CALCULUS REVISITED
PART 3
A Self-Study Course

STUDY GUIDE
Block 3
Selected Topics
in Linear Algebra

Herbert I. Gross
Senior Lecturer

Center for Advanced Engineering Study
Massachusetts Institute of
Technology

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BLOCK 3:
SELECTED TOPICS IN LINEAR ALGEBRA

Pretest

1. Let $V = [u_1, u_2, u_3]$ and suppose $\alpha_1 = (1,2,3)$, $\alpha_2 = (2,4,6)$, $\alpha_3 = (3,7,8)$, $\alpha_4 = (1,3,2)$, and $\alpha_5 = (1,-2,7)$. What is the dimension of the space spanned by $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$?

2. Let dim V = 4 and assume that $\{u_1, u_2, u_3, u_4\}$ is the coordinate system being used for denoting the elements of V as 4-tuples. Let W be the subspace of V spanned by (1,1,3,4), (2,3,7,9), (3,-2,4,7), and (4,-5,3,7). Express x_4 as a linear combination of x_1, x_2, and x_3 if it is known that $(x_1, x_2, x_3, x_4) \in W$.

3. Let $V = [u_1, u_2, u_3]$ and let $f: V \rightarrow V$ be the linear transformation defined by

$$\begin{cases} f(u_1) = u_1 + 2u_2 + 3u_3 \\ f(u_2) = 2u_1 + 5u_2 + 8u_3 \\ f(u_3) = u_1 + 4u_2 + 7u_3 \end{cases}$$

Describe the set $\{v: f(v) = 0\}$.

4. Use row-reduction techniques to evaluate

$$\begin{vmatrix} 1 & 1 & 1 & 2 & 3 \\ 2 & 3 & 2 & 4 & 5 \\ 2 & 5 & 3 & 4 & 7 \\ 3 & 4 & 4 & 4 & 8 \\ 4 & 9 & 2 & 3 & 9 \end{vmatrix}$$

5. Let $V = [u_1, u_2, u_3, u_4]$ and let $f: V \rightarrow V$ be the linear transformation defined by

$$f(u_1) = 8u_1 + 4u_3$$

$$f(u_2) = 9u_1 + 2u_2 + 6u_3$$

$$f(u_3) = -9u_1 - 4u_3$$

$$f(u_4) = 2u_2 + 3u_4$$

(a) What are the characteristic values of f?

(b) Describe $V_2 = \{v : f(v) = 2v\}$.

6. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{bmatrix}$.

Find a matrix B such that

$$BAB^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

7. Define f on $(-\pi, \pi)$ by

$$f(x) = \begin{cases} -1, & \text{if } -\pi < x \leqslant 0 \\ 1, & \text{if } 0 < x < \pi \end{cases}$$

(a) What is the Fourier representation of f(x)?

(b) Use (a) to evaluate $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$.

Preface

In some ways, this Block may be viewed as optional material. Certainly, in terms of the traditional curriculum, the applications of this material to calculus can be made without an in-depth knowledge of abstract linear algebra.

On the other hand, linear algebra contributes so much to our study of mathematical structure that it is finding its way into more and more areas which previously made no use of the subject. Thus, there is a high probability that you will encounter linear algebra somewhere along the line in your future investigations. Since the overall concept of abstract mathematical structure requires a certain amount of maturity and experience, it is crucial that you get some sort of feeling for the subject <u>before</u> the time comes that you really need it. It is for this reason that we have included this Block as part of our course.

We also recognize that it is highly likely that this Block, more than any other, is not really review material. For this reason we have elected to introduce various topics without pursuing them to sophisticated depths (although we have included sufficient depth so as not to have the material seem easier than it really is). In particular, we have contented ourselves to present the material for its own sake, with little if any attempt to give practical applications. For one thing, it is hard enough to get a feeling for the new concepts in their own right, and for another thing, what is called an application depends on your area of research. Our hope in this Block is to provide you with an initial overview and mastery of fundamental concepts in the hope that you will find the subject more accessible to you at the time that you are called upon to know it in more detail.

From a more aesthetic point of view, it is also our feeling that a course which was so heavily predicated on the concept of mathematical structure should conclude on the theme of mathematical structure. In this sense, Block 3 supplements our earlier treatment of mathematical structure and allows us to fill in some of the gaps in our earlier treatment.

The first three Units in this Block are for the most part a review of the mathematical structure of vector spaces as presented in Calculus Revisited, Part 2, especially in Blocks 1, 2, and 4. We have supplemented our earlier discussion by including a more formal definition of a vector space as well as explaining the meaning of basis vectors and the role of linear independence (which earlier had been introduced in our study of linear differential equations). While we have split the material into three sections, the student who is actually using this Block as a review might find the first three Units being treated as a single, albeit lengthy, entity.

Unit 4 introduces the notion of the linear transformation and tends to formalize the structure of linearity, used so often in our course. Indeed, it is more than coincidence that more and more in the modern curriculum, linear algebra is being integrated with advanced calculus and differential equations; for the use of linearity in these courses is more than enough "excuse" for teaching the student linear algebra.

Unit 5 is an attempt to trace the concept of determinants in general and to show how this study plays an important role in linear algebra. For the student who is already familiar with determinants or who is willing to accept the results without question, it should be noted that one can omit Unit 5 and proceed directly to Unit 6 as an extension of Unit 4. More precisely, Unit 6 talks about special properties of linear transformations known as eigen vectors or characteristic vectors.

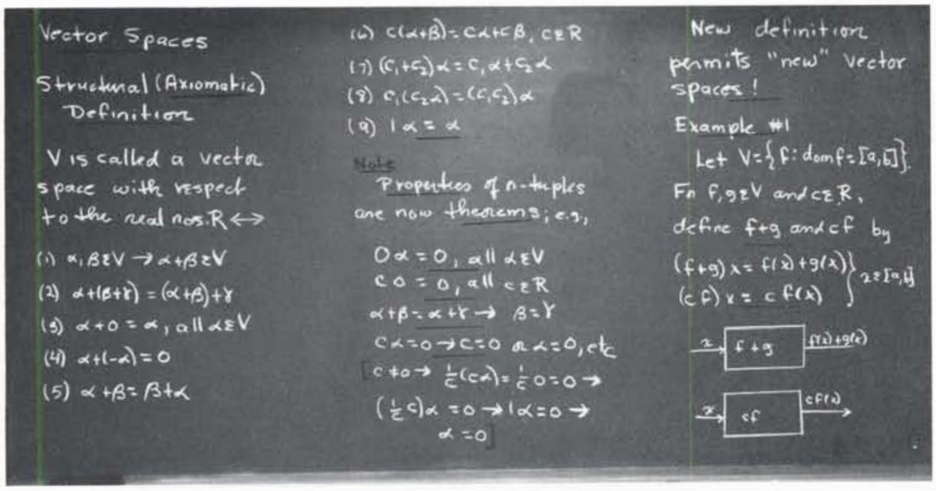
Unit 7 for the first time discusses the concept of a generalized dot product which <u>may</u> be defined on a vector space. It is in this context that we motivate the concept of orthogonal functions and we conclude our course in Unit 8 with a special application of the study of orthogonal functions. Namely, we give a brief introduction to the concept of Fourier series.

Unit 1: The Case Against n-Tuples

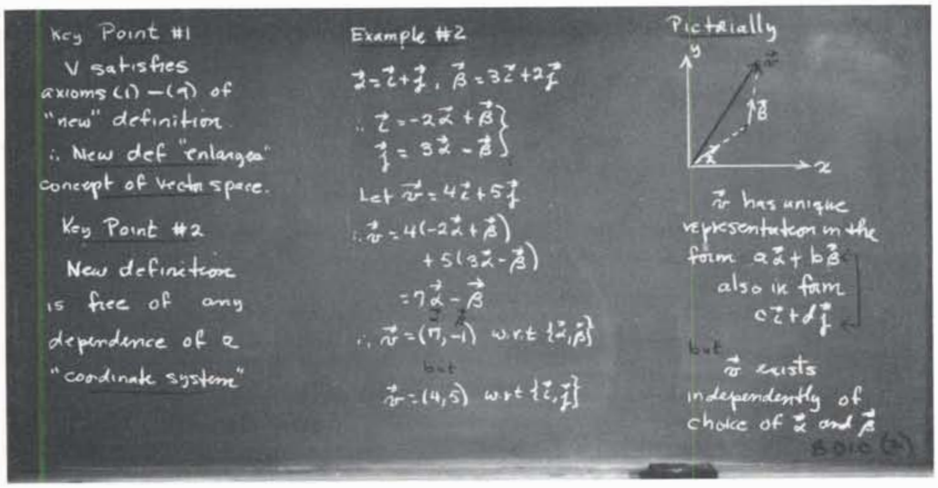
1. Overview

Our main aim in this unit is to give you a better idea of the nature of a vector space, and why it is best not to get too involved, at least initially, with an overemphasis on the n-tuple notation which we have stressed in the past. We present our case in Lecture 3.010 and refine our arguments through the exercises. Because of a somewhat different approach from that used in the text we prefer not to assign portions of the text as assignments at this time. At the conclusion of this Block, however, we shall present a format whereby you may integrate our approach with that of the text.

2. Lecture 3.010



a.



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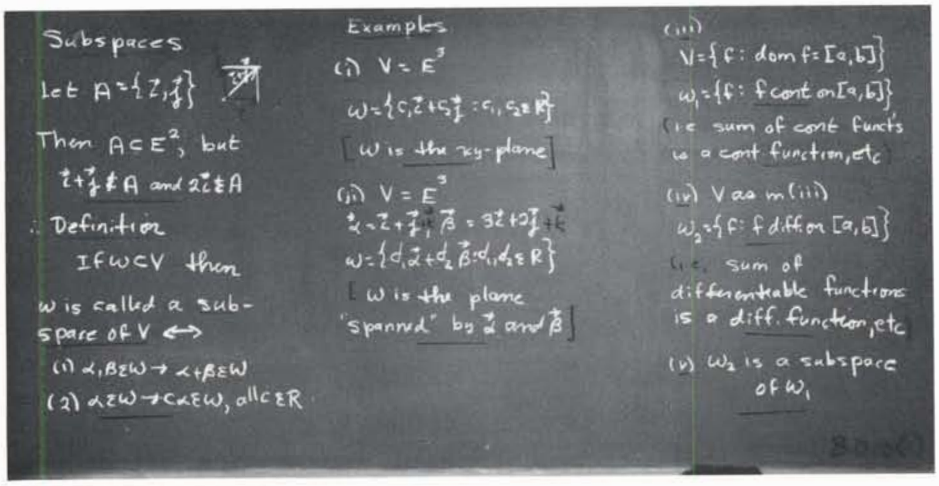

c.

3. Exercises:

3.1.1(L)

Using the usual abbreviation that for "arrows" in the plane, (x,y) denotes the vector $x\vec{i} + y\vec{j}$, let $\vec{\gamma} = (5,4)$. Now let $\vec{\alpha} = (3,4)$ and $\vec{\beta} = (2,3)$. Show that relative to $\vec{\alpha}$ and $\vec{\beta}$ coordinates $\vec{\gamma} = (7,-8)$. Does the fact that $(5,4) = (7,-8)$ contradict our previous axiom that $(x_1,y_1) = (x_2,y_2) \leftrightarrow x_1 = x_2$ and $y_1 = y_2$? Explain.

3.1.2(L)

a. Rewrite the following 3 by 6 matrix in row-reduced form:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 3 & 4 & 0 & 1 & 0 \\ 3 & 5 & 8 & 0 & 0 & 1 \end{bmatrix} .$$

b. Use the result of part (a) to express $\vec{i}$, $\vec{j}$, and $\vec{k}$ as linear combinations of $\vec{\alpha}$, $\vec{\beta}$, and $\vec{\gamma}$, where

$$\vec{\alpha} = \vec{i} + \vec{j} + \vec{k}$$

$$\vec{\beta} = 2\vec{i} + 3\vec{j} + 4\vec{k}$$

$$\vec{\gamma} = 3\vec{i} + 5\vec{j} + 8\vec{k} .$$

c. Let $\vec{\xi} = 5\vec{i} + 3\vec{j} - 2\vec{k}$. Use part (b) to express ξ as a linear combination of $\vec{\alpha}$, $\vec{\beta}$, and $\vec{\gamma}$.

d. Express 1, x, and x^2 as linear combinations of the second degree polynomials $1 + x + x^2$, $2 + 3x + 4x^2$, and $3 + 5x + 8x^2$.

e. Write $5 + 3x - 2x^2$ as a linear combination of the three second degree polynomials defined in (d).

3.1.3(L)

Suppose every vector in E^3 may be expressed uniquely as a linear combination of the three vectors α_1, α_2, and α_3. Let $\beta_1 = \alpha_1 + \alpha_2 + \alpha_3$, $\beta_2 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3$, and $\beta_3 = 3\alpha_1 + 5\alpha_2 + 8\alpha_3$. Suppose now that $\gamma = x\beta_1 + y\beta_2 + z\beta_3$. Show how we may use matrix algebra to express γ as a linear combination of α_1, α_2, and α_3.

3.1.4

a. Invert the matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 4 & 9 & 9 \end{bmatrix}.$$

b. Given that $\beta_1 = \alpha_1 + 2\alpha_2 + 3\alpha_3$, $\beta_2 = 2\alpha_1 + 5\alpha_2 + 7\alpha_3$, and $\beta_3 = 4\alpha_1 + 9\alpha_2 + 9\alpha_3$; express γ_1 as a linear combination of α_1, α_2, and α_3 if $\gamma_1 = 3\beta_1 - 2\beta_2 + \beta_3$.

c. With the β's and α's as in (b) suppose $\gamma_2 = 3\alpha_1 - 2\alpha_2 + \alpha_3$. Express γ_2 as a linear combination of β_1, β_2, and β_3.

3.1.5(L)

Use our axiomatic definition of a vector space to prove the following theorems.

a. If $\alpha, \beta, \gamma \varepsilon V$ and $\beta + \alpha = \gamma + \alpha$ then $\beta = \gamma$.

b. If $\alpha, \beta, \gamma \varepsilon V$ and $\alpha + \beta = \alpha + \gamma$ then $\beta = \gamma$.

c. $0\alpha = \vec{0}$

d. $c\vec{0} = \vec{0}$

(c, d) We must distinguish between the number 0 and the vector 0. $\vec{0}$ refers, of course, to the vector 0. However, after once making this distinction, we shall assume that it is clear from context whether 0 means the number or the vector.

(continued on next page)

3.1.5(L) continued

e. $\alpha + (-1)\alpha = 0$.

f. $(-1)\alpha = -a$

g. If $\alpha + \beta = \alpha$, then $\beta = 0$.

3.1.6(L)

a. Which of the nine axioms of a vector space (as listed in our lecture) apply to any subset of V?

b. Which properties (axioms) need not be true if S is an arbitrary subset of V?

c. From (a) and (b) show that a necessary and sufficient set of conditions that S be a subspace of V is that

$\alpha, \beta \varepsilon S \rightarrow \alpha + \beta \varepsilon S$

and

$c \varepsilon R, \alpha \varepsilon S \rightarrow c\alpha \varepsilon S$ (where, as usual, R = real numbers).

3.1.7(L)

In this exercise $V = E^2 = \{(x_1,x_2): x_1,x_2 \varepsilon R\}$. Tell which of the following subsets of V are also subspaces. In each case give reasons for your choice.

a. $S = \{(0,0)\}$

b. $S = \{(x_1,x_2): x_1 = 0 \text{ or } x_2 = 0\}$

c. $S = \{(x_1,x_2): x_2 = x_1 + 1\}$

d. $S = \{(x_1,x_2): x_2 = 3x_1\}$

3.1.8

In this exercise V is the space of all functions defined on [0,1]. Tell which of the following subsets of V are also subspaces of V.

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3.1.8 continued

a. $S = \{f \varepsilon V: \ f(x) = f(1 - x)\}$

b. $S = \{f \varepsilon V: \ f(0) = 2\}$

3.1.9(L)

Let α_1 and α_2 belong to V. Define the set $W = S(\alpha_1, \alpha_2)$ to be the set consisting of all linear combinations of α_1 and α_2.

a. Show that W is a subspace of V (it is called the space generated [spanned] by α_1 and α_2).

Suppose every vector in V is a unique linear combination of u_1, u_2 and u_3 so that we may use (x,y,z) as an abbreviation for $xu_1 + yu_2 + zu_3$. Define α_1, $\alpha_2 \varepsilon$ V by $\alpha_1 = (1,2,3)$ and $\alpha_2 = (3,5,5)$. We now define the space spanned by α_1 and α_2 (see the next exercise for more details) to be the set

$$W = S(\alpha_1, \alpha_2) = \{x_1\alpha_1 + x_2\alpha_2: x_1, x_2 \varepsilon R\}$$

That is W is the set of all linear combinations of α_1 and α_2.

b. Describe the 3-tuples that make up W.

c. Does (2,3,2) belong to W? Explain.

d. Does (3,4,3) belong to W? Explain.

e. For what value(s) of c does (3,4,c) belong to W? In this case, how is (3,4,c) described as a linear combination of α_1 and α_2?

f. Letting V denote the usual 3-space, describe W geometrically.

3.1.10 (optional)

a. Verify that the set $W = S(\alpha_1, \alpha_2) = \{x_1\alpha_1 + x_2\alpha_2: x_1, x_2 \varepsilon R\}$ where α_1, $\alpha_2 \varepsilon V$ is a subspace of V.

b. Prove that if S and T are subspaces of V so also is $S \cap T$.

c. Given that S and T are subspaces of V, define $S + T = \{s + t: s \varepsilon S, t \varepsilon T\}$. Show that S + T is also a subspace of V.

3.1.11 (optional)

a. By defining $c\alpha$ to be 0 for all $c \in R$ and $\alpha \in V$ prove that the axiom $1\alpha = \alpha$ cannot be derived from our other eight axioms.

b. By computing $(1 + 1)(\alpha + \beta)$ in two different ways, show that we may deduce the result that $\alpha + \beta = \beta + \alpha$ from the other axioms.