{u}_r &= 5\vec{i} + 12\vec{j}\\ 13\vec{u}_\theta &= -12\vec{i} + 5\vec{j}\end{aligned}$$

Therefore

$$\left.\begin{aligned}12(13\,\vec{u}_r) &= 12(5\vec{i} + 12\vec{j})\\ 5(13\,\vec{u}_\theta) &= 5(-12\vec{i} + 5\vec{j})\end{aligned}\right\} \text{ Therefore } \quad 156\,\vec{u}_r + 65\,\vec{u}_\theta = 169\vec{j}$$

or

$$\vec{j} = \frac{12}{13}\vec{u}_r + \frac{5}{13}\vec{u}_\theta \tag{4}$$

4. continued

Similarly

$$\left.\begin{aligned} 5(13\,\vec{u}_r) &= 5(5\vec{i} + 12\vec{j}) \\ -12(13\,\vec{u}_\theta) &= -12(-12\vec{i} + 5\vec{j}) \end{aligned}\right\} \text{therefore } 65\vec{u}_r - 156\,\vec{u}_\theta = 169\vec{i}$$

therefore

$$\vec{i} = \frac{5}{13}\vec{u}_r - \frac{12}{13}\vec{u}_\theta \qquad (5)$$

Utilizing equations (4) and (5), we have

$$\vec{v} = 3\vec{i} + 4\vec{j}$$

$$= 3\left(\frac{5}{13}\vec{u}_r - \frac{12}{13}\vec{u}_\theta\right) + 4\left(\frac{12}{13}\vec{u}_r + \frac{5}{13}\vec{u}_\theta\right)$$

$$= \frac{63}{13}\vec{u}_r - \frac{16}{13}\vec{u}_\theta \qquad (6)$$

c. Since $\vec{R} = r\,\vec{u}_r$ and $\frac{d\vec{u}_r}{d\theta} = \vec{u}_\theta$ we know that

$$\vec{v} = \frac{d\vec{R}}{dt} = \frac{dr}{dt}\vec{u}_r + r\frac{d\theta}{dt}\vec{u}_\theta.$$

From equation (6) it follows that $\frac{dr}{dt} = \frac{63}{13}$ and $r\frac{d\theta}{dt} = -\frac{16}{13}$. Since at $t = 0$, $|\vec{R}| = 13$, and $r = -|\vec{R}|$, $r = -13$. Hence

$$r\frac{d\theta}{dt} = -\frac{16}{13} \rightarrow -13\frac{d\theta}{dt} = -\frac{16}{13}$$

therefore

$$\frac{d\theta}{dt} = \frac{16}{169} \qquad (7)$$

4. continued

d. Here we recall the formula for curvature in terms of velocity and acceleration. Namely

$$\kappa = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3} \tag{8}$$

Now

$$\left.\begin{aligned}\vec{v} &= 3\vec{i} + 4\vec{j}\\ \vec{a} &= \vec{i} + 6\vec{j}\end{aligned}\right\} \rightarrow$$

1. $|\vec{v}| = 5$, therefore $|\vec{v}|^3 = 125$
2. $\vec{v} \times \vec{a} = 18\vec{k} - 4\vec{k}$, therefore $|\vec{v} \times \vec{a}| = 14$.

Putting these results into (8) yields

$$\kappa = \frac{14}{125}$$

5. $\vec{R} = r\vec{u}_r \rightarrow$

$$\vec{v} = \frac{dr}{dt}\vec{u}_r + r\frac{d\theta}{dt}\vec{u}_\theta \tag{1}$$

Differentiating $\vec{v}$ with respect to t (recalling that $\frac{d\vec{u}_r}{dt} = \frac{d\vec{u}_r}{d_\theta}\frac{d\theta}{dt} = \frac{d\theta}{dt}\vec{u}_\theta$ and $\frac{d\vec{u}_\theta}{dt} = \frac{d\vec{u}_\theta}{d\theta}\frac{d\theta}{dt} = -\frac{d\theta}{dt}\vec{u}_r$), we obtain

$$\vec{a} = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\vec{u}_r + \left[r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right]\vec{u}_\theta \tag{2}$$

6. a. Since $x = r\cos\theta$ and $y = r\sin\theta$, we multiply both sides of $r = \sin\theta + \cos\theta$ by r to obtain

$$r^2 = r\sin\theta + r\cos\theta^* \qquad (1)$$

or

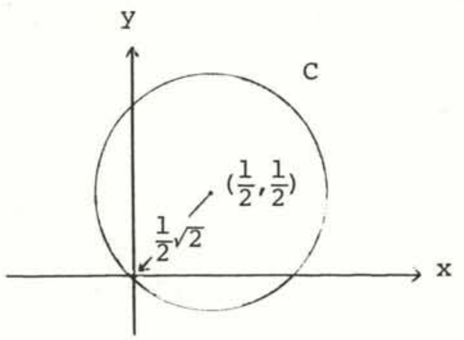
$$x^2 + y^2 = y + x$$

therefore $(x^2-x) + (y^2-y) = 0$

therefore $(x^2-x+\frac{1}{4}) + (y^2-y+\frac{1}{4}) = \frac{1}{2}$

therefore $(x-\frac{1}{2})^2 + (y-\frac{1}{2})^2 = \frac{1}{2}$

therefore C is a circle centered at $(\frac{1}{2}, \frac{1}{2})$ with radius $\sqrt{\frac{1}{2}} = \frac{1}{2}\sqrt{2}$



*The only subtlety here is that multiplying by r guarantees that the origin (r=0) satisfies (1). This causes no problem (i.e., extraneous point) if the origin also satisfies the original equation. In our problem $r = \sin\theta + \cos\theta \rightarrow r = 0$ wherever $\tan\theta = -1$ (i.e., $\sin\theta + \cos\theta = 0$). Therefore $(0, \frac{3\pi}{4})$ [the origin] is on the given curve.

5. continued

In this problem $r = \sin 2\theta$ and $\theta = 3t^2$ therefore

$\frac{d\theta}{dt} = 6t, \frac{d^2\theta}{dt^2} = 6$,

Also $r = \sin 2\theta = \sin 6t^2$ therefore

$$\frac{dr}{dt} = 12t \cos 6t^2$$

$$\frac{d^2r}{dt^2} = 12 \cos 6t^2 + 12t[-12t \sin 6t^2]$$

$$= 12 \cos 6t^2 - 144t^2 \sin 6t^2$$

Putting these results into (2) yields

$$\vec{a} = [12 \cos 6t^2 - 144t^2 \sin 6t^2 - 36t^2 \sin 6t^2]\vec{u}_r$$

$$+ [6 \sin 6t^2 + 2(12t \cos 6t^2)6t]\vec{u}_\theta$$

$$= (12 \cos 6t^2 - 180t^2 \sin 6t^2)\vec{u}_r + (6 \sin 6t^2 + 144t^2 \cos 6t^2)\vec{u}_\theta \quad (3)$$

When $t = \frac{\sqrt{3\pi}}{6}$, $t^2 = \frac{3\pi}{36} = \frac{\pi}{12}$ therefore $6t^2 = \frac{\pi}{2}$.

Equation (3) then becomes

$$\vec{a} = (12 \cos \tfrac{\pi}{2} - 180 [\tfrac{\pi}{12}] \sin \tfrac{\pi}{2})\vec{u}_r + (6\sin\tfrac{\pi}{2} + 144[\tfrac{\pi}{12}]\cos\tfrac{\pi}{2})\vec{u}_\theta$$

$$= -15\pi\ \vec{u}_r + 6\ \vec{u}_\theta$$

Therefore

$a_r = -15\pi$ and $a_\theta = 6$

6. continued

b. $x^2 + y^2 = r^2$

$y = r \sin \theta$

therefore $(x^2+y^2)^3 = y^4 \rightarrow$

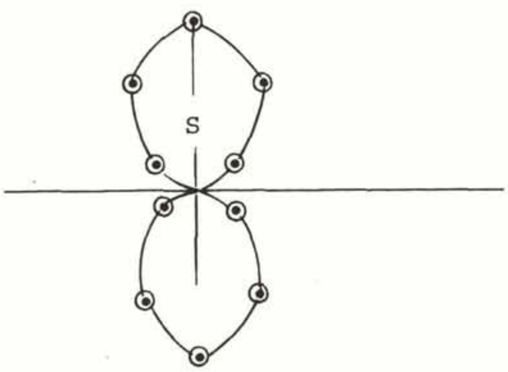
$r^6 = r^4 \sin^4\theta$

therefore $\underline{r^2 = \sin^4\theta}$ (2)

Equation (2) is equivalent to the pair of equations

$r = \sin^2\theta$ and $r = -\sin^2\theta$

and these two equations describe the same curve, namely:



$r = \sin^2\theta$
traces top part first

$r = -\sin^2\theta$
traces bottom part first

Letting S be the region enclosed by $r = \sin^2\theta$, $0 \leqslant \theta \leqslant \pi$ we have that

$$A_S = \frac{1}{2} \int_0^{\pi} r^2 d\theta = \frac{1}{2} \int_0^{\pi} \sin^4\theta \, d\theta \quad ,$$

and since the area enclosed by the curve is twice the area of S (since the top and bottom parts are congruent), we have that the desired area is

6. continued

$$\int_0^{\pi} \sin^4\theta \, d\theta =$$

$$\int_0^{\pi} (\sin^2\theta)^2 \, d\theta =$$

$$\int_0^{\pi} \left(\frac{1-\cos 2\theta}{2}\right)^2 d\theta =$$

$$\frac{1}{4}\int_0^{\pi} (1-2\cos 2\theta+\cos^2 2\theta)\,d\theta =$$

$$\frac{1}{4}\int_0^{\pi} \left(1-2\cos 2\theta+\left[\frac{1+\cos 4\theta}{2}\right]\right) d\theta =$$

$$\frac{1}{8}\int_0^{\pi} (3-4\cos 2\theta+\cos 4\theta)\,d\theta =$$

$$\frac{1}{8}\left[3\theta-2\sin 2\theta+\frac{1}{4}\sin 4\theta\right|_{\theta=0}^{\pi} =$$

$$\frac{3\pi}{8}$$

7. $r = \sin^2\theta$

therefore $\frac{dr}{d\theta} = 2\sin\theta\cos\theta$

7. continued

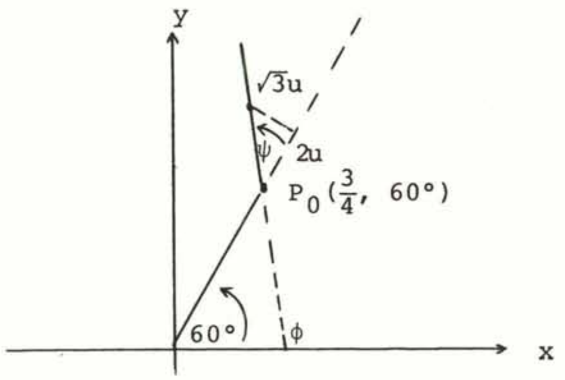
therefore

$$\tan\psi = \frac{r}{\frac{dr}{d\theta}} = \frac{\sin^2\theta}{2\sin\theta\cos\theta} = \frac{\sin\theta}{2\cos\theta} = \frac{1}{2}\tan\theta$$

(provided $\sin\theta \neq 0$, otherwise we cannot divide by it).

In particular when $\theta = 60°$, $\tan\psi = \frac{1}{2}\tan 60° = \frac{1}{2}\sqrt{3}$.

Therefore $\psi = \tan^{-1}\frac{\sqrt{3}}{2} \approx 41°$, and this is the angle that the tangent line to C at $(\frac{3}{4}, 60°)$ makes with R. Pictorially



However, the slope of the line is, by definition, tan ϕ and $\phi = \psi + 60°$.

Therefore

$$\tan\phi = \frac{\tan\psi + \tan 60°}{1 - \tan\psi\tan 60°} = \frac{\frac{1}{2}\sqrt{3} + \sqrt{3}}{1 - \frac{1}{2}\sqrt{3}\sqrt{3}} = \frac{\frac{3}{2}\sqrt{3}}{1 - \frac{3}{2}} = -3\sqrt{3}$$

8. 1. Clearly the origin belongs to each curve. For example it belongs to C_1 in the form $(0,\pi)$, and to C_2 in the forms $(0,0)$, $(0,2\pi)$, and $(0,4\pi)$.

2. To find the other solutions, we next look at simultaneous solutions and this leads to

8. continued

$$1 + \cos\theta = \sin\frac{\theta}{2} \qquad (1)$$

Recalling that $\cos\theta = \cos^2(\frac{\theta}{2}) - \sin^2(\frac{\theta}{2})$ $(=1-2\sin^2\frac{\theta}{2}$ or $2\cos^2\frac{\theta}{2}-1)$, equation (1) yields

$$2\sin^2\frac{\theta}{2} + \sin\frac{\theta}{2} - 2 = 0 \qquad (2)$$

or

$$\sin\frac{\theta}{2} = -\frac{1 \pm \sqrt{17}}{4} \qquad (3)$$

Since $|\sin\frac{\theta}{2}| \leqslant 1$ we may discard the negative sign in $\pm\sqrt{17}$ in (1) to conclude

$$\sin\frac{\theta}{2} = \frac{\sqrt{17} - 1}{4} \qquad (4)$$

therefore $\sin\frac{\theta}{2} \approx 0.78$

therefore $\frac{\theta}{2} \approx 52°$ or $\frac{\theta}{2} \approx 128°$

therefore $\theta \approx 104°, 256°$

Hence (0.78, 104°) and (0.78, 256°) are simultaneous points of intersection.

3. Since (r,θ) and $(r,\theta+2\pi n)$ name the same point, the "simultaneous-technique" in (2) should be broadened to include

$$1 + \cos\theta = \sin\left(\frac{\theta+2\pi n}{2}\right) = \sin\left(\frac{\theta}{2} + \pi n\right)$$

or

$$1 + \cos\theta = \cos\pi n \sin\frac{\theta}{2} \qquad (5)$$

8. continued

When n is even, equation (5) is the same as equation (1) (since $\cos \pi n = 1$ for n even), but when n is odd, equation (5) becomes

$$1 + \cos \theta = - \sin \frac{\theta}{2} \quad (6)$$

Letting $\cos \theta = 1 - 2 \sin^2 \frac{\theta}{2}$ in (6), we obtain

$$2 - 2 \sin^2 \frac{\theta}{2} = - \sin \frac{\theta}{2}$$

or

$$2 \sin^2 \frac{\theta}{2} - \sin \frac{\theta}{2} - 2 = 0 \quad (7)$$

Solving (7) yields

$$\sin \frac{\theta}{2} = \frac{1 - \sqrt{17}}{4} \approx - 0.78 \quad (8)$$

Equation (8) yields the same points (but by different names) as those obtained from equation (4).

4. The final possibility involves the fact that

$$(r,\theta) = (-r,\theta+\pi+2\pi n).$$

Hence we look at

$$- \sin \frac{\theta}{2} = 1 + \cos(\theta+\pi+2\pi n)$$

$$= 1 + \cos(\theta+\pi)$$

$$= 1 - \cos \theta$$

therefore $- \sin \frac{\theta}{2} = 1 - (1-2 \sin^2 \frac{\theta}{2}) = 2 \sin^2 \frac{\theta}{2}$

8. continued

therefore $\sin \frac{\theta}{2} (2 \sin \frac{\theta}{2} + 1) = 0$

therefore $\sin \frac{\theta}{2} = 0$ or $\sin \frac{\theta}{2} = -\frac{1}{2}$.

If $\frac{\theta}{2} = 0$ we obtain the origin, and if $\sin \frac{\theta}{2} = -\frac{1}{2}$, we have $\frac{\theta}{2} = \frac{7\pi}{6}$, $\frac{11\pi}{6}$. Therefore $\theta = \frac{7\pi}{3}$ or $\theta = \frac{11\pi}{3}$ (this is not strange since the full curve $r = \sin \frac{\theta}{2}$ requires that $0 \leq \theta \leq 4\pi$).

When $\theta = \frac{7\pi}{3}$, $r = \sin \frac{\theta}{2} \rightarrow r = \sin \frac{7\pi}{6} = -\frac{1}{2}$. Therefore

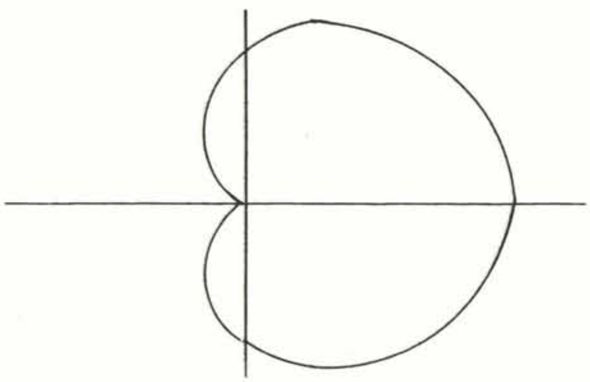
$(-\frac{1}{2}, \frac{7\pi}{3})$ is a point of intersection. [In the form $(-\frac{1}{2}, \frac{7\pi}{3})$ it satisfies C_2; in the form $(\frac{1}{2}, \frac{10\pi}{3})$ is satisfies C_1; and in "simplest" form it is $(\frac{1}{2}, \frac{4\pi}{3}) = (\frac{1}{2}, 240°)$].

From $\theta = \frac{11\pi}{3}$ we obtain that $(\frac{1}{2}, \frac{2\pi}{3}) = (\frac{1}{2}, 120°)$ is also a point of intersection.

[We would have obtained these same two points had we solved $1 + \cos \theta = -\sin (\frac{\theta+\pi}{2})$].

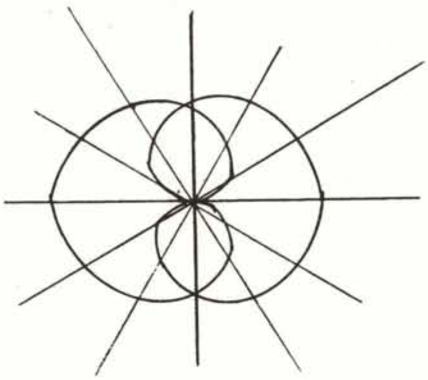
In any event there are five points of intersection = the origin, $(0.78, 104°)$, $(\frac{1}{2}, 120°)$, $(\frac{1}{2}, 240°)$, and $(0.78, 256°)$.

Pictorially,



$r = 1 + \cos \theta$

8. continued



Left portion traced out as θ goes from 0 to 2π

Right portion as θ goes from 2π to 4π

Notice this curve intersects itself!

Superimposing the diagrams, we have