Unit 8: Green's Theorem

1. Overview

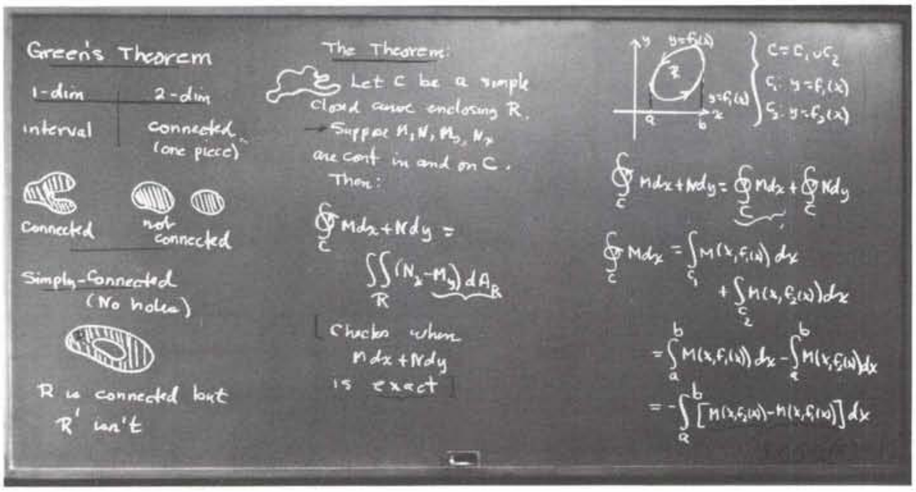
So far we have presented two different types of integration, both of which may be viewed as "natural outgrowths" of the ordinary definite integral, depending on what interpretation we have in mind. In this unit which concludes the present Block of material, we present Green's Theorem which shows a rather amazing connection between certain types of line integrals and double integrals.

Our main aim in this unit will be to exploit the formula stated in the theorem, and we shall leave the proof of the theorem as an optional exercise in the study guide.

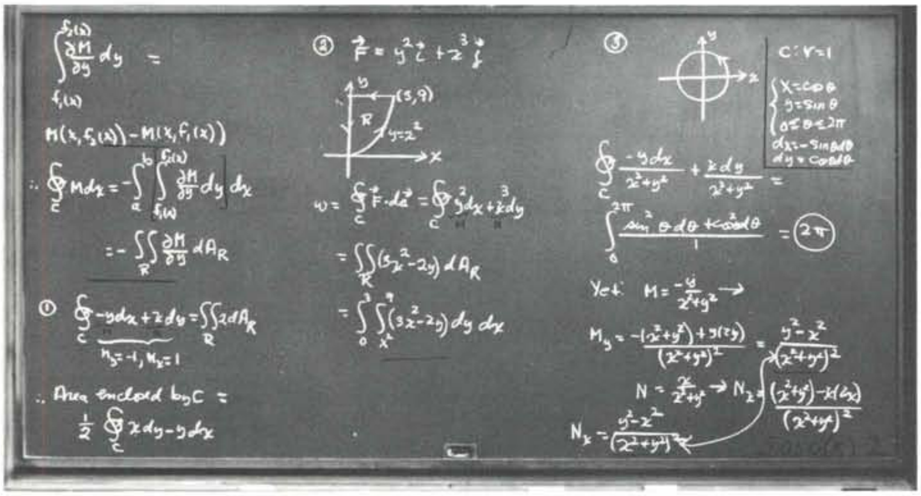
While it is not our aim to stress the subject known as vector analysis in our present course, it should be pointed out that Green's Theorem and some of its extensions do play a vital role in the calculus of vector analysis and it is our hope that the rather brief treatment here will be sufficient for the immediate needs of most students, and that it will serve as an introduction of the subject to those students who wish to pursue the ideas further.

2. Lecture 5.050

a.



Lecture 5.050 continued



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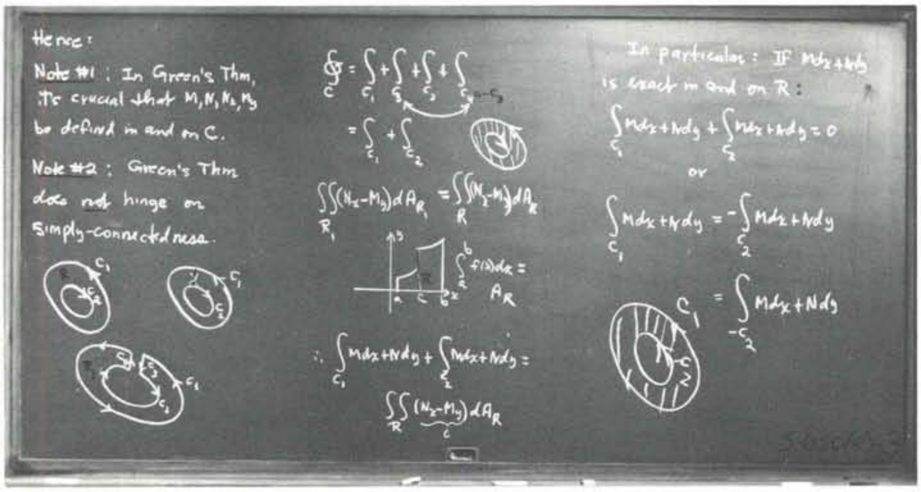

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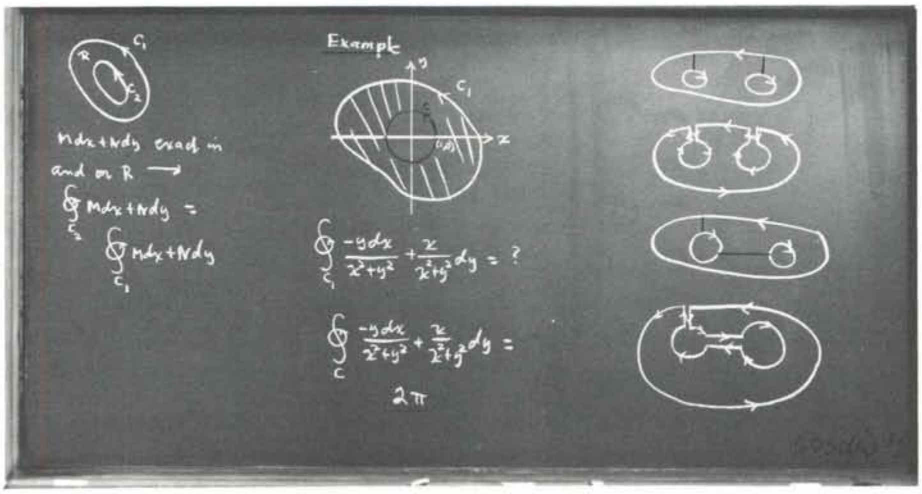

d.

3. Read: Thomas, Section 17.5. (Optional: Read any other portions of Chapter 17 in the text that are of interest to you. For the most part, the discussion in the text is adequate to give you a pretty decent idea of the extensions of Green's Theorem, especially in terms of a 3-dimensional analog.)

4. Exercises

5.8.1(L)

a. Without Green's Theorem compute

$$\oint_C - x^2y\, dx + y^2x\, dy$$

where C is the oriented curve* which is the boundary of the unit disc centered at the origin.

b. Use Green's Theorem to obtain the same result as in part (a).

5.8.2

Let C be the oriented boundary of the rectangle R with vertices at (0,0), (3,0), (3,2), and (0,2). Compute

$$\oint_C 2ydx - 3xdy$$

first without use of Green's Theorem and then with the use of Green's Theorem.

5.8.3(L)

a. Show that $\int_a^b f(x)dx$ is a special case of a line integral (hint: think of the curve C as being the interval [a,b]).

b. Show that if C is any closed curve

$$\oint_C f(x)dx = 0.$$

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*By oriented we mean that we traverse C so that the enclosed region R appears on our left. Thus, in this case the circle is traversed in the counterclockwise direction.

5.8.3(L) continued

a. Show that $\int_C (M_1 + M_2)\,dx + (N_1 + N_2)\,dy$ is equal to $\int_C M_1 dx + \int_C M_2 dx + \int_C N_1 dy + \int_C N_2 dy$.

b. Let R be a region in the xy-plane with oriented boundary C. Then in Newtonian mechanics

$$\iint_R (x^2 + y^2)\,dA_R$$

is called the moment of inertia of R about the z-axis. Use Green's Theorem to show that this moment is also given by

$$\frac{1}{3} \oint_C - y^3 dx + x^3 dy.$$

5.8.4(L)

a. Let C be the oriented boundary of the region R. Apply Green's Theorem to show that

$$A_R = \frac{1}{2} \oint_C -ydx + xdy.$$

b. Show geometrically why the result established in (a) is plausible.

5.8.5

Use the result of part (a) of Exercise 5.8.4 to compute the area of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

5.8.6 (optional)

[This is a rehash of our discussion that a multiply-connected domain may be viewed as a simply-connected domain if suitable "cuts" are made. If you understand this idea (or prefer to accept the result on faith) you may omit this exercise and

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5.8.6 continued

simply apply the principle involved to the following two exercises.]

Suppose the region enclosed by the oriented curve C_1 is contained in the region enclosed by the oriented curve C_2. Suppose further that Mdx + Ndy is exact in the region on and between C_1 and C_2. Show that in this case

$$\oint_{C_1} Mdx + Ndy = \oint_{C_2} Mdx + Ndy.$$

5.8.7(L)

Let C be the ellipse

$$\frac{x^2}{4} + \frac{y^2}{9} = 1.$$

Compute

$$\oint_C \frac{-\,ydx + xdy}{x^2 + y^2}.$$

5.8.8

a. In what region is

$$\frac{-\,ydx + (x - 2)dy}{(x - 2)^2 + y^2}$$

exact.

b. Compute

$$\oint_{C_1} \frac{-\,ydx + (x - 2)dy}{(x - 2)^2 + y^2}$$

if C_1 is the oriented boundary of the circle centered at (0,0) with radius 1.

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5.8.8 continued

a. Compute

$$\oint_{C_2} \frac{-\,ydx + (x-2)\,dy}{(x-2)^2 + y^2}$$

if C_2 is the oriented circle centered at (0,0) with radius 4, and show that

$$\oint_{C_1} \frac{-\,ydx + (x-z)\,dy}{(x-2)^2 + y^2} \neq \oint_{C_2} \frac{-\,ydx + (x-z)\,dy}{(x-2)^2 + y^2}.$$

b. Why isn't (c) a contradiction of the result obtained in Exercise 5.8.6(L)?

5.8.9

Suppose R is the region enclosed by the simple closed curve C. Let v and w both be twice-differentiable functions of x and y such that in R: $v_{xx} + v_{yy} = w_{xx} + w_{yy} \equiv 0$. Suppose that $v \equiv w$ on C. Prove that $v \equiv w$ <u>throughout R</u> (this says that any function v such that $v_{xx} + v_{yy} \equiv 0$ in R is completely determined by its behaviour on C). <u>Hint</u>: Let $u = v - w$ and apply Green's Theorem to

$$\oint_C -\,u_y u\,dx + u_x u\,dy.$$

5.8.10 (optional)

[This is for those students who would like a more thorough outline of the proof of Green's Theorem to reinforce the proof in the text.]

a. Prove Green's Theorem for the special case in which the closed curve C is the rectangle with vertices at (a,c), (b,c), (b,d) and (a,d) where $a < b$ and $c < d$.

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5.8.10 continued

a. Do the same as in part (a) only for the curve C which consists of the smooth curve $y = f(x)$, $a \leq x \leq b$ which connects (a,c) to (b,d), followed by the line which connects (b,d) to (a,d), followed by the line which joins (a,d) to (a,c).

5.8.11 (optional)

Mimic the proof in the previous exercise and derive the appropriate form of Green's Theorem that applies when M and N are functions of the polar coordinates r and θ. That is, write $M(r,\theta)\,dr + N(r,\theta)\,d\theta$ as a double integral in the case that C is the region bounded between $r = a$ and $r = b$, $(a < b)$ and $\theta = \alpha$ and $\theta = \beta$ $(\alpha < \beta)$.

[The point of this exercise is to emphasize the difference between concepts and their representation in different coordinate systems. For example (and hopefully by this time in the course we have seen enough examples where the following argument is faulty, so that we are on guard) one might induce from the form

$$\oint_C M(x,y)dx + N(x,y)dy = \iint_R (N_x - M_y)dA_R$$

that

$$\oint_C M(r,\theta)d\theta + N(r,\theta)d\theta = \iint_R (N_r - M_\theta)dA_R$$

simply by replacing x and y by r and θ, respectively. Among other things this exercise shows that the above "solution" is incorrect.]

Quiz

1. Write

$$\int_0^3 \int_{x^2}^{3x} xy^2 \, dy \, dx$$

as an equivalent double integral with the order of integration reversed. Check your answer by computing both integrals.

2. (a) Find the mass of the plate R if R is the region enclosed by the cardioid $r = 1 + \cos\theta$ and the density of R at any point is equal to its distance from the origin.

   (b) With R as in part (a), find the volume of the portion of the cylinder whose cross section is R bounded between the xy-plane and the surface $z = \sqrt{x^2 + y^2}$.

3. Let c denote the parallelogram with vertices at (0,0), (2,-1), (-3,2), and (-1,1); and let R denote the region enclosed by c.

   (a) Given the linear mapping that carries R into the unit square S of the uv-plane whereby (2,-1) is mapped into (1,0); (-3,2) into (0,1); and (-1,1) into (1,1), find how u and v are related to x and y.

   (b) Use the result of (a) to compute

$$\iint_R e^{2x + 3y} \cos(x + 2y) \, dA_r.$$

4. Find the volume cut from the sphere $x^2 + y^2 + z^2 = 4a^2$ by the cylinder $x^2 + y^2 = a^2$.

5. (a) Show that $2xydx + (x^2 + \cos y)dy$ is exact. In particular find $f(x,y)$ such that $df = 2xydx + (x^2 + \cos y)dy$.

   (b) Let c be any smooth path which connects (0,0) to (1,1). Use the result of part (a) to compute

$$\int_c 2xydx + (x^2 + \cos y)dy.$$

6. Let c_1 be the straight line that joins (0,0) to (1,1) and let c_2 be the portion of the parabola $y = x^2$ which joins (0,0) to (1,1).

   (a) Compute $\int_{c_1} (\sin x - y^3)dx + (x^3 - e^y)dy$.

   (b) Compute $\int_{c_2} (\sin x - y^3)dx + (x^3 - e^y)dy$.

   (c) From (a) and (b) what can we conclude about $(\sin x - y^3)dx + (x^3 - e^y)dy$? Explain.

7. Compute

$$\oint_c (\sin x - y^3)dx + (x^3 - e^y)dy$$

where c is the circle centered at the origin with radius 1.