Unit 6: The Jacobian

The role of the Jacobian in the study of calculus of several variables cannot be over-emphasized. However, many of us will never use the Jacobian matrix except in cases where it seems intuitive to us. Very few of us will use the Jacobian in the advanced contexts often studied by the mathematician and physical scientists.

Accordingly, this unit is meant as a buffer between the casual mention of the Jacobian in the previous unit and the very rigorous treatment of the Jacobian and its consequences in advanced calculus books.

By and large this unit can be omitted without loss of continuity in what follows. Moreover, if omitted, it can always be returned to as a separate, self-contained unit if so desired.

Nevertheless, it is recommended that you at least read Chapter 7 in the supplementary notes, even casually, so that you get an overview of the role of the Jacobian in the structure of vector calculus.

1. Read Supplementary Notes, Chapter 7.

2. Exercises

4.6.1

Let $\underline{f}:E^2 \to E^2$ be defined by $\underline{f}(x,y) = (u,v)$ where

$$\begin{cases} u = x^2 - y^2 \\ v = 2xy. \end{cases}$$

a. By comparing $\underline{f}(x,y)$ and $\underline{f}(-x,-y)$, show that if R is any neighborhood of $(x,y) = (0,0)$. Then $\underline{f}$ is not 1-1 on R.

b. Let S be the region described in polar coordinates by $1 \leq r \leq 2$, $-\pi \leq \theta \leq \frac{\pi}{2}$. Show that $\underline{f}$ is not 1-1 on S even though $|\underline{f}'(\underline{a})| \neq 0$ for each $\underline{a} \in S$.

4.6.2

Let $\underline{f}: E^2 \rightarrow E^2$ be defined by

$$\begin{cases} u = x^3 - y^3 \\ v = 2xy \end{cases} .$$

a. Show that $\underline{f}$ is locally invertible in the neighborhood of any point (x,y) provided only that the point is not on the line $y = -x$.

b. Explain what goes wrong if the point does lie on the line $y = -x$.

4.6.3

In what sense may we view

$$\frac{\partial(x_1,\ldots,x_n)}{\partial(x_1,\ldots,x_n)}$$

as if it were an "ordinary" fraction?

4.6.4

Let

$$\begin{cases} u = x^3 - y^3 \\ v = 2xy \end{cases} .$$

a. What is

$$\frac{\partial(u,v)}{\partial(x,y)} ?$$

b. Invert the given system of equations (i.e., express dx and dy in terms of du and dv) to determine

$$\frac{\partial(x,y)}{\partial(u,v)} .$$

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4.6.2 continued

c. Compute

$$\left[\frac{\partial(u,v)}{\partial(x,y)}\right]\left[\frac{\partial(x,y)}{\partial(u,v)}\right]$$

provided $y \neq -x$.

d. How are

$$\left|\frac{\partial(u,v)}{\partial(x,y)}\right|$$

and

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$$

related?

4.6.5

a. Let

$$u = \frac{x^2 - y^2}{x^2 + y^2}$$

and

$$v = \frac{xy}{x^2 + y^2}.$$

Use

$$\frac{\partial(u,v)}{\partial(x,y)}$$

to conclude that u and v are functionally dependent.

b. Find $\underline{f}:E^2 \to E^2$ such that $f \neq 0$ but $f(u,v) \equiv 0$.

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4.6.5 continued

c. Interpret part (b) geometrically.

4.6.6

a. Under what condition does the system

$$\left.\begin{array}{l} f(x,y,z,u,v) = 0 \\ g(x,y,z,u,v) = 0 \end{array}\right\}$$

determine u and v as continuously differentiable functions of x,y, and z [where f and g are themselves continuous differentiables]?

b. Show how the answer to (a) was "implied" by our previous methods.

4.6.7

Consider the system

$$\begin{cases} x + y + z = 0 \\ x^2 + y^2 + z^2 + 2xz - 1 = 0. \end{cases}$$

a. Under what conditions does this system allow us to conclude that y and z are differentiable functions of x? Compute dy/dx and dz/dx in this case.

b. Does this system ever determine x and z as differentiable functions of y? Explain.

4.6.8

Consider the system

$$\left.\begin{array}{r} x - y + 2z + u = 0 \\ 2x + 2y - 3z + 2u = 0 \\ 3x + y - z + u^2 = 0 \end{array}\right\}.$$

(continued on next page)

4.6.8 continued

a. Use the row-reduction technique to show that this system is compatible if and only if $u = 0$ or $u = 3$.

b. In the event that $u = 0$ or $u = 3$, show explicitly how we may solve for x and y in terms of z.

c. Does the given system determine x, y, and z as functions of u? Explain.

d. Use the Jacobian method to show that our given system determines x, y, and u in terms of z, but that we cannot express x, y, and z in terms of u.

4.6.9

a. Show that the system

$$\begin{cases} x + y + z = 0 \\ \frac{1}{3}x^3 + x - \frac{1}{3}y^3 - z^2y = 0 \end{cases}.$$

determines x and y as differentiable functions of z.

b. Compute $\frac{dx}{dz}$.