Unit 7: Line Integrals

5.7.1(L)

a. We have

$$\int_C xy\,dx + (x^2 + y^2)\,dy$$

$$= \int_{t_0}^{t_1} \left[xy \frac{dx}{dt} + (x^2 + y^2) \frac{dy}{dt}\right] dt \qquad (1)$$

and since

$x = \cos t$ and $y = \sin t$ where t varies continuously from 0 to $\frac{\pi}{2}$, equation (1) becomes

$$\int_0^{\frac{\pi}{2}} [\cos t \sin t\,(-\sin t) + (\cos^2 t + \sin^2 t) \cos t]\, dt$$

$$= \int_0^{\frac{\pi}{2}} (-\sin^2 t \cos t + \cos t)\, dt$$

$$= \left(-\frac{1}{3}\sin^3 t + \sin t\right) \Bigg|_{t=0}^{\frac{\pi}{2}}$$

$$= \left(-\frac{1}{3} + 1\right) - 0$$

$$= \frac{2}{3}.$$

b. Notice that another parametric form for C is given by

$$\begin{aligned} x &= t \\ y &= \sqrt{1 - t^2} \end{aligned} \quad \text{as } t \text{ varies from 1 to 0} \qquad (2)$$

(i.e., $y = \sqrt{1 - x^2}$).

5.7.1(L) continued

From (2) we have that

$$\frac{dx}{dt} = 1 \quad \text{and} \quad \frac{dy}{dt} = \frac{-t}{\sqrt{1 - t^2}}$$

so that now (1) becomes

$$\int_1^0 \{t\sqrt{1 - t^2} + [t^2 + (\sqrt{1 - t^2})^2]\frac{(-t)}{\sqrt{1 - t^2}}\} dt$$

$$= \int_1^0 [t\sqrt{1 - t^2} + \frac{-t}{\sqrt{1 - t^2}}] dt$$

$$= \int_0^1 [\frac{t}{\sqrt{1 - t^2}} - t\sqrt{1 - t^2}] dt$$

$$= [-\sqrt{1 - t^2} + \frac{1}{3}(1 - t^2)^{\frac{3}{2}} \Big|_{t=0}^{1}]$$

$$= 0 - [-1 + \tfrac{1}{3}]$$

$$= \tfrac{2}{3}.$$

The key point that we want to make here is that it is intuitively clear that the value of the line integral should not depend on the equation chosen to represent C. Yet it is not abstractly clear that the value of the line integral is independent of the parametric form for C. After all, as we change the equation of C both the integrand and the limits of integration change, and, as a result, the possibility exists that our answer changes as well. In the next unit we shall show in terms of someting called Green's Theorem that the line integral does not depend on the equation of C, but in this exercise we are using a less rigorous device.

5.7.1(L) continued

Namely, we have shown in parts (a) and (b) that the line integral does not depend on which of the two equations for C is used. Of course, this is a far cry from proving that the integral is the same for all equations for C, but it at least adds plausibility to our statement; and this is sufficient for our present aims.

c. But just to "play it safe" we compute the same integral a third way. We now solve $x^2 + y^2 = 1$ for x in terms of y. That is

$$\left.\begin{aligned} x &= \sqrt{1 - t^2} \\ y &= t \end{aligned}\right\} \text{ t varies from 0 to 1}$$

so that

$$\frac{dx}{dt} = \frac{-t}{\sqrt{1 - t^2}}, \quad \frac{dy}{dt} = 1$$

and equation (1) becomes

$$\int_0^1 [xy \frac{dx}{dt} + (x^2 + y^2) \frac{dy}{dt}]\, dt \tag{3}$$

$$= \int_0^1 \{\sqrt{1 - t^2}\, t [\frac{-t}{\sqrt{1 - t^2}}] + [(\sqrt{1 - t^2})^2 + t^2](1)\}\, dt$$

$$= \int_0^1 (-t^2 + 1)\, dt$$

$$= -\frac{1}{3} t^3 + t \Big|_{t=0}^{1}$$

$$= (-\frac{1}{3} + 1) - 0$$

$$= \frac{2}{3}$$

5.7.1(L) continued

and our answer agrees with that of parts (a) and (b).

5.7.2(L)

In the previous exercise we tried to demonstrate that $\int_c Mdx + Ndy$ does not depend on the equation which expresses the curve c.

This is far different than saying that $\int_c Mdx + Ndy$ does not depend on the curve c. In other words, in terms of the physical interpretation of work, the work done in going from P_0 to P_1 along the path (curve) c "most likely" should depend on the choice of the path but not on the equation by which a given path is described.

In this exercise we observe that the end points of c as well as the integrand are the same as in the previous example in which we evaluated $\int_c xydx + (x^2 + y^2)\,dy$.

Now, however, the curve c is described by

$$\left.\begin{array}{l} x = 1 - t \\ y = t \end{array}\right\} \text{ t varies from 0 to 1.} \qquad (1)$$

In this case, $\frac{dx}{dt} = -\frac{dy}{dt} = 1$ and we have

$$\int_c xydx + (x^2 + y^2)\,dy$$

$$= \int_{t=0}^{1} \left[xy\frac{dx}{dt} + (x^2 + y^2)\frac{dy}{dt}\right] dt*$$

*Notice that this looks exactly like equation (3) in part (b) of Exercise 5.7.1. What has changed is the curve c and since the line integral is defined on c (i.e., x, y, dx/dt, and dy/dt are computed for points on c) the value of the integral need not be the same in this case as it was in equation (3) of the previous exercise.

5.7.2(L) continued

$$\int_0^1 \{(1 - t)(- t) + [(1 - t)^2 + t^2]\}\, dt$$

$$= \int_0^1 [-t + t^2 + (1 - 2t + t^2) + t^2]\, dt$$

$$= \int_0^1 (3t^2 - 3t + 1)\, dt$$

$$= t^3 - \frac{3}{2} t^2 + t \Big|_0^1$$

$$= 1 - \frac{3}{2} + 1$$

$$= \frac{1}{2},$$

and this is unequal to $\frac{2}{3}$ (the answer in the previous exercise).

In terms of the work interpretation we are saying that if a particle moves under the influence of the force $xy\vec{i} + (x^2 + y^2)\,\vec{j}$ from (1,0) to (0,1) then the work done is $\frac{2}{3}$ if the path is the portion of the circle $x^2 + y^2 = 1$ which joins these two points, while the work is $\frac{1}{2}$ if the path is the straight line which joins the two points. In other words, once the path is chosen, the work depends on the end points of the path alone (i.e., the limits of integration), but different paths through the same end points produce different work.

As a final point on this exercise, let us check that our answer did not depend on the equation of c once the path was chosen. For example, another parametric form for c is

$$\left.\begin{aligned} x &= t^3 \\ y &= 1 - t^3 \end{aligned}\right\} \text{ t varies from 1 to 0}$$

so that

5.7.2(L) continued

$\frac{dx}{dt} = 3t^2 = -\frac{dy}{dt}$.

Consequently

$$\int_C xydx + (x^2 + y^2)\, dy$$

$$= \int_1^0 \{t^3(1 - t^3)3t^2 + [t^6 + (1 - t^3)^2](-3t^2)\}\, dt$$

$$= \int_0^1 \{3t^2[t^6 + (1 - t^3)^2] - t^3(1 - t^3)3t^2\}dt$$

$$= \int_0^1 [3t^2(2t^6 - 2t^3 + 1) - 3t^5 + 3t^8]\, dt$$

$$= \int_0^1 (9t^8 - 9t^5 + 3t^2)\, dt$$

$$= t^9 - \frac{3}{2}t^6 + t^3 \Big|_{t=0}^{1}$$

$$= 1 - \frac{3}{2} + 1 = \frac{1}{2} ,$$

which checks with our previous result.

5.7.3

a. C is given by

$$\left.\begin{aligned} x &= t \\ y &= t^3 + 1 \end{aligned}\right\} \text{ t varies from 0 to 1.}$$

Hence, $dx/dt = 1$ and $dy/dt = 3t^2$. Therefore,

5.7.3 continued

$$\int_c (x + y)\ dx + xydy$$

$$= \int_0^1 [(t + t^3 + 1) + t(t^3 + 1)3t^2]\ dt$$

$$= \int_0^1 (t + t^3 + 1 + 3t^6 + 3t^3)\ dt$$

$$= \int_0^1 (3t^6 + 4t^3 + t + 1)\ dt$$

$$= \frac{3}{7}t^7 + t^4 + \frac{1}{2}t^2 + t \Big|_{t=0}^{1}$$

$$= \frac{3}{7} + 1 + \frac{1}{2} + 1$$

$$= 2\ \frac{13}{14}.$$

b. Now c is given by

$$\left.\begin{array}{l} x = t^3 \\ y = t^9 + 1 \end{array}\right\}\ \text{t varies from 0 to 1.}$$

Hence,

$$\frac{dx}{dt} = 3t^2 \text{ and } \frac{dy}{dt} = 9t^8.$$

Therefore,

$$\int_c (x + y)\ dx + xydy$$

$$= \int_0^1 [(t^3 + t^9 + 1)3t^2 + t^3(t^9 + 1)\ 9t^8]dt$$

5.7.3 continued

$$\int_0^1 (3t^5 + 3t^{11} + 3t^2 + 9t^{20} + 9t^{11})\, dt$$

$$= \int_0^1 (9t^{20} + 12t^{11} + 3t^5 + 3t^2)\, dt$$

$$= \frac{3}{7} t^{21} + t^{12} + \frac{1}{2} t^6 + t^3 \Big|_{t=0}^{1}$$

$$= 2\ \frac{13}{14}.$$

[This should check with part (a) since in both parts of this exercise c is a parametric form of the portion of the curve $y = x^3 + 1$ from (0,1) to (1,2)]

c. With c given by

$$\left.\begin{aligned} x &= t \\ y &= t + 1 \end{aligned}\right\} \text{ where t varies from 0 to 1, i.e., } y = x + 1$$

we have $dx/dt = dy/dt = 1$ and, therefore,

$$\int_c (x + y)\, dx + xydy$$

$$= \int_0^1 [(2t + 1) + (t^2 + t)]dt$$

$$= \int_0^1 (t^2 + 3t + 1)\, dt$$

$$= \frac{1}{3} t^3 + \frac{3}{2} t^2 + t \Big|_{t=0}^{1}$$

$$= \frac{1}{3} + \frac{3}{2} + 1$$

$$= 2\ \frac{5}{6}.$$

5.7.3 continued

d. With c given by $y = x^2 + 1$ where x varies from 0 to 1 we have $dy/dx = 2x$, hence

$$\int_c (x + y)\,dx + xy\,dy$$

$$= \int_0^1 \left[(x + y) + xy\frac{dy}{dx}\right] dx$$

$$= \int_0^1 [(x + x^2 + 1) + x(x^2 + 1)2x]\,dx$$

$$= \int_0^1 (2x^4 + 3x^2 + x + 1)\,dx$$

$$= \frac{2}{5}x^4 + x^3 + \frac{1}{2}x^2 + x \Big|_{x=0}^{1}$$

$$= \frac{2}{5} + 1 + \frac{1}{2} + 1$$

$$= 2\,\frac{9}{10}.$$

5.7.4(L)

This problem is the same as the previous one except that our path is now a union of smooth curves (i.e., the equation of the path is piecewise differentiable).

Letting c_1 denote the straight line from (0,1) to (0,6); c_2, the straight line from (0,6) to (1,8); and c_3 the straight line from (1,8) to (1,2) we have that

5.7.4(L) continued

$$\int_c (x + y)\ dx + xydy$$

$$= \int_{c_1} (x + y)\ dx + xydy$$

$$+ \int_{c_2} (x + y)\ dx + xydy$$

$$+ \int_{c_3} (x + y)\ dx + xydy. \qquad (1)$$

[This is in keeping with the definition that if $c = c_1 \cup \ldots \cup c_n$ then

$$\int_c Mdx + Ndy = \int_{c_1} Mdx + Ndy + \ldots + \int_{c_n} Mdx + Ndy.$$

In terms of work, this formula says that to find the total work done as we move along a piecewise-smooth curve we need only sum the amounts of work done over the individual smooth pieces.

In any event, c_1 may be written as

$$\left.\begin{array}{l} x = 0 \\ y = t \end{array}\right\} \text{ where t varies from 1 to 6.}$$

Hence, $dx/dt = 0$ and $dy/dt = 1$. Therefore

$$\int_{c_1} (x + y)\ dx + dxdy$$

$$= \int^6 [(x + y)\ \frac{dx}{dt} + xy\ \frac{dy}{dt}]\ dt$$

5.7.4(L) continued

$$\int_1^6 [t(0) + 0(t)1]\, dt$$

$$= \int_1^6 0dt = 0. \qquad (2)$$

c_2 may be written in the form $y = 2x + 6$ where x varies from 0 to 1.

Hence, $dy/dx = 2$ and

$$\int_{c_2} (x + y)dx + xydy$$

$$= \int_0^1 [(x + y) + xy \frac{dy}{dx}]\, dx$$

$$= \int_0^1 [(x + 2x + 6) + x(2x + 6)2]\, dx$$

$$= \int_0^1 (4x^2 + 14x + x + 6)\, dx$$

$$= \frac{4}{3} x^3 + 7x^2 + \frac{1}{2} x^2 + 6x \Big|_{x=0}^{1}$$

$$= \frac{4}{3} + 7 + \frac{1}{2} + 6$$

$$= 14 \frac{5}{6}. \qquad (3)$$

Finally c_3 may be written as

$$\left.\begin{aligned} x &= 1 \\ y &= t \end{aligned}\right\} \text{ where t varies from 8 to 2.}$$

5.7.4(L) continued

Hence, $dx/dt = 0$, $dy/dt = 1$; and we have

$$\int_{c_3} (x + y)\,dx + xy\,dy$$

$$= \int_8^2 \left[(x + y)\frac{dx}{dt} + xy\frac{dy}{dt}\right] dt$$

$$= \int_8^2 [(t + 1)(0) + 1(t)(1)]\,dt$$

$$= \int_8^2 t\,dt$$

$$= -\int_2^8 t\,dt$$

$$= -\frac{1}{2}t^2 \Big|_2^8$$

$$= -32-(-2) = -30. \qquad (4)$$

Putting the results of (2), (3), and (4) into (1) we obtain

$$\int_c (x + y)\,dx + xy\,dy = 0 + 14\tfrac{5}{6} - 30$$

$$= -(15\tfrac{1}{6})$$

$$= -\frac{91}{6}.$$

Physically we may think of this problem as finding the work done as a particle moves from (0,1) to (1,2) along the curve c under the influence of the force $\vec{F} = (x + y)\vec{i} + xy\vec{j}$.

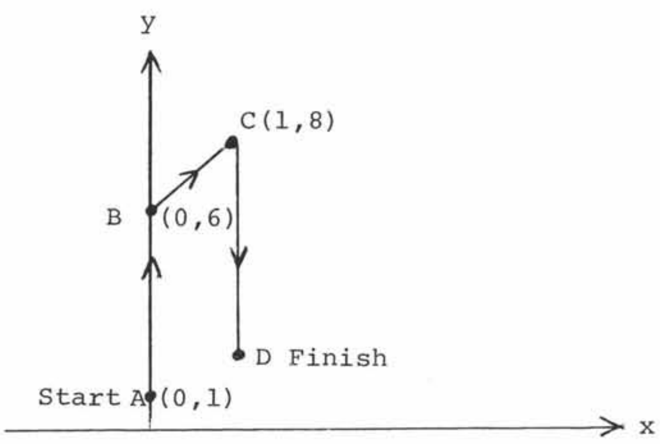
5.7.4(L) continued

Along the line $x = 0$, $\vec{F} = y\vec{i}$ so that the force is the direction of the x-axis and hence at right angles to the motion. Consequently, since the component of the force in the direction of motion is zero, no work is done in moving from (0,1) to (0,6) along the y-axis.

Along the line $y = 2x + 6$, $0 \leq x \leq 1$, notice that the $\vec{i}$ and $\vec{j}$ components of $\vec{F}$ are positive (since x and y are both positive there). Hence, the component of $\vec{F}$ in the direction of motion has the same sense as the motion so the force "assists" the motion, which accounts for the work being $+ 14\frac{5}{6}$.

Finally, in going from (1,8) to (1,2) along $x = 1$, we have that $\vec{F} = (y + 1)\vec{i} + y\vec{j}$, so since $2 \leq y \leq 8$ $\vec{F}$ "points" up and to the right. Consequently, the component of $\vec{F}$ in the direction of the motion has the opposite sense of the motion (since the motion is downward), and as a result the force is "against" the motion, which accounts for the work being -30.

Pictorially,



1. No work in going from A to B.
2. Particle has $14\frac{5}{6}$ units of work done on it in going from B to C.
3. Particle does 30 units of work in going from C to D.
4. Therefore, the net effect is that the particle does $15\frac{1}{6}$ units of work in going from A to D along the given path.

5.7.5(L)

Suppose there exists a function $F(x,y)$ such that $dF = Mdx + Ndy$, i.e., $F_x = M$ and $F_y = N$. Now let c by <u>any</u> (piecewise) smooth curve which joins $P_o(x_o,y_o)$ to $P_1(x_1,y_1)$. Let C be given by

$$\left.\begin{matrix} x = g(t) \\ y = h(t) \end{matrix}\right\} \text{ where } t \text{ varies from } t_o \text{ to } t_1$$

[i.e. $(x_o,y_o) = (g(t_o), h(t_o)$ and $(x,y) = (g(t_1), h(t_1))$]

Then

$$\left.\begin{aligned} &\int_C M(x,y)\,dx + N(x,y)\,dy \\ &= \int_{t_o}^{t_1}\{M[g(t), h(t)]g'(t) + N[g(t),h(t)]h'(t)\}\,dt \\ &= \int_{t_o}^{t_1}\{F_x[g(t), h(t)]g'(t) + F_y[g(t), h(t)]h'(t)\}\,dt. \end{aligned}\right\} \quad (1)$$

Notice that our final integrand in (1) is, <u>by the chain rule</u>, precisely

$$\frac{dF(g(t), h(t))}{dt}.$$

To see this, notice that if we let

$f(t) = F(g(t), h(t))$

then

$f(t) = F(x,y)$, where $x = g(t)$ and $y = h(t)$ (20)

5.7.5(L) continued

Applying the chain rule to (2) yields

$$f'(t) = F_x \frac{dx}{dt} + F_y \frac{dy}{dt}$$

$$= F_x g'(t) + F_y h'(t)^* \qquad (3)$$

In any event this means that (1) may be written as

$$\int_c M(x,y)\,dx + N(x,y)\,dy$$

$$= \int_{t_o}^{t_1} \left[\frac{dF(g(t),h(t))}{dt}\right] dt$$

$$= F(g(t),\ h(t)) \Big|_{t=t_o}^{t_1}$$

$$= F[g(t_1),\ h(t_1)] - F[g(t_o),\ h(t_o)]$$

$$= F(x_1,y_1) - F(x_o,y_o). \qquad (4)$$

The crucial fact concerning (4) is that F in no way depends on C. That is, F is determined by $F_x = M$ and $F_y = N$ without regard for the curve c which joins P_o to P_1. In fact, the only effect that the choice of c has is in determining $g(t)$, $h(t)$, t_o, and t_1 but $F(g(t_1), h(t_1)) = F(x_1,y_1)$ and $F(g(t_o), h(t_o)) = F(x_0,y_o)$ regardless of how g,h,t_o, and t_1 are defined.

*Notice that we are using the smoothness of c when we assume that g'(t) and h'(t) exist.

5.7.5(L) continued

In other words, if $Mdx + Ndy$ is an exact differential then the line integral $Mdx + Ndy$ along any curve which joins P_o to P_1 is independent of the choice of path, but rather depends only on the end points P_o and P_1.

Least we not fully appreciate what this says, what we are saying is that if $Mdx + Ndy$ is not exact and we pick a curve c that joins P_o and P_1, then certainly the line integral <u>for that particular path</u> is a function of P_o and P_1 alone; but different paths may lead to different functions of P_o and P_1. On the other hand, for an exact differential, once we know the value of the line integral along one path which joins P_o and P_1, we know that it has the same value along any path that joins P_o and P_1. (For an important subtlety,see optional Exercise 5.7.9)

5.7.6

a. Given $\int_c (1 + 3x^2y + 5x^4y^2)\,dx + (x^3 + 5y^4 + 2x^5y)\,dy$ (1)
we first observe that

$$\frac{\partial(1 + 3x^2y + 5x^4y^2)}{\partial y} = 3x^2 + 10x^4y = \frac{\partial(x^3 + 5y^4 + 2x^5y)}{\partial x},$$

so our integrand is exact.

In fact,

$$d(x + x^3y + y^5 + x^5y^2 + c)$$
$$(1 + 3x^2y + 5x^4y^2)\,dx + (x^3 + 5y^4 + 2x^5y)\,dy. \qquad (2)$$

[By way of review we may derive (2) by setting $F_x = x + x^3y + x^5y^2$, whereupon $F = x + x^3y + x^5y^2 + g(y)$.

Hence, $F_y = x^3 + 2x^5y + g'(y)$ but since $F_y = x^3 + 5y^4 + 2x^5y$, it follows that $g'(y) = 5y^4$, so $g(y) = y^5 + c$, and as a result $F = x + x^3y + x^5y^2 + y^5 + c$.]

5.7.6 continued

Thus, equation (1) becomes

$$\int_{(0,0)}^{(1,1)} d(x + x^3y + y^5 + x^5y^2 + c)$$

$$= [x + x^3y + y^5 + x^5y^2 + c\ ]_{(0,0)}^{(1,1)}$$

$$= [1 + 1 + 1 + 1 + c] - [0 + c]$$

$$= 4.$$

b. 1. In the event c is y = x, where x varies from 0 to 1, then $dy/dx = 1$, and

$$\int_c (1 + 3x^2y + 5x^4y^2)\,dx + (x^3 + 5y^4 + 2x^5y)\,dy$$

$$= \int_0^1 [(1 + 3x^3 + 5x^6)1 + (x^3 + 5x^4 + 2x^6)1]\,dy$$

$$= \int_0^1 (1 + 4x^3 + 5x^4 + 7x^6)\,dx$$

$$= x + x^4 + x^5 + x^7 \Big|_{x=0}^{1}$$

$$= 4.$$

2. C is given by $y = x^3$ as x varies from 0 to 1.

Hence, $dy/dx = 3x^2$, and

5.7.6 continued

$$\int_C (1 + 3x^2y + 5x^4y^2)\,dx + (x^3 + 5y^4 + 2x^5y)\,dy$$

$$= \int_0^1 [(1 + 3x^5 + 5x^{10})\,1 + (x^3 + 5x^{12} + 2x^8)\,3x^2]\,dx$$

$$= \int_0^1 (15x^{14} + 11x^{10} + 6x^5 + 1)\,dx$$

$$= x^{15} + x^{11} + x^6 + x \Big|_{x=0}^{1} = 4$$

c. $$= \int_{(0,0,0)}^{(1,1,1)} x^2dx + y^2dy + z^2dz$$

$$= \int_{(0,0,0)}^{(1,1,1)} d\left(\frac{1}{3}x^3 + \frac{1}{3}y^3 + \frac{1}{3}z^3 + c\right)$$

$$= \frac{1}{3}x^3 + \frac{1}{3}y^3 + \frac{1}{3}z^3 \Big|_{(0,0,0)}^{(1,1,1)}$$

$$= \frac{1}{3} + \frac{1}{3} + \frac{1}{3}$$

$$= 1.$$

d. 1. Now c is the space curve $x = y = z = t$; t varies from 0 to 1 so that

$$\frac{dx}{dt} = \frac{dy}{dt} = \frac{dz}{dt} = 1.$$

5.7.6 continued

Hence,

$$\int_C x^2dx + y^2dy + z^2dz$$

$$= \int_0^1 \left(x^2 \frac{dx}{dt} + y^2 \frac{dy}{dt} + z^2 \frac{dz}{dt}\right) dt$$

$$= \int_0^1 (t^2 + t^2 + t^2)\, dt$$

$$= \int_0^1 3t^2 dt$$

$$= t^3 \Big|_{t=0}^{1}$$

$$= 1.$$

2. C is given by

$$\left.\begin{aligned} x &= t^2 \\ y &= t^3 \\ z &= t^4 \end{aligned}\right\} \quad \text{t varies from 0 to 1.}$$

Hence

$\frac{dx}{dt} = 2t$, $\frac{dy}{dt} = 3t^2$, $\frac{dz}{dt} = 4t^3$, so that

$$\int_C x^2dx + y^2dy + z^2dz$$

$$= \int_0^1 \left(x^2 \frac{dx}{dt} + y^2 \frac{dy}{dt} + z^2 \frac{dz}{dt}\right) dt$$

5.7.6 continued

$$= \int_0^1 (2t^5 + 3t^8 + 4t^{11})dt$$

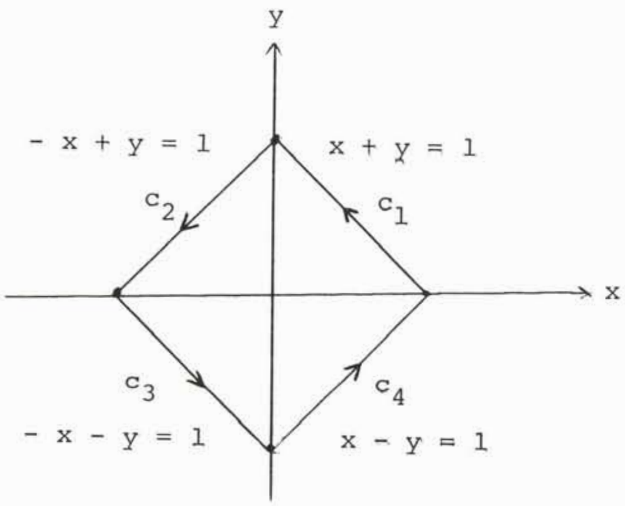
$$= \frac{1}{3}t^6 + \frac{1}{3}t^9 + \frac{1}{3}t^{12} \Big|_{t=0}^{1}$$

$$= 1.$$

5.7.7

Observe that

$$|x|+|y| = \begin{cases} x + y \text{ in 1st quadrant} \\ -x + y \text{ in 2nd quadrant} \\ -x - y \text{ in 3rd quadrant} \\ x - y \text{ in 4th quadrant} \end{cases}$$



$c = c_1 \cup c_2 \cup c_3 \cup c_4$

c_1: $y = 1 - x$ where x varies from 1 to 0; $dy/dx = -1$.
c_2: $y = 1 + x$ " " " " 0 to -1; $dy/dx = 1$.
c_3: $y = -1 - x$ " " " " -1 to 0; $dy/dx = -1$
c_4: $y = x - 1$ " " " " 0 to 1; $dy/dx = 1$.

5.7.7 continued

Hence,

$$\int_{c_1} ydx - xdy = \int_1^0 [y - x \frac{dy}{dx}] dx$$

$$= \int_0^1 (x \frac{dy}{dx} - y) dx$$

$$= \int_0^1 [- x -(1 - x)] dx$$

$$= \int_0^1 -1 dx$$

$$= -1. \qquad (1)$$

$$\int_{c_2} ydx - xdy = \int_0^{-1} (y - x \frac{dy}{dx}) dx$$

$$= \int_0^{-1} (1 + x) -x(1)] dx$$

$$= \int_0^{-1} dx$$

$$= -1. \qquad (2)$$

$$\int_{c_3} ydx - xdy = \int_{-1}^0 [y - x \frac{dy}{dx}] dx$$

$$= \int_0^{-1} (x \frac{dy}{dx} - y) dx$$

5.7.7 continued

$$= \int_0^{-1} [-x-(-1-x)]\ dx$$

$$= \int_0^{-1} (-\ x + 1 + x)\ dx = -1 \qquad (3)$$

$$\int_{c_4} ydx - xdy = \int_0^1 [y - x \frac{dy}{dx}]\ dx$$

$$= \int_0^1 [(x - 1) - x]\ dx$$

$$= \int_0^1 -\ 1\ dx = -1. \qquad (4)$$

Combining (1), (2), (3), and (4) yields

$$\int_c ydx - xdy = \sum_{i=1}^{4} \int_{c_i} ydx - xdy$$

$$= -1 + (-1) + (-1) + (-1)$$

$$= -4.$$

(The integral is not zero even though we have integrated along a closed path because ydx - xdy is not exact.)

5.7.8(L)

$$\int_c \vec{F} \cdot d\vec{s} \tag{1}$$

may be written as

$$\int_c [xy\vec{i} + (x^2 + y^2)\vec{j}] \cdot [dx\vec{i} + dy\vec{j}]$$

$$= \int_c xydx + (x^2 + y^2)\, dy$$

and this was the integral we evaluated in Exercise 5.7.1 and found to be $\frac{2}{3}$.

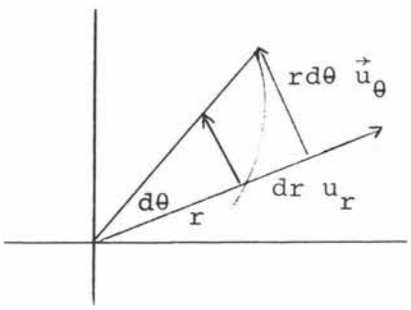
What we wish to do in this exercise is use polar coordinates and show that $\int_c \vec{F} \cdot d\vec{s}$ does not require that we use Cartisian coordinates.

Now in polar coordinates the basic unit vectors are $\vec{u}_r$ and $\vec{u}_\theta$ where

$$\left.\begin{aligned} \vec{u}_r &= \cos\theta\vec{i} + \sin\theta\vec{j} \\ \vec{u}_\theta &= -\sin\theta\vec{i} + \cos\theta\vec{j} \end{aligned}\right\} \tag{2}$$

and $d\vec{s} = dr\,\vec{u}_r + rd\theta\,\vec{u}_\theta$ (3)

[Pictorially we may remember (3) as



]

From (2) it follows that

5.7.8(L) continued

$$\vec{i} = \cos\theta\,\vec{u}_r - \sin\theta\,\vec{u}_\theta$$

$$\vec{j} = \sin\theta\,\vec{u}_r + \cos\theta\,\vec{u}_\theta\,.$$

Hence, in polar coordinates

$$\vec{F} = xy(\cos\theta\,\vec{u}_r - \sin\theta\,\vec{u}_\theta) + (x^2 + y^2)(\sin\theta\,\vec{u}_r + \cos\theta\,\vec{u}_\theta)$$

$$= (r\cos\theta)(r\sin\theta)(\cos\theta\,\vec{u}_r - \sin\theta\,\vec{u}_\theta) + r^2(\sin\theta\,\vec{u}_r + \cos\theta\,\vec{u}_\theta)$$

$$= r^2(\sin\theta\cos^2\theta + \sin\theta)\vec{u}_r - r^2(\sin^2\theta\cos\theta - \cos\theta)\vec{u}_\theta \qquad (4)$$

and we see in (4) that $\vec{F}$ could have been expressed without reference to Cartesian coordinates.

If we now use (3) and (4) to compute $\vec{F}\cdot d\vec{s}$ we have (since $\vec{u}_r\cdot\vec{u}_\theta = 0$ and $\vec{u}_r\cdot\vec{u}_r = \vec{u}_\theta\cdot\vec{u}_\theta = 1$)

$$\vec{F}\cdot d\vec{s} = r^2(\sin\theta\cos^2\theta + \sin\theta)\,dr - r^3(\sin^2\theta\cos\theta - \cos\theta)\,d\theta \qquad (5)$$

and since C is defined by $r = 1$, θ varies from 0 to $\frac{\pi}{2}$; we see from (5) that

$$\int_C \vec{F}\cdot d\vec{s} = \int_0^{\frac{\pi}{2}} [(\sin\theta\cos^2\theta + \sin\theta)(0) - (\sin^2\theta\cos\theta - \cos\theta)]d\theta$$

$$= -\frac{1}{3}\sin^3\theta + \sin\theta\,\Big|_{\theta=0}^{\frac{\pi}{2}}$$

$$= -\frac{1}{3} + 1$$

$$= \frac{2}{3}$$

which agrees with our answer to Exercise 5.7.1.

The main reason that one elects to use Cartesian coordinates and write $\int Mdx + Ndy$ is that the basic unit vectors in this

5.7.8(L) continued

case are i and j which are constant vectors while the basic unit vectors in other coordinate systems need not be. For example while $\vec{u}_r$ and $\vec{u}_\theta$ always have unit magnitude they change direction and hence are not constant vectors.

5.7.9 (optional)

a. In the statement of Exercise 5.7.5, we mentioned that Mdx + Ndy was not only exact on C but also in a region that contained C. The main reason for this additional requirement is that when we talk about the various curves which join two given points, we are admitting that the integrand must be defined at all points on each of the curves under consideration. In other words, we are assuming that Mdx + Ndy is exact in a region containing C and that all curves which join the two points under consideration lie in this region.

Now, at first glance it appears that

$$\frac{y}{x^2 + y^2}\,dx - \frac{x}{x^2 + y^2}\,dy$$

is exact. Indeed

$$\frac{\partial\left(\frac{y}{x^2 + y^2}\right)}{\partial y} = \frac{(x^2 + y^2) - y(2y)}{(x^2 + y^2)^2}$$

$$= \frac{x^2 - y^2}{(x^2 + y^2)^2} \qquad (1)$$

while

$$\frac{\partial}{\partial x}\left[\frac{-x}{x^2 + y^2}\right] = \frac{-(x^2 + y^2) + x(2x)}{(x^2 + y^2)^2}$$

$$= \frac{x^2 - y^2}{(x^2 + y^2)^2} \qquad (2)$$

5.7.9 (optional) continued

so that it indeed appears that

$$\frac{\partial\left(\frac{y}{x^2 + y^2}\right)}{\partial y} = \frac{\partial\left(\frac{-x}{x^2 + y^2}\right)}{\partial x},$$

and, accordingly, that

$$\frac{ydx}{x^2 + y^2} - \frac{xdy}{x^2 + y^2} \tag{3}$$

is exact. [In fact with a little "luck" we might even have observed that

$$d(\text{arc tan } \frac{x}{y}) = \left[\frac{1}{1 + (\frac{x}{y})^2}\right] d(\frac{x}{y})$$

$$= \left[\frac{y^2}{x^2 + y^2}\right]\left[\frac{ydx - xdy}{y^2}\right]$$

$$= \frac{ydx - xdy}{x^2 + y^2}$$

$$= \frac{ydx}{x^2 + y^2} - \frac{xdy}{x^2 + y^2}\ ]$$

<u>However</u>, notice that (1) and (2) are undefined at (0,0).

In other words, expression (3) is exact in any region R <u>which excludes the origin</u>.

b. In our particular problem, we are computing

$$\int_c \frac{ydx}{x^2 + y^2} - \frac{xdy}{x^2 + y^2}$$

along two curves c which between them enclose the origin. Hence, it should not be too shocking in this case if the integral is dependent on the path even though the integrand is exact and defined for each point along the <u>two</u> given paths.

5.7.9 (optional) continued

In any event we have that c_1 is given by

$$\left.\begin{array}{l} x = \cos t \\ y = \sin t \end{array}\right\} \text{ as t varies from 0 to}$$

so that on c_1 $\frac{dx}{dt} = -\sin t$, $\frac{dy}{dt} = \cos t$, and $x^2 + y^2 = 1$. Hence

$$\int_{c_1} \frac{ydx}{x^2 + y^2} - \frac{xdy}{x^2 + y^2}$$

$$= \int_0^{\pi} [\frac{y}{x^2 + y^2} \frac{dx}{dt} - \frac{x}{x^2 + y^2} \frac{dy}{dt}] dt$$

$$= \int_0^{\pi} [\frac{\sin t}{1}(- \sin t) - \frac{\cos t}{1} (\cos t)] dt$$

$$= \int_0^{\pi} (- \sin^2 t - \cos^2 t) dt$$

$$= - \int_0^{\pi} dt$$

$$= - \pi.$$

On the other hand c_2 may be viewed as being defined by

$$\left.\begin{array}{l} x = \cos t \\ y = \sin t \end{array}\right\} \text{ as t varies from } 2\pi \text{ to } \pi.$$

Thus,

5.7.9 (optional) continued

$$\int_{c_2} \frac{ydx}{x^2 + y^2} - \frac{xdy}{x^2 + y^2}$$

$$= \int_{2\pi}^{\pi} (-\sin^2 t - \cos^2 t)\, dt$$

$$= -\int_{2\pi}^{\pi} dt$$

$$= \pi.$$

Since $\pi \neq -\pi$ we see that

$$\int_{c_1} \frac{ydx}{x^2 + y^2} - \frac{xdy}{x^2 + y^2} \neq \int_{c_2} \frac{ydx}{x^2 + y^2} - \frac{xdy}{x^2 + y^2}$$

even though the integrand is exact on both c_1 and c_2, and the curves have the same endpoints.

What we can positively conclude, however, is that if R is any region which does not include the origin and if P_o and P_1 are points in R then

$$\int_{c_1} \frac{ydx - xdy}{x^2 + y^2} = \int_{c_2} \frac{ydx - xdy}{x^2 + y^2}$$

along <u>every</u> pair of curves c_1 and c_2 <u>which lie in R</u> and extend from P_o and P_1.

5.7.10 (optional)

This exercise is meant as an "excuse" to give a definition of line integral which is independent of the notion of work or dot product.

5.7.10 (optional) continued

First of all let us observe that

$$\int_c \vec{F} \cdot d\vec{s} \tag{1}$$

can be written in a somewhat more scalar* form

$$\int_c (\vec{F} \cdot \vec{u})\ ds \tag{2}$$

where $\vec{u}$ is a unit tangent vector to the curve c at the point in question.

Notice also that (2) can be obtained from (1) mechanically by defining the vector increment ds to be the scalar increment ds multiplied by u; that is,

$d\vec{s} = (ds)\ \vec{u}.$

If we recall that $\vec{F} \cdot \vec{u}$ is the projection of $\vec{F}$ in the direction of $\vec{u}$ (i.e., in the direction of $d\vec{s}$) we see that the integrand in (2) does represent the component of the force in the direction of the motion.

Our first point is that formula (2) is well defined independently of the work concept. For example, if we think of s as a parameter we may write that

$$\int_c Mdx + Ndy$$

$$= \int_c \left(M \frac{dx}{ds} + N \frac{dy}{ds}\right) ds. \tag{3}$$

*We say scalar since $\int_c (\vec{F} \cdot \vec{u})\, ds$ indicates an ordinary integral (since s is a scalar) and $\vec{F} \cdot \vec{u}$ is also a scalar.

5.7.10 (optional) continued

If we now observe that

$$M \frac{dx}{ds} + N \frac{dy}{ds} = (M\vec{i} + N\vec{j}) \cdot \left(\frac{dx}{ds}\vec{i} + \frac{dy}{ds}\vec{j}\right)$$

and that $\frac{dx}{ds}\vec{i} + \frac{dy}{ds}\vec{j} = \vec{u}$, a unit tangent vector to c, then if we let $K = K(x,y) = M(x,y)\,\vec{i} + N(x,y)\,\vec{j}$, formula (3) becomes

$$\int_c (\vec{K} \cdot \vec{u})\, ds \tag{4}$$

and clearly (4) and (2) have an identical structure.

Notice also that since $\vec{K}$ and $\vec{u}$ are both vector functions of x and y (in Cartesian coordinates) we may let $H = H(x,y) = K(x,y) \cdot u(x,y)$ and in this event formula (4) becomes

$$\int_c H ds. \tag{5}$$

Written as in (5) our line integral is free of any vector interpretation, including that of the dot product.

Our final aim is to free the definition even more by not making it dependent upon arc length. This is very similar to our treatment of calculus of a single-variable in which we start with the "standard" form

$$\int_a^b f(x)\, dx \tag{6}$$

known as a Riemann or definite integral, and then proceed to talk about a more general integral

$$\int_a^b f(x)\, dg(x) \tag{7}$$

called the Riemann-Stieltjes integral.

Among other places we encounter the Riemann-Stieltjes integral (often called more concisely, the Stieltjes integral) in the formula for integration by parts when we write

5.7.10 (optional) continued

$\int u\,dv = uv - v\,du.$

Observe that the Stieltjes integral is a generalization of the definite integral in the sense that $\int_a^b f(x)dx$ is the special case of $\int_a^b f(x)dg(x)$ in which $g(x) = x$.

Omitting the rigorous details, recall that the Stieltjes integral is evaluated by the familiar "change-of-variables" technique. Namely,

$$\int_a^b f(x)dg(x) = \int_a^b f(x)g'(x)dx \qquad (8)$$

where definition (8) is perfectly meaningful provided that f is (piecewise) continuous on [a,b] and that g(x) is (piecewise-) continuously differentiable on [a,b] (since in that case $\int_a^b f(x)g'(x)dx$ exists).

It is from the point of view of the Stieltjes integral that one can define a line integral as an extension of the ordinary definite integral and this is the aim of this exercise.

Namely:

Let c be a (piecewise) smooth curve in E^2 with parametric form

$$\left.\begin{aligned} x &= f(t) \\ y &= g(t) \end{aligned}\right\} \quad a \leq t \leq b$$

and assume that $H = H(x,y)$ is any function which is (piecewise-) continuous on c. Then, if $\gamma(t)$ is any (piecewise-) differentiable function defined on [a,b] we define the line integral $\int_c H d\gamma$ to be the "ordinary" Riemann integral

$$\int_a^b H(f(t), g(t))\, \gamma'(t)\, dt.$$

5.7.10(optional)continued

Observe that in the special case that $\gamma(t)$ denotes arc length the line integral coincides with our more intuitive physical definition.

b. Since c is given by

$$\left.\begin{aligned} x &= t^2 [= f(t)] \\ y &= t^4 + 1\ [= g(t)] \end{aligned}\right\} \quad 0 \le t \le 1,$$

$H(x,y) = x^2 + y^2$, and by $\gamma(t) = t^3$ for $0 \le t \le 1$, we have

$$\int_c H d\gamma = \int_0^1 H(t^2, t^4 + 1) \frac{d(t^3)}{dt} dt$$

$$= \int_0^1 [(t^2)^2 + (t^4 + 1)^2] \, 3t^2 \, dt$$

$$= \int_0^1 [t^4 + t^8 + 2t^4 + 1] \, 3t^2 dt$$

$$= \int_0^1 (9t^6 + 3t^{10} + 3t^2) dt$$

$$= \frac{9}{7} t^7 + \frac{3}{11} t^{11} + t^3 \Big|_{t=0}^{1}$$

$$= \frac{9}{7} + \frac{3}{11} + 1$$

$$= 2 \, \frac{43}{77} .$$