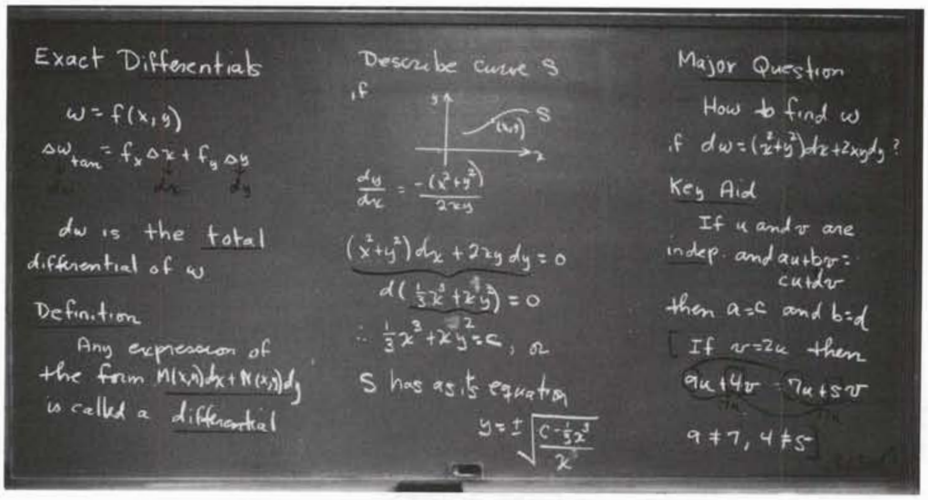
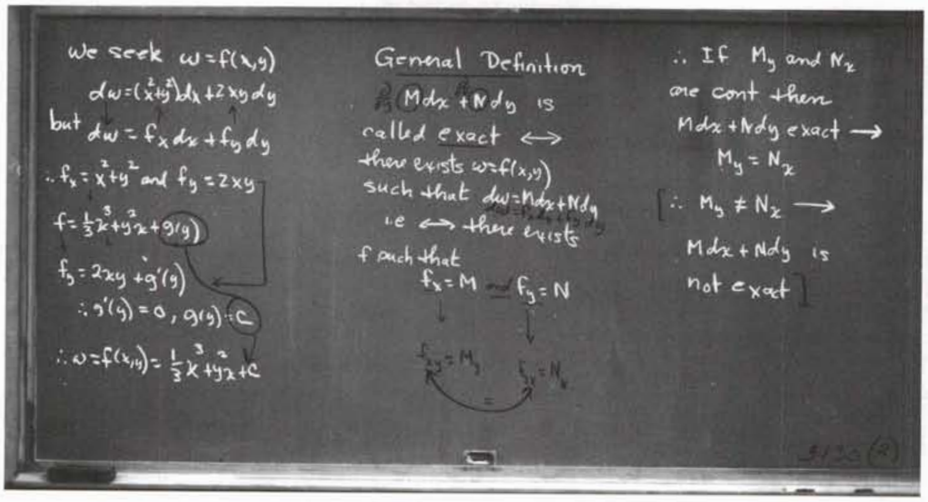
Unit 8: The Total Differential

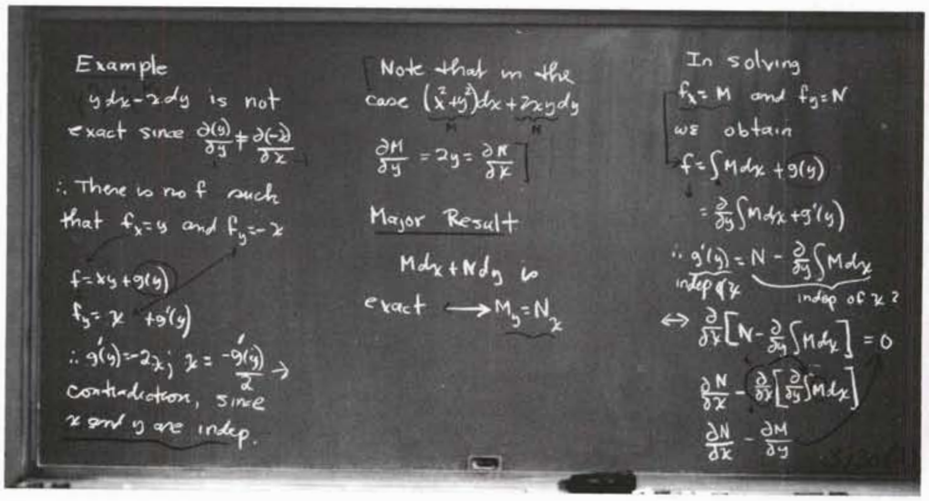
1. Lecture 3.060



a.



b.



c.

2. Read: Thomas 15.8 and 15.13 (In 15.13 disregard the discussion of simply connected regions. While such regions are extremely important, especially in terms of line integrals which we study in Block 5, it is not necessary to mention them within the context of Section 15.13).

3. Exercises:

3.8.1 (L)

a. Given an expression of the form $f(x,y) = 0$, in what sense is this expression an equation and in what sense is it an identity? Illustrate your remarks concretely in terms of letting $f(x,y) = x^2 - y$.

b. Suppose that $f(x,y) = 0$ determines y as a differentiable function of x, say $y = g(x)$. Letting $w = f(x,y)$, we have that if $y = g(x)$ then $w = f(x,g(x)) = h(x)$. Show that the value of dw is the same whether we compute it from $w = f(x,y)$ or from $w = h(x)$.

c. Use the previous results to justify the concept of implicit differentiation. Namely, assuming that $f(x,y) = 0$ determines y as a differentiable function of x, then we can compute $\frac{dy}{dx}$ from $f_x dx + f_y dy = 0$, i.e., from $df = 0$.

d. Illustrate the results of part (c) using the example $f(x,y) = x^2 + y^2 - 1$.

3.8.2 (L)

Let $w = f(x,y,z)$ where f is a continuously differentiable function of x, y, and z.

a. In what sense is $dw = 0$ an identity?

b. If $f(x,y,z) = 0$ determines z as a continuously differentiable function of x and y, determine $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

c. Suppose $f(x,y,z) = 0$ where $f(x,y,z) = z^5 + x^2y^3$. Express dw in the form $M(x,y)dx + N(x,y)dy$ on the set $S = \{(x,y,z): z^5 + x^2y^3 = 0\}$ and show that $M = N = 0$.

3.8.3 (L)

a. Show that $(e^xy^3 + 2x \sin y + 4x^3y^5)dx + (3e^xy^2 + x^2\cos y + 5x^4y^4)dy$ is exact and then find $f(x,y)$ such that $f_x(x,y) = e^xy^3 + 2x \sin y + 4x^3y^5$ and $f_y(x,y) = 3x^ey^2 + x^2 \cos y + 5x^4y^4$.

(continued on next page)

3.8.3 (L) continued

b. Try to construct $f(x,y)$ such that $f_x(x,y) = 4x^3 \sin y$ and $f_y = x^4 \cos y + x$. Then explain how we could have determined in advance that no such f existed.

c. The curve C passes through (0,2) and its slope at each point (x,y) is given by

$$\frac{dy}{dx} = -\frac{(e^x y^3 + 2x \sin y + 4x^3 y^5)}{3e^x y^2 + x^2 \cos y + 5x^4 y^4}$$

Determine the (Cartesian) equation of C.

3.8.4

Find the equation of the curve C if C passes through the origin and its slope at each point (x,y) is given by

$$\frac{dy}{dx} = \frac{-(2xe^y + e^x)}{(x^2 + 1)\, e^y}$$

3.8.5

$M(x,y,z)dx + N(x,y,z)dy + P(x,y,z)dz$ is called an exact differential if there exists $f(x,y,z)$ such that $df = Mdx + Ndy + Pdz$. Show that if M,N, and P are each continuously differentiable then $M_y = N_x$, $M_z = P_x$ and $N_z = P_y$ if $Mdx + Ndy + Pdz$ is exact.

3.8.6

a. Is it possible that $M(x,y,z)dx + (xz - e^x \sin y)dy + (xy + z)dz$ is exact? If so, how must $M(x,y,z)$ be defined?

b. Let $h(x)$ be any differentiable function of x. Find a function $f(x,y,z)$ such that $f_x(x,y,z) = yz + e^x \cos y + h(x)$, $f_y(x,y,z) = xz - e^x \sin y$, and $f_z(x,y,z) = xy + z$.

c. Find all functions $f(x,y,z)$ such that $df = e^{y + 2z}\, dx + xe^{y + 2z} dy + 2xe^{y + 2z} dz$.

3.8.7

In thermodynamics, the five quantities S,T,u,p, and v are related in such a way that any two may be chosen at random (i.e. may be viewed as independent variables) whereupon the remaining three are then determined in terms of the other two. The quantities are related by the equation $TdS \equiv du + pdv$.

Compute $d(St - u)$ to conclude that $\left(\frac{\partial p}{\partial T}\right)_v = \left(\frac{\partial S}{\partial v}\right)_T$

Quiz

1. We are given that

$$\begin{cases} u = x^3 - 3xy^2 \\ v = 3x^2y - y^3 \end{cases}$$

where x and y are independent and $(x,y) \neq (0,0)$. Assuming that x and y are differentiable functions of u and v, use implicit differentiation to compute $\left(\frac{\partial x}{\partial u}\right)_v$.

2. We are given that $w = f(u,v)$ where f is a continuously differentiable function of u and v and $w_{uv} = w_{vu}$. Suppose that $u = 3x + 2y$ and $v = 8x + 5y$.

   (a) Determine $\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}$ in terms of u and v.

   (b) Check your result in (a) in terms of the specific example $f(u,v) = u^3 + v^2 + uv$.

3. The plane M is tangent to the surface

$$z^5 + 6xyz + x^4y^5 = 8$$

at the point (1,1,1). At what point does M intersect the line $z = 2y = 4x$?

4. We are given the surface $w = f(x,y)$ where f is a continuously differentiable function of x and y. At (2,3), the directional derivative $\frac{dw}{ds}$ in the direction toward (5,7) is 4 and in the direction toward (6,6) is 10. In what direction from (2,3) is $\frac{dw}{ds}$ maximum and what is this maximum value?

5. f(x) is defined by $f(x) = \int_a^x (x - y)h(y)dy$.

Find a differential equation which is satisfied by f(x).

6. (a) Find $w = f(x,y)$ such that

$$dw = (3x^2y + e^x \cos y)dx + (x^3 - e^x \sin y)dy.$$

(b) Solve the differential equation

$$\frac{dy}{dx} = \frac{3x^2y + e^x \cos y}{e^x \sin y - x^3}.$$