Unit 7: Line Integrals

1. Overview

From a structural point of view, one cannot distinguish between area and length in 1-dimensional space. What we mean by this is that in n-space one defines an n-dimensional rectangle to be a set of points (n-tuples) S such that S is composed precisely of those n-tuples $(x_1,\ldots,x_n)$ for which

$$\begin{array}{ccccc} a_1 & \le & x_1 & \le & b_1 \\ a_2 & \le & x_2 & \le & b_2 \\ \vdots & & \vdots & & \vdots \\ a_n & \le & x_n & \le & b_n \end{array}$$

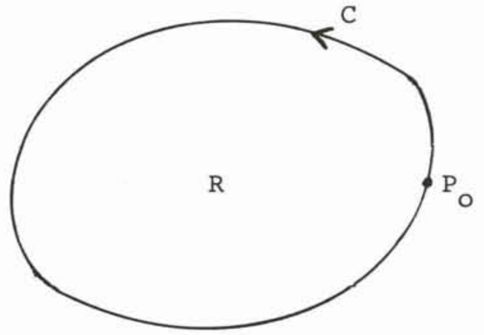
where the a's and b's are given constants and $a_i \le b_i$ for $i = 1,2,\ldots,n$.

One then defines the area of S to be the product $(b_1 - a_1)(b_2 - a_2)\ldots(b_n - a_n)$.

Notice that for n = 2 this definition of area coincides with the usual geometric interpretation of the area of a rectangle, while if n = 3 our n-dimensional rectangle is what we refer to geometrically as a parallelepiped and its area is what we think of as being volume. If n = 1, however, our abstract definition of area defines what we ordinarily think of as being length. In other words, in 1-space we may think of b - a as being either a length (geometrically) or an area (abstractly). In fact, when we spoke about sets of content measure zero, the measure of the line segment connecting (a,0) and (b,0) is zero if we view the line as a subset of 2-space, but it has measure (b - a) when viewed as a subset of 1-space.

What does all this have to do with the lesson in this unit? The answer is that when we are dealing with a region in 2-space there are two very different but equally natural ways of defining a definite integral. (These two different ways exist even in 1-space but because we cannot structurally distinguish between area and length in 1-space the two interpretations although conceptually very different yield the same result.)

For example, consider the simple closed curve C shown below, and let the region enclosed by C be denoted by R. On the one hand we might be interested in the mass of R*, in which case we must consider the usual double integral discussed in detail thus far in the present Block. On the other hand, we might be interested in the work done by a particle moving along the curve C under the influence of a given force.



1. The mass of R involves a double integral obtained from an element $\rho\Delta x_i \; \Delta y_j$.

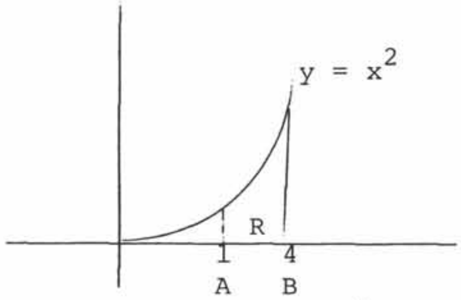
2. The work done by (or on) a particle moving along C from P_o back to P_o under the influence of a given force involves a <u>single</u> integral obtained by integrating with respect to arc length.

Clearly these two interpretations are quite different in 2-space. The second interpretation is known as a <u>line</u> integral to indicate that we are computing an integral with respect to a curve (line). The two interpretations also exist in 1-space but yield the same answer. By the way of illustration consider

$$\int_1^4 x^2 dx. \qquad (1)$$

*Notice again that when we refer to mass it is not important whether we talk about R together with C or with C excluded, since in 2-space the area of C is zero.

On the one hand we may think of it 2-dimensionally as representing the area of that portion of the parabola $y = x^2$ bounded on the right by the line $x = 1$,on the left by the line $x = 4$, and below by the x-axis.



$$A_R = \int_1^4 x^2 dx = \frac{1}{3}x^3 \Big]_1^4 = 21$$

On the other hand we may think of this as being the work done by a particle moving along the x-axis from A(1,0) to B(4,0) under the influence of the horizontal force $f(x) = x^2$.

Notice that our second interpretation can be written very nicely in the language of vectors. For example our force $f(x) = x^2$ is really a vector since we have specified the direction in which it acts (horizontally, i.e., parallel to the x-axis). Thus, perhaps we should have written that the force was $\vec{f}(x)$ where $\vec{f}(x) = x^2\vec{1}$. Also our segment of AB represents a displacement and hence it too may be treated as a vector. That is, perhaps we should have written $d\vec{x}=dx\ \vec{1}$ rather than dx. With this notation in mind, the definite integral

$$\int_0^1 f(x)\ dx = \int_0^1 x^2 dx$$

is really

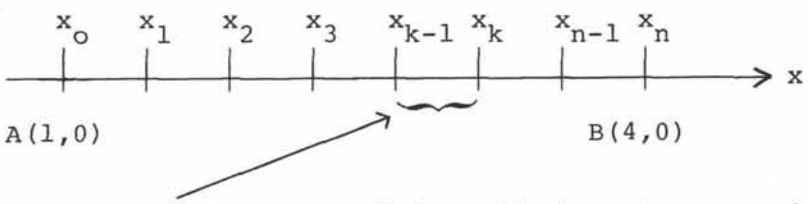
$$\int_{(1,0)}^{(4,0)} \vec{f}(x) \cdot d\vec{x}. \qquad (3)$$

Written as in (3) our integral does suggest a line integral, but computationally (1) and (3) are equivalent.

From a physical point of view, we partition our interval AB into segments $x_1, \ldots, x_n$, we let c_k denote a point in the k^{th} partition $[x_{k-1}, x_k]$ and we compute

$$\sum_{k=1}^{n} f(c_k)\Delta x_k = \sum_{k=1}^{n} c_k^{\,2}\,\Delta x_k \qquad (2)$$

and we compute the limit of (2) as max $\Delta x_k \to 0$. That is, we assume that on a sufficiently small segment $[x_{k-1}, x_k]$, f(x) is essentially constant (this is where f being continuous is important) and hence that $f(c_k)\,\Delta x_k$ is approximately the work done as the particle moves from $(x_{k-1}, 0)$ to $(x_k, 0)$; and we then sum over all the small segments.

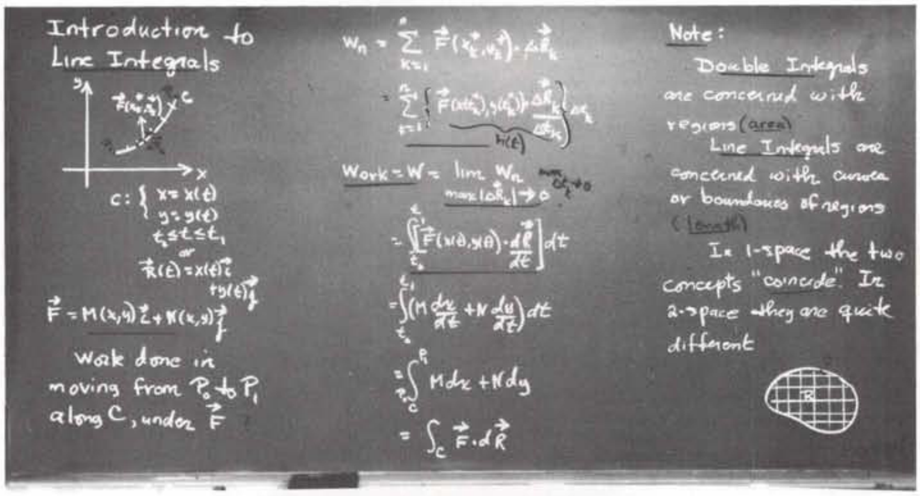


We pick c_k in here and from $f(c_k)\ x_k$ to approximate the work.

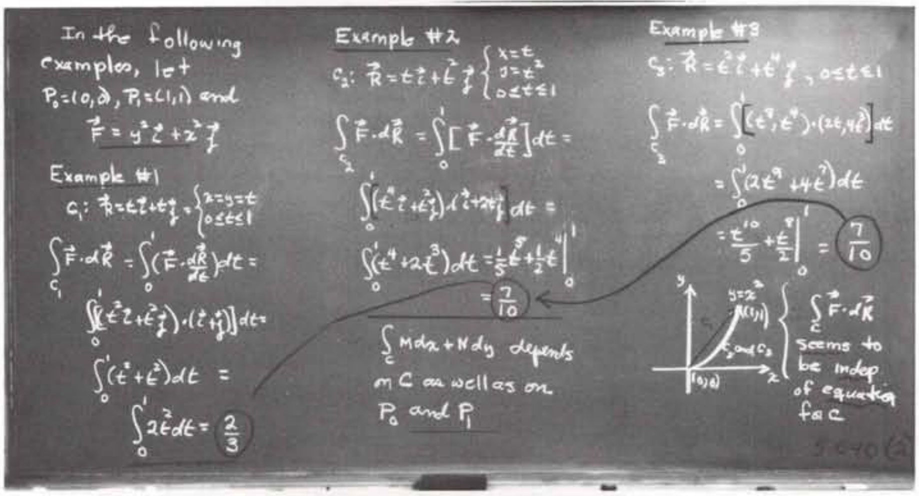
In summary then, in 1-space the integrals $\int_a^b f(x)\,dx$ and $\int_a^b f(x) \cdot dx$ are conceptually different but numerically the same. In 2-space, however, it certainly makes both a conceptual difference and a computational difference depending on whether our integral is viewed with respect to dA_R or with respect to ds_c, where ds_c refers to an element of arc length along the curve C.

The aim of the lecture in this unit is to emphasize these remarks in somewhat more detail, while the aim of the reading material and the exercises is to help you get a better quantitative idea as to how double integrals and single integrals (line integrals) are very different.

2. Lecture 5.040



a.



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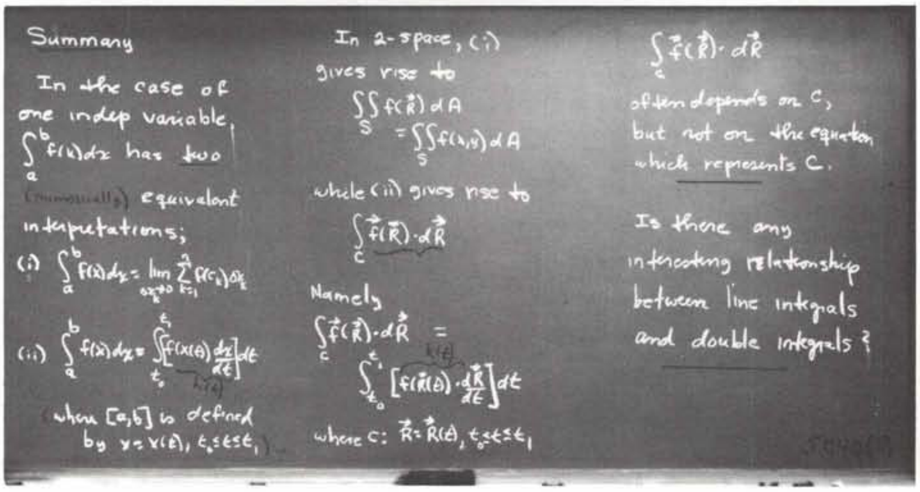

c.

3. Read: Thomas; Section 17.3

4. Exercises

5.7.1(L)

Compute $\int_c xy\,dx + (x^2 + y^2)\,dy$ where c is the portion in the first quadrant of the circle of radius 1 centered at the origin, traversed in the counter clockwise direction for each of the following equations for c.

a. $\left.\begin{matrix} x = \cos t \\ y = \sin t \end{matrix}\right\}$ t varies from 0 to $\frac{\pi}{2}$.

b. $y = \sqrt{1 - x^2}$, x varies from 1 to 0.

c. $x = \sqrt{1 - y^2}$, y varies from 0 to 1.

5.7.2(L)

Compute $\int_c xy\,dx + (x^2 + y^2)\,dy$ where c is the straight line segment which goes from (1,0) to (0,1).

5.7.3

Compute $\int_c (x + y)\,dx + xy\,dy$ where

a. C is given by

$$\begin{cases} x = t \\ y = t^3 + 1 \end{cases}$$

where t varies from 0 to 1.

b. C is given by

$$\begin{cases} x = t^3 \\ y = t^9 + 1 \end{cases}$$

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5.7.3 continued

where t varies from 0 to 1.

a. C is given by $y = x + 1$ where x varies from 0 to 1.

b. C is given by $y = x^2 + 1$ where x varies from 0 to 1.

5.7.4(L)

Compute $\int_c (x + y)\,dx + xy\,dy$ where c is the path composed of the straight line from (0,1) to (0,6), followed by straight line from (0,6) to (1,8), followed by the straight line from (1,8) to (1,2).

5.7.5(L)

Suppose Mdx + Ndy is an exact differential in a region R and C is a piecewise smooth curve <u>in R</u> joining the point (x_0, y_0) and (x_1, y_1). Use the chain rule to show that

$$\int_c Mdx + Ndy = \int_{(x_0,y_0)}^{(x,y)} dF = F(x,y) - F(x_0,y_0)$$

where dF = Mdx + Ndy.

5.7.6

a. Find a function F such that $F_x = 1 + 3x^2y + 5x^4y^2$ and $F_y = x^3 + 5y^4 + 2x^5y$ and use this to compute

$$\int_c (1 + 3x^2y + 5x^4y^2)\,dx + (x^3 + 5y^4 + 2x^5y)\,dy$$

where c is any (pieceview) smooth curve which connects (0,0) to (1,1).

b. Verify the result of (a) by direct computation in the case that:

1. C is $y = x$ where $0 \le x \le 1$
2. C is $y = x^3$ where $0 \le x \le 1$.

c. Evaluate $\int_{(0,0,0)}^{(1,1,1)} x^2dx + y^2dy + z^2dz$ along any (piecewise smooth) path which joins (0,0,0) and (1,1,1), by using the

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5.7.6 continued

fact that $x^2dx + y^2dy + z^2dz$ is an exact differential.

a. Evaluate the integral in (c) directly for the cases

1. C is given by $x = y = z = t$, $0 \le t \le 1$.
2. C is given by

$$\left.\begin{array}{l} x = t^2 \\ y = t^3 \\ z = t^4 \end{array}\right\} \quad 0 \le t \le 1.$$

5.7.7

Evaluate $\int_c ydx - xdy$ where c is the closed curve $|x| + |y| = 1$ traversed in the counter-clockwise direction starting at (1,0).

5.7.8(L)

Let $\vec{F} = xy\vec{i} + (x^2 + y^2)\vec{j}$ and $d\vec{s} = dx\vec{i} + dy\vec{j}$. Use polar coordinates to compute $\int_c \vec{F} \cdot d\vec{s}$ where c is the first quadrant of the unit circle from (1,0) to (0,1).

5.7.9 (optional)

The main aim of this exercise is to show that it is crucial that Mdx + Ndy be exact in a region R which contains C and that it is not enough that Mdx + Ndy be exact just on the curve C itself.

a. Check to see whether $ydx/(x^2 + y^2) - xdy/(x^2 + y^2)$ is exact.

b. Compute

$$\int_{c_1} \frac{ydx}{x^2 + y^2} - \frac{xdy}{x^2 + y^2} \quad \text{and} \quad \int_{c_2} \frac{ydx}{x^2 + y^2} - \frac{xdy}{x^2 + y^2}$$

where c_1 is the semicircle $x^2 + y^2 = 1$ in the counterclockwise sense from (1,0) to (-1,0) and c_2 is the semicircle $x^2 + y^2 = 1$ in the clockwise sense from (1,0) to (-1,0).

5.7.10 (optional)

Our aim here is to give an alternative, more general definition of a line integral.

a. Let c be given by

$$\left.\begin{array}{l} x = f(t) \\ y = g(t) \end{array}\right\} \quad a \leq t \leq b$$

and suppose H(x,y) is (piecewise) continuous on c. Let γ be any continuous differentiable function defined on [a,b].

a. Show that the definition $\int_c H d\gamma = \int_a^b H(f(t), g(t))\, \gamma'(t)\, dt$ is consistent with $\int_a^b f(t) dg(t) = \int_a^b f(t) g'(t) dt$.

b. Compute $\int_c H d\gamma$ where

$$c: \left.\begin{array}{l} x = t^2 \\ y = t^4 + 1 \end{array}\right\} 0 \leq t \leq 1$$

$$H(x,y) = x^2 + y^2$$

and

$$\gamma(t) = t^3 \text{ for } t \in [0,1].$$