HERBERT GROSS: Hi, today, we're going to emphasize the role of power series in the solution of linear differential equations. And let me just say at the outset that we have paid no attention to nonlinear differential equations in this course. And we won't pay any attention to the nonlinear equation simply because the nonlinear equation is very, very difficult to handle. We usually tackle it only by special cases. And, even of more importance, most of the problems that we have to tackle are at least reasonably well approximated by linear differential equations.

Now what I'd like to do before we launch into power series today is to pull together everything that we've said so far about linear differential equations. And, rather than to drone on here saying that, I thought that I would write the summary on the board, and we could go through this fairly rapidly together.

At any rate, the lecture for today is power series solutions, but the summary to date is this. Given the general linear differential equation L of y equals f of x with non-constant coefficients, this equation always has a general solution, provided p, q, and f are continuous. Not only does the solution exist, but the solution is always given by y sub h plus y sub p where y sub h is the general solution of L of y equals 0, and y sub p is any solution, in other words, a particular solution, of L of y equals f of x.

And then you see the rest of our study has been how do you find y sub h and how do you find y sub p. The point is it's very easy to find y sub h if L sub y has constant coefficients. That was the first case that we tackled.

If L of y has constant coefficient, then y sub h has one of these three forms, depending on what? This is the case that holds if the roots of the characteristic equation are real and unequal. This is the form that holds if the roots are real but equal. And this is the equation that holds if the reals are non-- if the roots are non-real where alpha is the real part of the root, and beta is the imaginary part of the root, OK?

As far as y sub p was concerned, recall that the method of undetermined coefficients yields y sub p when L of y has constant coefficients and when f of x has the very special form e to the mx or x to the n or cosine mx or sine mx. We then turned our attention to variation of parameters, and the key point about variation of parameters was that, by use of variation of parameters, we could always obtain a y sub p when y sub h was known.

By the way, in particular, notice that, if we are dealing with constant coefficients, then we do know y sub h. That's what we were just talking about earlier. So, in particular, the variation of parameters method will work whenever we have constant coefficients because we know y sub h in that case.

But quite in general, you see, even if the coefficients aren't constant, variation of parameters works. And, in fact, once we know y sub h-- say y sub h is c1 u1 plus c2 u2, then y sub p is g1 u1 plus g2 u2 where g1 prime and g2 prime satisfy this pair of equations.

The other important thing about variation of parameters was that it reduces the order of L of y equals 0 once one solution of the equation is known. And that, you see, completes our summary. What we want to tackle next is what's still lacking. And that is that the success of the method known as variation of parameters hinges on the fact that we have the general solution of the homogeneous equation.

In other words, our next problem is to find the general solution of L of y equals 0 when the coefficients are not constant. In other words, if I can find this general solution of L of y equals 0 when coefficients are not constant, then, you see, I can use the method of variation of parameters, which always yields y sub p once y sub h is known, to find the particular solution. Then y sub p plus y sub h will be my general solution.

At any rate, it's into this environment that we introduce the concept of power series. And the key theorem in using power series is this. Let's suppose that p of x and q of x are analytic. And, by the way, I've use that word for complex valued functions.

Analytic meant that all of the derivatives existed. It meant that the function could be expanded or represented by a convergent power series. That's the meaning of analytic, even in the real case. When I say that p and q are analytic for absolute value of x minus x0 less than some value R, what I mean is that p and q can be represented by convergent power series for all x, which are within R of some fixed point x0.

But, at any rate, all I'm saying is, if p and q are analytic in this interval, then every solution of this equation, every solution which exists, which is defined at x equals x0, is itself analytic, at least in the same interval that p and q were analytic. In other words, what this means is that, in the first part of our review, we said, lookit, as long as p, q, and f are continuous, there will always be a general solution.

Now we're going one step further or maybe several steps further. We're saying, lookit, as long as p and q are not only just continuous, but they also happen to be analytic, we can not only guarantee that there'll be a general solution, but we can guarantee, more to the point, that whatever solution there is will be found in the form of a convergent power series. And that gives us the step that we need to solve problems by this technique.

Now what I'm going to do is this. I am going to start with an example that we already knew how to solve before. First of all, notice that the equation y double prime plus y equals 0, first of all, has constant coefficients. And notice that the lead in to today's lesson was we we're going to tackle situations where we didn't have constant coefficients. This does have constant coefficients. And, even more to the point, solution is known.

In fact, what is the solution here? It's y equals an arbitrary constant times sine x plus an arbitrary constant times cosine x. I picked one with constant coefficients where the solution was known simply so that we could illustrate what the method is.

By the way, notice that, in this particular problem, p of x, which is the coefficient of y prime, is 0. q of x, the coefficient of y, is 1. And, certainly, 0 and 1 are analytic functions. In fact, their power series-- the power series for 0 is 0. And the power series for 1 is 1. So, certainly, our coefficients meet the requirement.

And what we do is we start off saying, OK, if any solution exists, it must look like a power series. Let's go find it. Notice this is a glorified version of undetermined coefficients. We say, lookit, let's assume that our solution has the form of a power series, in other words, summation, n goes from 0 to infinity, a sub n x to the n where all I have to know now are the values of the a sub n's. And, if you don't like the sigma notation here, all we're saying is we're trying for a solution in the form y equals a0 plus a1 x plus a2 x squared, et cetera, where this series is a convergent power series.

And why is convergent power series important? Well, the answer, recall from part one of our course that, in the interval of convergence, the power series behaves precisely like a polynomial. We can add power series term by term. We can subtract them term by term. We can multiply them the way we do polynomials.

We can integrate term by term. We can differentiate term by term. We can rearrange the terms the way we wish, et cetera. In other words, it has all of the niceties of a finite polynomial expression.

At any rate, what I then do is, because I'm assuming this is a convergent power series, I differentiate this thing term by term. And, if I differentiate this, notice what I'm doing. I differentiate each term here, but, because I can-- because I can differentiate term by term, a very nice generic way of doing this is notice that this is the n-th term. The derivative of this term would be what? n an x to the n minus 1, n an x to the n minus 1.

Notice that, every time you differentiate, a term drops out. Like, when I differentiate this, the a0 term drops out. When I differentiate this, the a1 one term drops out. So what I do is I differentiate the general expression inside the sigma sign and then start the index of summation one up further. In other words, instead of going from 0 to infinity, I now go from 1 to infinity.

In a similar way, y double prime, I differentiate this term by term. That's n times n minus 1 times a sub n times x to the n minus 2. And now my sum goes from 2 to infinity. That's what I've written over here.

Now I'm trying to find out what the a's are equal to. I could actually call these y sub p's. See, I'm looking for a particular solution perhaps, not particular in the sense of a solution, but this is a trial.

Maybe it should have been a T over here, y sub T. I'm looking for a trial solution over here, a trial solution. I take yT double prime, yT, plug it in here, and see what this means. And, simply replacing y double prime by this and y by this, I see that this is the equation that must be satisfied identically.

Now the point is the only way a power series can be identically equal to 0 is if coefficient by coefficient the power series is 0. Now what has me bugged a little bit here is that the exponents don't match up. You see, notice that the general term here has an exponent n, and the general term here has an exponent n minus 2.

A very nice trick that I can use here is I say, you know, why don't I jack up n by 2 every place I see it? In other words, let me replace n by n plus 2. I'll jack it up by 2. And, to compensate for that, I'll lower the summation of index-- the index summation by 2 every place I see it.

Now let me show you what that means so that we don't get too confused on that. Let's suppose I had written down here summation n goes from 0 to infinity a sub n. This means what? a0 plus a1 plus a2, et cetera.

Suppose, for some reason, I would like to either lower or raise the index here. Let's suppose I want to lower the index. In other words, I want each-- I want the subscript here to be k less than what it was before. In other words, I'm going to replace a sub n by a sub n minus k.

I claim that all I have to do is start my counting process at n equals k now, instead of n equals 0. In fact, look what this says. When n is k, n minus k is 0. So the first term here is a0. The next term would be k plus 1. k plus 1 minus k is 1. So the next term would be a1, et cetera. And this is just another way of saying this.

So the idea is, if I want to get this to be an n, that means I have to raise everything by 2 inside the summation sign. So I will lower everything by 2 outside the summation sign.

Now, to do that, notice what that's going to do. My sum here will now go from 0 to infinity. This will become n plus 2. This will become n plus 1.

You see, I'm raising it by 2. Every place I'm seeing an n, I'm replacing it by n plus 2. This becomes a sub n plus 2, and this will become n. In summary, then this expression is the same as summation, n goes from 0 to infinity, n plus 2 times n plus 1 a sub n plus 2 x to the n plus summation, n goes from 0 to infinity, an x to the n. And that must be identically zero.

Now the point is that the exponents now line up. I'm starting the sum in both series at the same place. And, consequently, because these are assumed to be convergent power series, I can add them term by term. If I add them term by term, that means I come inside the summation sign here.

x to the n is a common factor. I factor that out. And what's left is what? n plus 2 times n plus 1 times a sub n plus 2 plus a sub n, that's this expression here. And this is what must be identically zero.

Now the point is that the only way that a convergent power series can be identically zero is if each coefficient is 0. And that's how I now handle the undetermined coefficients. I come to the conclusion that, because this is to be identically zero, every one of these terms must be itself 0. Setting this equal to zero and solving for a sub n plus 2 in terms of a sub n, I find that a sub n plus 2 must be minus an over n plus 2 times n plus 1.

And what this tells me is that I now know what a term looks like as soon as I know the one that comes two before it. You see, notice that this tells me how to find a sub n plus 2 once I happen to know what a sub n is. Now, lookit, before I can get to two away from a particular subscript, I must be at least at the second subscript. In other words, it seems that this recipe will tell me how to find a sub 2 in terms of a sub 0, how to find a sub 4 in terms of a sub 2, how to find a sub 3 in terms of a sub 1, but it gives me no hold on what a sub 0 and a sub 1 must be.

So I say, OK, what I will do is I will use this recipe, recognizing that I am free to pick a0 and a1 completely arbitrarily. Now look at what happens here. If I pick a0 and a1 arbitrarily, what does the recipe tell me? If I pick n to be 0, the recipe says what? a sub n plus 2, a sub 2, is minus a0 over n plus 2 times n plus 1. That's just 2 times 1. a sub 2 is just minus a0 over 2 factorial.

In a similar way, the recipe tells me that a sub 4 is minus a sub 2 over 4 times 3. You see, in this case, I take n to be 2. So n plus 2 is 4. And, if I now say a sub n plus 2 is minus a sub n over n plus 2 times n plus 1, this just says that a4 is minus a sub 2 over 4 times 3.

I know what a sub 2 is from here. So I can now express a sub 2 in terms of a sub 0. In fact, it now turns out that a sub 4 is a sub 0 over 4 factorially-- 4 factorial.

Similarly, a sub 6 is minus a sub 4 over 6 times 5. a sub 4 is a0 over 4 factorial. So, substituting that value of a4 in here, I get that a6 is minus a0 over 6 factorial. And, without beating this thing to death, notice that I can find every even subscript of a, a2, a4, a6, a8, et cetera, in terms of a0 times something. In fact, I think you can see what's starting to develop over here.

In a similar way, picking n to be 1 so that n plus 2 is 3, the recipe a sub n plus 2 equals minus a sub n over n plus 2 times n plus 1 says that a3 is minus a1 over 3 times 2. That's minus a1 over 3 factorial.

Similarly, a5 is minus a3 over 5 times 4. Noticing that a3 is minus a1 over 3 factorial, I see that a5 is a1 over 5 factorial. And, without going on further here, notice that a3, a5, a7, a9, a11, et cetera, will all be expressible in terms of a1.

And, in fact, I think you see what's happening here, that the terms will alternate in sign, and the coefficients will be things like what? 1, 1 over 3 factorial, 1 over 5 factorial, 1 over 7 factorial, et cetera. But I'll leave those details to you because I think that's the easy part to meditate upon.

But now what we're saying is I now know what all these coefficients look like in terms of a0 and a1. Remember what I'm looking for. I'm looking for a power series solution that's y equals a 0 plus a1 x plus a2 x squared, et cetera.

We're assuming that this is a convergent power series. That means I can group the terms any way that I want. In particular, since all of the even subscripts seem to be expressible all in terms of a0, and all the odd subscripts seem to be expressible in terms of a1, let me group the terms with even powers together and the terms with odd powers together.

And now, remembering what a2 and a4 are-- remember, a2 was minus x squared-- was minus a0 over 2 factorial. a4 was a0 over 4 factorial, et cetera. I now factor out a0 from here. And, substituting in the values of a2, a4, a6, et cetera, that we found from before, this tells me that the coefficient that multiplies a0 is 1 minus x squared over 2 factorial plus x to the fourth over 4 factorial, et cetera.

And, similarly, since a3 is minus a1 over 3 factorial, and a5 was a1 over 5 factorial, et cetera, I can now express all of these in terms of a1. I factor out a1. And what's left this x minus x cubed over 3 factorial plus x to the fifth over 5 factorial, et cetera. And now it appears that my solution is this plus this.

Keep in mind that a0 and a1 are arbitrary constants. By the ratio test, even if I didn't know what these series represented, I can show that these series converge absolutely for all finite values of x, but that's not too crucial here. The important point is that, even though it may be implicit, this does name some function of x, whatever it happens to be. This names some function of x, whatever it happens to be.

I can find particular solutions by choosing a0 and a1 to be certain fixed constants that I want them to be. These two power series are not constant multiples of one another, as you can see by a glance. Therefore, these two solutions are linearly independent. And, consequently, I now have a general solution here, even though this may be awkward for me.

Well, the reason that I picked constant coefficients and y double prime plus y equals 0 was that I wanted our first experience to be pleasant, at least in the sense that we would get a feeling for what the answer meant. Keep in mind that, if I had never heard of sine x and cosine x, this solution here would make sense.

But, if I happen to recall that the power series expansion for cosine x is precisely this, it just happens, you see, as an afterthought that this is cosine x. This also happens to be sine x. Notice now what this tells me is that the solution to I've found is a0 cosine x plus a1 sine x. And that just happens to agree with the result that I already knew.

Now, rather than have you feel that I've wasted your time by giving you something that you've already known, let me go one step further and give you the same kind of a problem. Only now, I'm going to give you one that you couldn't solve before and, in fact, I don't think you could solve now if you don't use power series.

And the example I have in mind is almost the same as the one we just did. Instead of y double prime plus y equals 0, I pick y double prime plus xy equals 0. And now you see I'm back-- not back, but now I have arrived at the case of non-constant coefficients. You see the coefficient of y is x, and that's not a constant. So you see now I'm at a bona fide problem where I couldn't find y sub h in the trivial way that I could if the coefficients all happened to be constant.

Now the way I tackle this is precisely the same way as before. What I do is I again try for a solution in the form y equals summation, n goes from 0 to infinity, a sub n x to the n. And what I must do now is to determine what the a sub n's are.

And, again, as before, I find y prime the same mechanical way as before. I differentiate term by term, meaning, generically, I differentiate this to get this. And, similarly, to get y double prime, I differentiate this term to get this one, OK?

And now what I do, in terms of my so-called trial solution-- see, remember, these are trial solutions-- I'm going to try to feed these back into here and see if that will tell me what the coefficients a0, a1, a2, a3, et cetera have to look like. Now, if I do that, you see, I get what? y double prime is just this term here.

And now I want x times y. That means I have to take this y and multiply it by x. Multiplying by x on the outside, since we're assuming that this is a convergent series, that means I can multiply term by term. Therefore, I can bring the x inside, and that makes this x to the n plus 1, rather than x to the n. And so I wind up with the fact that I now want to find the constants a0, a1, et cetera, from this particular relationship.

You see what's happened here? This is just y double prime. This is y multiplied by x. That's why that n plus 1 is in here. And that must be identically zero.

Now what I'm going to do is the same trick as before. What I'm going to do is I want to add these term by term. And I notice that their exponents are different. In fact, this exponent is three larger for each value of n than this one, right? n plus 1 minus the quantity n minus 2 is 3.

So what I want to do is I want to lower this subscript-- this exponent by 3. So I'm going to inside the integral-- I'm sorry, inside the summation sign, I am going to replace n by n minus 3. And, in line with our previous note, to compensate for this, instead of my summation going from 0 to infinity here, it will now go from 3 to infinity. In other words, I am now simply going to replace this by the equivalent sum, n equals 3 to infinity, a sub n minus 3 x to the n minus 2, all right?

And you see, if I do that, what I will end up with is simply this expression here. And now a new wrinkle comes up that didn't happen before, but I just wanted to make sure a few funny things happened here so I can tell you directly what's going on. And then we can drill with the remaining fine points in the exercises.

But the key point here is, lookit, I've got the exponents lined up now. But look at the indices of summation. One begins at 2, and the other begins at 3. And it seems that they're out of phase.

All I want you to observe is that, to make this start at 3, I could split off separately the term that corresponds to n equals 2. In other words, notice that, when n equals 2, what I have is what? 2 times 1 times a sub 2 times x to the 0. In other words, all I have is 2 a sub 2, 2 a sub 2 when n is equal to 2.

So what I can do is I'll split that n equals 2 term off separately. That's 2 a sub 2. Then what's left is the sum as n goes from 3 to infinity.

You see, splitting off these terms isn't hard at all. All you do is, if these don't match up, take the smaller index, and split off the number of terms necessary so that the resulting sum will begin at a higher index matching this one. Since they only differed by 1 here, I only have to split off the n equals 2 term.

Now what I have is what? I have 2 a2 plus this summation, now going from 3 to infinity, plus this summation, which also goes from 3 to infinity. And now, since both the exponents and the indices match up, now I can add term by term for these convergent power series. I bring these inside one integral sign. See, the sigma of a sum is the sum of the sigmas. I add these up term by term.

So that leaves me what? I have my 2 a2 still outside here. And then, inside the sigma notation, from 3 to infinity, I have what? n times n minus 1 times a sub n plus a sub n minus 3, that whole quantity, times x to the n minus 2. And that must be identically zero.

Now the only way that this can be identically zero is if each coefficient is 0. By the way, notice that, since the first term that appears here corresponds to n equals 3, when n equals 3, the exponent here is 1. So notice that this is my only constant term. All the terms in here begin with at least a factor of x in them. Since my power series, to be identically zero, must have all of its coefficients identically zero, it means that not only must each of these be 0, but 2 a2, which is my constant term on the left-hand side, that must also be 0.

So, in other words, to summarize this up, what we're saying so far is that, for this to be identically zero, a2 must be 0. And, once n is at least as big as 3, this condition here must be obeyed. What condition? That the expression in brackets must be 0 for each n once n is at least as big as 3.

In other words, then, in summary, a2 is 0. And, for n at least as big as 3, a sub n is minus a sub n minus 3 over n times n minus 1. And keep in mind the way I got that quite trivially was I set this equal to 0. I transposed the a sub n minus 3, divided through by n times n minus 1, and solved for a sub n.

What this tells me now is that, once I know a particular a, look at what this tells me. This says, to find a sub n, once I know what the a was three before this one, I'm home free. In other words, to find a sub 4, all I have to know is a sub 1. To find a sub 5, all I have to know is a sub 2. You see? To find a sub 6, all I have to know is a sub 3, et cetera.

So the idea is I can now pick a0 and a1 at random, just as before. And now, using this recipe with n equal to 3, I have that a sub 3 is minus a sub 0 over 3 times 2. a sub 6 is minus a sub 3 over 6 times 5.

And, remembering what a3 is in terms of a0, this tells me that a6 is a0 over 5 times 6 times 3 times 2. And this is not a misprint that the four is missing here. This is not a factorial. You see, I have a way of generating what these coefficients look like, but now I've picked a problem where I may not remember or recognize what power series I'm getting.

That's irrelevant again. All I'm saying is I can determine a3, a6, a9, et cetera, just knowing what a0 is. And, just knowing what a1 is, I can determine a4. Just by using this recipe with n equal to 4, a4 is minus a1 over 4 times 3. a7 is minus a4 over 7 times 6, using that same recipe.

Putting in the value of a4 from here into here, I have that a7 is a1 over 7 times 6 times 4 times 3. And this keeps on going. I can now find a10 in terms of a7. That will give me a10 in terms of a1, et cetera, et cetera, et cetera.

Finally, to find a5 or a8, notice that, using the same recipe, a5 is minus a2 over 5 times 4. But, since a2 is 0, a5 will be 0. And, similarly, because a5 is 0, a8 will be 0 and so will a11 and a14, et cetera.

Now the idea is the series that I was looking for, being a convergent series, can be rearranged. You see, the idea in this problem was that I would like to group the exponents according to whether they are divisible by 3, leave a remainder of 1 when I divide by 3, or leave a remainder of 2 when I divide by 3.

So I group the terms this way-- see, a0 plus a sub 3 x cubed plus a sub 6 x to the sixth, et cetera, plus a1 x plus a4 x to the fourth, et cetera, plus a2 x squared plus a5 x fifth, et cetera-- the idea being that I know how to express a3, a6, a9 in terms of a0. I know how to express a4, a7, a10, et cetera, in terms of a1. And I know how to express a5, a8, et cetera, in terms of a2. In fact, in this simple case, which turned out nicely, they all turned out to be 0, these coefficients.

So what's left is what? Replacing a4, a7, et cetera, by what they look like in terms of a1 and a3, a6, et cetera, by what they look like in terms of a0, I wind up with y equals a0 times 1 minus x cubed over 3 times 2 plus x to the sixth over 6 times 5 times 3 times 2, et cetera, plus a1 times this thing here.

Now the only difference between this problem and the previous one was that, in the previous one, I happened to recognize what well-known function had this as a power series. In this case, I don't know that, but this is no less a bona fide solution than we had in the previous case.

This is a convergent power series. We give it a name-- h of x, k of x. If this were the solution to an important enough problem, we would say, lookit, instead of calling it h of x and k of x, let's give it a special name because it's going to come up over and over again.

I'm not going to go into this. If I do go into it all, it'll be very lightly in the exercises. But things like that you may have heard, in terms of name dropping, Bessel functions and the like, Legendre polynomials, these were all special power series solutions to very special differential equations where the equation itself was so important an application that the power series that represented the equation was given a special name.

So you see, in terms of power series solutions, one extremely powerful use of power series is that it's used to help us find solutions of the homogeneous equation L of y equals 0. And, once we have the general solution of the equation L of y equals 0, we can then use variation of parameters to find a solution of L of y equals f of x. Then we add these two solutions together to get the general solution.

And that ends our theory on linear differential equations, except for one more topic called the Laplace transform that I would like to introduce for you to see. And so I will have one more lecture in the guise of linear differential equations in order that I can bring up the Laplace transform, but that won't be until next time. Until next time then, goodbye.

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