HERBERT GROSS: Hi. Today we tackle another facet of applications of complex numbers in our survey of our complex variable mini course. And in particular, what we're going to study today is the role of infinite series and sequences in complex numbers. And I call today's lesson, therefore, Sequences and Series.

Now, among other things, let's take a look at a purely pedagogical problem. For example, at this stage of the game, notice that I can't even talk meaningfully about what I mean by e to the z, nor by what I mean by sine z. I can not visualize z as an angle, because it may be imaginary. I can't visualize it as a length in the traditional sense that we view functions from real variables into real variables. And the question that comes up is, how could I define e to the z and sine z?

In fact, the cynic might say, why would you want to define e to the z and sine z? Now, one very cynical answer to the cynic would simply be that one of the most practical reasons for knowing how you define either the z and sine z is that they will be on the quiz at the end of this particular block. But that's hardly a fair motivation.

I will go into this in more detail later in the lecture. Let me simply point out that, every time we have an analytic function, its real and imaginary parts define a conformal mapping. And consequently, every time we can meaningfully invent an analytic function, we have automatically invented a new conformal mapping. As I say, we'll go into that in more detail as we go along.

But the question that now comes up is, how can I define e to the z? How can I define sine z, cosh z, all of these things? And one of the best ways of beginning is, again, to emphasize something that we said in one of our earlier elections on complex numbers-- that whatever definitions we invent, we want to make sure that the new definition does not contradict the old one. In other words, whatever definition I give of e to the z, I want to make sure that, if z happens to be real, if I replace z by x, then e to the x has the same meaning whether I view x as a real number or whether I view it as a complex number.

Now, one of the standard ways of doing something like this is the following. We, in part one of our course, had shown that the power series representation for e to the x was given by this infinite series where this infinite series converged to e to the x for all real values of x. Now, what we can therefore say is, look at-- if I just verbatim, every place I see an x, replace that by a z and now make up the definition, that e to the z will be summation n goes from zero to infinity, z to the n over n factorial, then no matter what else happens, I'm sure that my definition of e to the z must agree with my definition of e to the x whenever z and x are equal-- in other words, whenever z happens to denote a real number.

Why is that the case? Because I got the new definition by lifting it from the old, just replacing the x by a z. The question that comes up of course is, what do you mean by this expression? What do you mean by a series when you're dealing with complex numbers? In particular what it means is that, given a sequence of complex numbers, I must define what I mean by a limit and then go through this idea of defining a series to be a sequence of partial sums and going through the same bit structurally that I went through in the real case.

Now let's do something that I think is very interesting over here. Let's just write down what the definition was for a limit of a sequence a sub n to equal l as n went to infinity. What was that definition in the real case? Limit as n approaches infinity a sub n equals l means, given any epsilon greater than 0, there exists a capital N such that, whenever a little n exceeds capital N, the absolute value of a sub n minus l is less than epsilon. I hope this is very familiar to you. If not, feel free to review it.

But here is the interesting point. No place in here do I mention that a sub n and l have to be real numbers. In other words, when I made up this definition, I assumed that they were. But notice that no place in here in this definition do I say that a sub n and l are real. In fact, where are the only places that I use realness? That since epsilon is real, I would like to believe that, no matter what a sub n and l are, that the magnitude-- the absolute value of a sub n minus l-- must be less than epsilon means that the magnitude of a sub n minus l must be a real number, among other things-- in fact, preferably a positive real number.

Notice that the magnitude of a complex number is still a non-negative real. Consequently, even if a sub n and l are complex, this definition still holds. And in fact, the pictorial interpretation of the definition holds precisely the same as it did in the real case, except that an epsilon neighborhood of l is an interval in the real case. And it's a circle, a disk-- I shouldn't say a circle.

Remember, the circle was the thing we call the circumference. The disk is the inside of the circle. The neighborhood is a disk in the complex case. Namely, notice that to say that the absolute value of a sub n minus l is less than epsilon still means that a sub n is with an epsilon of l. The difference is that the domain of l is now the xy plane, the Argand diagram.

So to be with an epsilon of l, I now take a circle centered at l with radius epsilon and draw that circle. And what I'm saying is that if the limit of the sequence is l, after a certain number of terms, all the remaining terms are inside the circle. In other words, just as in the real case, in the complex case, the limit replaces an infinite number of points-- numbers, you see, an infinite number of points-- by a finite number plus one dot. In other words, there's a bunch of numbers that may lie outside the circle.

But after a point, after a certain term, all of the remaining terms lie inside this particular disk. And structurally, you see, that's exactly the same thing that happened in the real case. Consequently, since the structural definition is the same, since the geometric interpretation is the same except that disks have replaced intervals, I would expect that all the limit theorems held, just as in the real case. And indeed, they do. And we'll talk about those in the exercises.

In a similar way, though, that we went from sequences to series in the real case, we can now do the same thing in the complex case. Namely, suppose c1, c2, c3, et cetera represent a sequence of complex numbers by the infinite sum and goes from 1 to infinitely c sub n. I simply mean the limit as n goes to infinity c1, plus et cetera, cn, noticing that this is-- what? One number, it depends on n. And that's what you'll call the n-th partial sum of the series that was added up c1 through cn.

The next term would be obtained by adding on c sub n plus 1, et cetera. But notice again the similar structure. And consequently, the usual theorems for series that apply to the real variable case apply to the complex variable case. Again, what's the only difference? The only difference will be that intervals of convergence will be replaced by disks or circles of convergence.

In particular, if we let s denote the set of all complex numbers z for which the power series summation a nz to the n converges then, just as in the real case, one of three things must happen. Namely, either s consists solely of zero. See, obviously, if I replace z by zero, this thing converges. It's zero, in fact. Or it may be such that these terms go to zero so rapidly that this will converge for all complex numbers, c-- in other words, that the set S can be all complex numbers.

Usually, you'll get something in between these two extremes. And that's the case, like just in the real case, there exists a number capital R greater than zero such that the set S consists of all of those z's such that the absolute value of z is less than R.

And by the way, what does this mean geometrically? The absolute value of z is the magnitude of z. That's the distance of z from the origin. This says that z is within capital R of the origin. That means you're inside the circle, centered with the origin, with radius, R.

At any rate, going on the convergences both absolute and uniform in any interior disk-- in other words, inside the disk absolute value of z is less than or equal to little r where a little r is less than capital R. Again, in terms of a picture, what we're saying is, if the first two conditions don't hold given a power series, at the origin, I draw a circle. I don't draw a circle. What I'm saying is there exists a circle such that the power series will converge every place inside here.

It will diverge every place outside here. And what happens on the boundary must be investigated separately, again, just as in the real case. OK? That's what the key theorem is. And again, you'll have time to reread this and to digest it in your leisure before you try the exercises.

Now, once we've define what we mean by a power series-- and notice, by the way, that absolute convergence goes through word for word in the complex value case because, after all, in terms of absolute convergence, we look at absolute values. And absolute values are non-negative reals, regardless of whether the thing that we're looking at happened to be complex numbers or real numbers to begin with. Magnitudes are non-negative.

At any rate, one can then show, in precisely the same way that we did it in the real variable case, that this particular power series-- forget about this being e to the z right now. I make up this power series. I use the ratio test. And I can show, by the ratio test, that this series converges uniformly and absolutely for all real values of z, which-- I'm sorry, for all finite values of z. I don't mean real values. I mean finite values. Z does not have to be real.

Now what I say is, because this is uniformly convergent, this power series behaves like a polynomial. I can differentiate it term by term. I can integrate it term by term, et cetera, et cetera, et cetera. I'll now give this a name. And being very, very judicious, what name do I pick?

I pick e to the z, because that's what motivated my inventing this function, this series, in the first place. And what am I now sure of? I'm sure of the fact that, if I replace z by x-- in other words, if z is real-- that this must still be the same as what e to the x was before. And what's the best proof of that? Replace z by x.

What you get is summation. n goes from 0 to infinity, x to the n over n factorial. And we saw in part 1 of our course that that was precisely e to the x. Well, since exponentials may bother you a little bit more than sines and cosines, let's revisit the same problem in terms of sine z.

What I do is I define sine z to be this uniformly, absolutely, convergent power series. Again, using hindsight-- knowing that, in the case of sine x, this was the convergent power series-- I simply replace x by z up here. All right, I replace x by z. I now use the ratio test on this to show that this does converge for all finite z.

In fact, that's sort of redundant to do that. I know darn well that's going to happen, because it happened here. And structurally, these are the same. But I now then, after showing that, I say, OK, this converges for all z. Let's give it a name. I'll call it sine z. And what guarantee do I have that sine z is the same as sine x whenever z is real?

And again, let me point out something. Notice, in the real case, we started knowing what sine x was but not knowing what its power series expansion was. And we developed the power series expansion.

And by the way, notice that when you want to compute sine x to any number of decimal places, whether you're doing it by hand or by the computer, notice that it's the power series that you use. And you chop this off after a certain number of terms, if you wish. But one does use the power series idea.

What I am saying though is, had I wished to, from a purely pedagogical point of view I could have said, let me now-- forget about this definition of sine x and define sine x just by this particular power series. That would have been artificial to do at that time. What I'm saying now, though, is this-- start with sine x in the real case. Develop its convergent power series. Take that power series, replace x by z, and define that new complex powers series to be what sine z is. And that gets the job done for you.

Now, the interesting thing is that one can now show that all of the trigonometric functions, be they hyperbolic or circular functions, are related to e to the z. Well, why that's interesting is not very important right now. We'll mention that in more detail on other occasions, hopefully.

But the idea is this. By using the fact that these various series are absolutely and uniformly convergent so I can manipulate them term by term-- and these things are done in detail in the text, and I will give you other ones to do for homework so that you get the drill on this-- one can show the very amazing results that e to the iz is cosine z plus i sine z, a very amazing result that relates the cosine and the sine to an exponential. In fact, to stress this from a different point of view, if I take the special case that z is real, notice that this says-- what? e to the ix-- so you replace z by x-- is cosine x plus i sine x.

In particular, what this then says is, let's go back to the Argand diagram where we take the complex number, view it in polar coordinates as r comma theta-- which means what? r cosine theta plus ir sine theta. Factor out the r, so it's r times cosine theta plus i sine theta. But cosine theta plus i sine theta is e to the i theta.

And what that says is, that if you really wanted a nice name for this, r comma theta does turn out in the language of polar coordinates using complex variables without Cartesian reference is re to the i theta. Which, by the way, would explain very nicely why, when you multiply complex numbers, you multiply the magnitudes and add the arguments. Namely, notice that the magnitudes are these scalar factors in front of the exponential.

The arguments occur as exponents. And when you multiply two numbers to the base e, you add the exponents. In fact, when you multiply numbers to any base, you add the exponents. And that's why you add the angles over here.

Again, a very interesting aside, which is worth absolutely nothing-- I think it was the first thing that ever turned me on in mathematics that got my metaphysical mathematical dander up-- was, if you replace x by pi over here, what does that mean if I replace x by pi? The magnitude is one. The angle is pi. e to the i pi-- e to the i pi is this another way of writing minus 1.

What is e to the i pi? The argument is pi. That means 180 degrees. The argument is one. And the number, which is one unit from the origin in the Argand diagram at an angle 180 degrees, is the real number minus 1. I've always been mystified by the three most remarkable numbers.

Constants in mathematics seem to be ei and pi. And e to the i pi turns out to be minus 1. I just mentioned this to show you how complicated things can get if you try to read too much meaning into things. But don't worry about that right now. I couldn't resist the comment.

What I would like to do though is to show you now how this is used to define other functions and get us other results. Among other things, notice that the first bad thing that happens with the polar coordinate representation is, if the complex number z can be written as re to the i theta, it can also be written as re to the i theta plus 2 pi k because, every time I changed my angle by 2 pi or 360 degrees, I come back to the same point again.

At any rate, forgetting about that problem for the time being, notice how I can now define a logarithm for a complex number. In fact, the logarithm will turn out to be the inverse of e to the z, the same as before. But we won't worry about that here either.

Let's see how we could define log z from this now. We would like the log to have the usual logarithmic properties. The log of a product should be the sum of the logs. So log z should be log r plus log e to the i theta plus 2 pi k.

Now, r is a positive real number. We already know that the natural log applies there. So what we say is, OK, the log of r is just natural log r. Log e to anything-- the log in the e-- are inverse of one another. So they should cancel. And this will just be i theta plus 2 pi k.

In other words, to find the log of the complex number z, you simply take the log of its magnitude and add on, as the imaginary part, what? The argument-- noticing, by the way, that the argument is multi-valued. See, log z is multi-valued. Consequently, we must get into the idea of principal values.

And one usually assumes, unless otherwise stated, that the argument is no greater than pi but greater than minus pi. If we don't mean that, we have to state that separately. Again, we will drill on that in the exercises.

By the way, another application that relates to hyperbolic functions to the circular functions-- notice that, if we wanted to define hyperbolic cosine of ix mimicking what happened in the real case, this would be e to the ix plus e to the minus ix over 2. We already saw that e to the ix was cosine x plus i sine x. e to the minus ix is e to the i of minus x-- that's cosine minus x-- plus i sine minus x.

By the way, I hope that's clear to you. When I say that e to the ix is cosine x plus i sine x, what I really mean is-- or even up here. What I really mean is that e to the i anything is cosine of that anything plus i sine of that anything. In particular, if I replace z by minus x, I get e to the i minus x. In other words, minus ix is cosine minus x plus i sine minus x.

Cosine of minus x equals cosine x. Sine minus x is minus sine x. These terms cancel. This becomes 2 cosine x. 2 cosine x divided by 2 is cosine x. And we have the rather amazing identity that cosh ix is cosine x-- in other words, a very interesting relationship between the hyperbolic and the circular functions.

And by the way, this shouldn't be too surprising. Notice that the circular functions come from studying x squared plus y squared equals 1. The hyperbolic functions come from studying x squared minus y squared equals 1.

And x squared minus y squared can be written as x squared plus the quantity iy squared. So what would be a hyperbole in the xy plane would be a circle in the x iy plane, whatever that might be. But again, that's just in the form of an aside.

And now we'll come back to what we mean by saying that we can use these new analytic functions to do conforming mappings. Let's come back to the question that we raised at the very beginning of our lecture about e to the z. Certainly, because z is x plus iy, each of the z will be e to the x plus iy because of the properties that the exponential will have.

And by the way, all of these exponential properties can be proven from the power series. You see, we don't need the real interpretation. The power series will give us all these results. But again, we'll drill on those in the exercises. I just want to get the highlights here.

This is e to the x times e to the iy. But notice here, you see that y is real. Or even if it wasn't real, it doesn't make any difference. e to the iy is cosine y plus i sine y. Multiplying this out, each e to the z is e to the x cosine y plus ie to x sine y.

The important thing being that, because y is real, e to the x cosine y and e to the x sine y are both real. And consequently, if I let you denote the real part and v denote the imaginary part, as usual, what I have is that u equals e to the x cosine y, v equals e to x sine y. Must be a real conformal mapping.

Why? Because the real and the imaginary parts-- the real and the imaginary parts-- make up an analytic function. How do I know the function is analytic? Because its power series exists. And the fact that its power series exists means that all of the derivatives must exist.

Now, again, I didn't write this in here. But maybe it's a worthwhile aside to show you what this conformal mapping looks like. For example, notice that, if I square both sides of each equation here and add-- if I square and add, what do I get?

On one side, I get u squared plus v squared. On the other side, I get e to the 2x cosine squared y plus e to the 2x sine squared y. I factor out e to the 2x. And that gives me e to the 2x sine squared y plus cosine squared y, which is 1. So this is just u squared plus v squared is e to the 2x.

To eliminate x from the equations, I can divide the bottom by the top. And I get that v over u is equal to tangent y. And by the way, notice what this thing tells me? This tells me-- see, what are the basic lines in the xy plane that aligns x equal to constant, y equal to constant?

Notice that, if x is a constant, this says that u squared plus v squared equals e to a constant power. That's u squared plus v square is a constant. That's a circle, centered at the origin. In other words, the lines x equal to constant map into concentric circles.

On the other hand, if y is a constant, this says that v over u is a constant. In the uv plane, v over u is a constant is a straight line that goes through the origin. See, v is a constant times u.

Now, what that means pictorially-- let's see if I have some room to squeeze this in here. What this means pictorially is that this kind of a network in the xy plane-- this kind of a network in the xy plane is mapped into what? This network is mapped into a what? The vertical lines go into concentric circles. And the horizontal lines go into lines through the origin here.

Notice, by the way, that every line through the origin meets the circle at right angles. Notice that these lines were at right angles here. Notice the conformal property here. In other words, lines that mean at right angle in the xy plane must meet at right angles in the uv plane. And let me separate this out from anything else that I've done so that we don't get too mixed up with this.

By the way, one of the ways that this is used in application is with the inverse mapping-- that, in certain cases where we have to deal with regions that have this kind of a shape, we back map this region. And what will it go into? It will go into a rectangle here. And for example, in solving Laplace's equation, it might be a lot easier to solve Laplace's equation on the boundary of a rectangle than on the boundary of a region of this particular type.

Again, I don't want to go into too much detail here, because it obscures what the main overview is-- namely, every time we invent an analytic function like e to the z, sine z, cosh z, et cetera, we have invented a new conformal mapping. Whether that conformal mapping will help us in a particular application or not depends on the application. But what the meaning of it is is independent of any application.

Now let's take one more important reason. You see, what I'm afraid of now is I'm giving you the impression that complex numbers have value only if you're doing conformal mappings. You see, complex numbers give you a degree of consciousness, so to speak, above the real numbers. Let me give you an example of another problem that used to hang me up when I was an undergraduate student learning calculus.

I'm going to apply complex value results to real series now. Remember the geometric series 1 over 1 minus u is 1 plus u plus u squared, et cetera-- in other words, summation n goes from 0 to infinity, u to the n. And that converges for the absolute value of u less than 1.

Let's suppose now I replace u by minus x squared. If I do that, I get 1 over 1 plus x squared is 1 minus x squared plus x to the fourth minus x to the sixth, et cetera. In other words, summation n goes from 0 to infinity. I want my sines to alternate with the first one being positive. So I put in minus 1 to the n, x to the 2n. And that converges for the absolute value of x less than 1.

Now let me just focus my attention on a question for you here. And I'll come back to this. I want to come back here in a second. But the idea is, what goes wrong when x equals plus or minus 1? You see, let me show you what I mean by that.

When I saw this series for the first time, and somebody says, look at, you got to be careful when the absolute value of u is equal to 1-- in other words, be careful when u equals 1 or minus 1. Well, minus 1 didn't bother me that much. But 1, I could see right away what went wrong. Namely, when u is 1 here, 1 over 1 minus u is 1 over 0, which is undefined. And I expect trouble over here.

On the other hand, when the absolute value of x equals 1 here, that means that x squared is 1. I mean, whether x is 1 or minus 1, x squared is 1. Look at-- nothing goes wrong over here. 1 over 1 plus x squared is perfectly well defined at x equals 1. I would expect-- because my upbringing was such that I always thought that the bad points occurred where the denominator blew up.

But see, this denominator never blows up, because 1 plus x squared is always at least as big as 1. Provided what, of course? That x is real, because a square of a real number can't be negative. That's the hint, by the way. The square of a complex number could certainly be negative.

And here's what I'm saying. If I now take this real-valued series and convert it to a complex-valued series by replacing x by z using the same ratio test, et cetera, that we did before, I can prove that this infinite series, this power series, converges to 1 over 1 plus z squared provided the absolute value of z is less than 1.

By the way, what does it mean to say that the absolute value of z is less than 1? It means that z is less than 1 unit from the origin. It means that, in the Argand diagram, z is inside the circle, centered at the origin, with a radius equal to 1.

But here's the key point. When I look at this, I'm not at all upset that something can go wrong on this circle. Namely, 1 over 1 plus z squared factors into 1 over z plus i times z minus i. And certainly, I am leery of letting z either equal minus i or i, because it's going to make my denominator vanish. And even in the complex numbers, as you recall from our first lecture on complex numbers, you cannot divide by 0.

In other words, at a glance, I recognize that z equals plus or minus i is trouble. In other words, going to the Argand diagram in this picture, I know I'm in trouble at these two points on the circle, absolute value of z equals 1. Remember, this is a circle. It's the locus of all points in the Argand diagram, one unit from the origin.

Now, remember what we said about the circle of convergence-- that that set S, where the power series converges-- there is a circle such that, inside the circle, the thing always converges. Outside, it always diverges. Consequently, the fact that I have two bad spots over here tells me that I cannot go beyond the circle in talking about what happened with the power series expansion for 1 over 1 plus z squared.

Notice that all I require is that something go wrong on the circle. There are infinitely many points on this circle. And the likelihood that the point at which things went bad happened to be at the point where the circle crossed the x-axis is particularly small.

In other words, speaking from another point of view, if I have the good fortune to be able to be standing in the complex number world looking down at this circle, I see what went wrong. If my perspective is limited to the x-axis, I can't see anything that went wrong. And I can't understand why I can't expand this circle further.

In other words, the bad spots are on the magnitude of z equals 1, but not at the point z equals 1 or z equals minus 1. These were good points. See, these were good points. These were the bad ones. But all you needed was one bad apple to spoil the whole circle, so to speak.

At any rate, I think that's enough to make my point-- the point being that, by studying complex series, we do two things. One is we get a larger hold on the set of analytic functions, which help us in things like conformal mappings and other things, which we haven't gone into. And secondly, it gives us a new perspective at series involving real power series, because we now have an extra dimension from which to view the power series.

At any rate, we'll talk about other aspects of complex variables-- namely, integration-- next time. For the time being, do the exercises in this unit. And until next time, goodbye.

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