HERBERT GROSS: Hi. Today, we begin our eighth and final block of material in our course. And since we began the course with an emphasis on structure, in particular vector spaces, it's only appropriate that we conclude on the same level. Now, you may recall, when we first started talking about vectors, we began with the rather simple interpretation of arrows. Then we got to 2-tuples and 3-tuples to revisit what arrows looked like numerically.

Then we generalized this to n-dimensional n-tuples so that we could talk about functions of n real variables. And what we did was we defined a vector space to be the n-tuples together with certain structural properties. And at the very end, we then did the usual thing that one does in a structure. We showed what a vector space would look like without reference to n-tuples.

Now what I'd like to do in this lecture, and emphasize this for the remainder of our course, is to show what vector spaces would have looked like, what enrichment we could have gained if we had stuck with our more generalized axiomatic definition in the first place. And rather than go on about this philosophically, let me get to the gist of our lecture today, which, as I say, is simply called vector spaces, but hopefully from a perspective different from what was had before, that the structural axiomatic definition, you'll recall, was that V was called a vector space with respect to the real numbers R.

And I should mention here that there are places where one might talk about a vector space with respect to the complex numbers C, or something like this. But, for our purposes, we are going to be content to stick with what is called real vector spaces, namely a vector space with respect to the real numbers. And what were the structural properties? Well, the sum of two vectors had to be a vector, that the sum had to obey the associative rule. There had to be an additive identity. There had to be additive inverses. And the sum had to be commutative.

We also had a scalar multiplication structure, if you'll recall, that said that if a scalar multiplied the sum of two vectors, the distributive rule held, namely that c times alpha plus beta was c alpha plus c beta, where c was a real number. And the distributive rule also held if the sum of two numbers was multiplying a vector, you see, where c1 and c2 here are real numbers.

And, finally, if a scalar times a scalar multiple of a vector were given, this product was also associative, namely that c1 times the vector c2 alpha is the same as the scalar multiple c1 c2 times alpha. And, finally, that the number 1 multiplied by the vector alpha, the scalar multiple 1 times alpha, was still alpha. In other words, that the scalar-- that, with respect to scalar multiplication, 1 still behaves like the multiplicative identity.

And what we noted was that the properties of n-tuples were now theorems with respect to the structural definition. You see, we could show that if alpha were an n-tuple, that 0 times alpha was 0 for all n-tuples in V. But now what we're saying is we also showed at that time that, axiomatically, one could prove all of these results, that the 0 scalar times a vector was always 0. The scalar c times the 0 vector was always the 0 vector for all scalars. That if alpha plus beta equals alpha plus gamma, beta equaled gamma. The cancellation rule held.

If c times alpha was 0, either c was 0 or alpha was 0, et cetera, meaning, we had all of these theorems. And, just again by way of refreshing your memories, I very quickly jotted down a semi-proof of this result, namely, if c wasn't 0, knowing that c times alpha was 0, we multiply both sides by 1 over c. By the associative property, I can group 1 over c with c. And 1 over c times c is 1. I also know that c times 0 is still 0. And in words, any real number time the 0 vector is still the 0 vector.

Putting all of this together, you see I have what? That regrouping this in this form, this must equal the 0 vector. But this is 1 times alpha equaling 0. But by our ninth axiom, 1 times alpha is alpha. Therefore, alpha equals 0. And we can go on in this particular way, re-deriving all of the properties that we previously talked about when we were dealing with n-tuples.

The question that comes up is, what's so great about doing the same thing in two different ways? Why do we have to come back to the structural definition when the n-tuples were working very nicely for us? And what I'd like to do is two things, and I'm going to introduce one new concept, and let it drop fairly rapidly, and then come back to a older concept from a new point of view.

In the first place, my claim is that the axiomatic definition of a vector space opens up a whole new avenue of things that we can call vector spaces that we couldn't have called vector spaces before, because they couldn't be represented as n-tuples, for example. And let me just say that-- so you see it written down here-- new definition permits new vector spaces.

By way of example, let me take the set of all functions subject only to the condition that they have a common domain. For example, let's take a set of all functions f whose domain of definitions, say, is the closed interval from a to b. In other words, all you need to belong to the set V is a function defined for all real numbers on the closed interval from a to b. Notice that, both in part 1 and part 2 of this course, we had already defined very special meanings for what it meant by the sum of two functions and the scalar multiple of a function, namely, by f plus g, where f and g were functions with a common domain, and by c times f. Recall our definitions were what? That f plus g of x is simply f of x plus g of x, and cf of x is simply c times f of x, where x is any number in the close interval from a to b.

Notice, in terms of a picture, what we're saying is that, for the f plus g machine, if the input is x, the output is f of x plus g of x. For the cf machine, if the input is x, the output is c times f of x. Notice that if f is defined on the closed interval from a to b, c times f is defined on the closed interval from a to b. And if f and g have a common domain, in particular if the domain happens to be the closed interval from a to b, notice that x belongs to both the domains of f and g, and that, consequently, this summation makes sense. In other words, if we had a number which belonged to the domain of f but not to the domain of g, this wouldn't even be defined.

All right. So here is a new type of set that doesn't look like n-tuples. In other words, there are a voluminous number of independent functions which are defined on some interval from a to b. But the key point is that this new vector-- that this new set V satisfies axioms 1 through 9 of our new definition of our structural definition. In other words, the set of all functions which are continuous-- I'm sorry-- which are defined on the interval from a to b, according to our structural definition, would also be a vector space. In other words, our key point is that the new definition enlarges the concept of a vector space, where, by enlarges the concept, I mean it allows new animals to get in to the definition, that we have enlarged what can now be called a vector space.

Let me pause here for a moment just to enforce one little detail for you, and that is keep in mind that I am appreciative of the fact that, probably, you are not used to visualizing a set of functions as being a vector space. Consequently, it's my feeling that if I continue harping on this particular example, much of what I say will be lost on you because you haven't had enough time to have this new concept sink in. So what I'm going to do is to leave a number of exercises for our study guide where you can get adequate drill on what we mean by this new type of vector space.

What I'd like to do for the next part of our lecture is to get away from this new concept, which, by the way, is very important, which we will emphasize as the remainder of this block continues. But for those who say, I'm not that interested in functions as vector spaces-- I can't visualize that-- I would like to point out a second key point. And that is that even when you deal with n-tuples, the structural definition has advantages over the n-tuple definition.

In particular, which I call key point number 2, the new definition-- and by new I mean the structural definition-- is free of any dependence of a coordinate system. Now, what do I mean by dependence of a coordinate system? Let's see if I can't make that clear by means of a particular example.

Let's suppose I'm dealing in the-- I'm in the xy plane, and we're used to the vectors i and j as being the basic vectors of the xy plane. Let's assume for the sake of argument that, for one reason or another-- and I'll clarify what one reason or another means as our course goes on. But, for one reason or another, let's suppose that, for the particular problems I'm interested in, the two vectors alpha and beta, where alpha is defined to be i plus j, and beta, say, is defined to be 3i plus 2j, let's suppose that the two vectors alpha and beta happen to be the two vectors that we, for some reason or other, happen to be more interested in than we are interested in i and j.

Now, the idea is something like this. First of all, observe that, because of the arithmetic, the axioms that a vector space satisfies, I can treat alpha, beta, i and j, the same as I would real variables, so to speak. And I can solve algebraically for i and j in terms of alpha and beta, which I've just done over here. And I leave the details for you. You just solve these as two equations and two unknowns. Solve for the i and j in terms of alpha and beta.

Now the idea is this. Let's suppose I'm given some vector v. For the sake of argument, let v be the vector 4i plus 5j. Now, notice that the vector v is well defined, and I'll show you that pictorially in a few moments. But that vector v exists independently of whether we talk about i and j components or not. In other words, once I tell you that v is 4i plus 5j, it exists as an arrow in the plane. And once I've drawn that arrow, that arrow is well defined even if I now erase what this definition is.

The point I wanted to make, though, is this. You will recall that when we dealt with 2-tuples, we picked i and j as being the important vectors, and we abbreviated v as 4 comma 5, indicating that the vector 4i plus 5j, if it originated at 0, 0, would terminate at the point 4 comma 5. Now suppose, for the sake of argument, we were interested in v not in terms of i and j but in terms of the new vectors alpha and beta. Knowing that i is minus 2 alpha plus beta, and j is 3 alpha minus beta, by direct substitution, we can replace i and j by what they're equal to in terms of alpha and beta. So just by direct substitution.

And I can conclude-- again, notice the arithmetic here. You see 4-- I can multiply this out-- minus 8 alpha plus 4 beta, because I have a rule that tells me that a scalar times the sum of two vectors is a scalar times the first vector plus a scalar times the second vector, et cetera. All the rules of arithmetic apply here. So I very quickly say 15 alpha minus 8 alpha is 7 alpha. 4 beta minus 5 beta is minus beta. So relative to alpha and beta, notice that v has the unique representation 7 alpha minus beta.

Now, suppose I wanted to abbreviate v with reference to the basis vectors alpha and beta, where the convention would be that I would list the alpha component first and the beta component secondly. Notice that the 2-tuple that names v would be 7 comma minus 1. See, 7 is the coefficient of alpha. Minus 1 is the coefficient of beta. 7 comma minus 1 would be the 2-tuple that represents v if I use alpha and beta as my representative vectors. Whereas the 2-tuple 4 comma 5 represents v with respect to the, in quotation marks, the "usual" representation in terms of i and j.

Notice that v is the same vector. That hasn't changed. But, certainly, the 2-tuple 7 comma minus 1 and the 2-tuple 4 comma 5 do not look alike. In other words, there is a big danger-- as, for one reason or another, we switch from different representative vectors to other, often a given set of representative vectors to another set of representative vectors-- that we can become confused as to which 2-tuple-- in this case, 2-tuple would be n-tuple in general-- but which representation goes with which 2-tuple.

See, the whole idea is that our new definition, structurally, never pays any attention to what the particular representation is. And from the mathematician's point of view, what we feel is that what should happen in the study of vector spaces should be consequences of the vectors, not the particular coordinate system. And if that phrase confuses you, I'm using the coordinate system in the following sense. When I talk about i and j, I think about i and j as being my coordinate system. When I talk about alpha and beta, I think of alpha and beta as being my coordinate system.

For example, given the vector v, let's suppose this is alpha and this is beta. Notice that v certainly is a linear combination of alpha and beta. In other words, it's this vector plus this vector. It's also a linear combination of i and j, certainly a different linear combination of alpha and beta than it is a linear combination of i and j. But also certain is the fact that the representation of v in terms of alpha and beta as our coordinate system is unique, just as the representation of v in terms of an i and j coordinate system is unique.

In other words, v has a unique representation as some scalar multiple of alpha plus a scalar multiple of beta. But it also has a unique representation in the form a scalar multiple of i plus a scalar multiple of j. You see, we do not mean that the representation is unique in the sense they can only be written as one set of linear combinations. What we mean is that, for each coordinate system, for each coordinate system, there is a unique representation as a linear combination of the coordinates, but different coordinate systems lead to different linear combinations.

But the key point from the mathematician's point of view is that v exists independently of the choices of alpha and beta. You see, again, we ran into this problem before when we talked about the distance between two points in a plane. Certainly, the distance between two points in a plane was independent of what coordinate system we used. But with respect to polar coordinates, the recipe for finding the distance between the two points did not look the same as the recipe that one used for finding the distance between the same two points when the points were expressed in Cartesian coordinates.

The same thing we're saying here. We would like our mathematics to depend only on the vectors, not on their representation. Well, enough said about that part of the theory. What we'd next-- again, the exercises will hopefully fill in any gaps that may be missing in the logical structure from your point of view. But the next thing we'd like to talk about is a thing called a substructure, that whenever one talks about a new type of structure, the concept of what we used to call a subset becomes too weak, and what one wants to talk about, then, is the concept of a substructure.

What I mean by that is the next topic I want to talk about is called subspaces. And to give you an idea of what I'm talking about, let's do the following. Let's go back into the plane and take just the two individual vectors i and j. Not the whole plane, just the two vectors-- the unit vector in the direction of the positive x-axis, the unit vector in the direction of the positive y-axis. Just these two vectors.

Now, certainly, the set consisting of just these two vectors is a subset of two-dimensional vector space. In other words, the set of all vectors in the xy plane certainly includes the i vector and the j vector. So it would certainly be fair to say that this set A is a subset of two-dimensional vector spaces. But the idea is that a vector space has a structure. We care much more about how you combine vectors and what you get than we worry about whether you just have a set.

Remember, mathematically, a structure is what? A set together with certain rules. And so what we're saying here is look what goes wrong with this. If we were looking at this thing structurally, one would say something like, gee, let's add i and j. If we add i and j, we get i plus j. i plus j is neither the vector i nor the vector j. In other words, in terms of a picture, here's i, here's j.

What is i plus j? i plus j is just the sum of these two vectors. What we're saying is i and j belong to the set A, but i plus j certainly is neither i nor j. Since they consist solely of i and j, the fact that i plus j is neither i nor j means that i plus j does not belong to A. And, similarly, neither does, for example, twice i. If I double i, I get the vector which has the same direction and sense as i but it's twice as long. Consequently, that vector is neither i nor j.

So, in particular, given the set whose only two elements are i and j, notice that if I add two vectors in the set A, I need not get a vector in the set A. And if I take a scalar multiple of any vector in the set A, I need not get a vector in the set A. And so, structurally, this is a bad thing, because notice that, structurally, we will be adding vectors. Consequently, if we start with a set and add two members in that set, and the resulting vector can then be a vector which isn't in that set, we don't have a very convenient structure.

So what we do is we isolate a very key point. And let me give you the definition abstractly first, and then show you in terms of familiar examples that we already knew this. Let's suppose that we have a vector space V and that W is simply a subset of V. So this just means subset so far. Then W is called a subspace of V provided that, one, the sum of any two elements in the subset W is also an element in W. In other words, that the subset W is close with respect to the addition operation that makes V a vector space.

And, secondly, that, for any element in the subset W, any scalar multiple of that element is also in W. In other words, W must also be close with respect to that same scalar multiplication with respect to which V is defined. And, by the way, I think if you think about that for a while, you will notice that this is true. For example, we have often talked about a line being a subset of a plane. But in terms of vectors, we could have made a stronger statement, namely a line is a subspace of the plane.

And, in a similar way, a plane is a subspace of three-dimensional space, namely, starting with a plane, if you take two vectors which lie in the plane and you add them, the sum of those two vectors is still in the plane. And if you take a scalar multiple of any vector in a plane, the resulting vector still lies in that plane. So by means of a few examples, then, if V now is usual three-dimensional Euclidean space, if I let W be the set of all linear combinations of i and j-- and, by the way, contrast that with the set W-- with the set A over here.

Notice that the set A over here consisted only of the vectors i and j. The set W here consists of all linear combinations of i and j. In other words, it's all vectors of the form c1 i plus c2 j, where c1 and c2 are real numbers. And notice that W is the entire xy plane. And, after all, any vector in the xy plane can be written as a linear combination of i and j.

Why is this a subspace? Well, leaving the details to you, notice, first of all, that it's clear that this should certainly be a subset, namely every vector in here is a vector that belongs to three-dimensional space even though it lies in a plane. But notice that the sum of any two linear combinations of i and j is again a linear combination of i and j. And any scalar multiple of a linear combination of i and j is again a linear combination of i and j. So W satisfies the criteria for being a subspace. In other words, W is not only a subspace in this case. W is the xy plane.

By the way, using the alpha and beta of the previous example, notice that if I let V again be three-dimensional space and pick alpha to be i plus j, and beta to be 3i plus 2j, notice that if I take the set of all linear combinations of alpha and beta, then W is also a plane, and it's called the plane spanned by or determined by alpha and beta. In other words, it's the plane that has alpha and beta as consecutive edges of a parallelogram.

By the way, let me make two comments here. First of all, if these d's look funny to you, they should, because just before the lecture began, these were c's. And I changed these c's to d's because the mathematician would have no danger in confusing the coefficients here with the coefficients here, but for the uninitiated I think it's very important to make this observation. Notice that alpha and beta, being linear combinations of i and j, both lie in the xy plane. Consequently, the plane determined by alpha and beta is also the xy plane.

This, in turn, means that, in both of these two examples, W is the xy plane. But, somehow or other, you wouldn't expect that W would be the same linear combination of alpha and beta as it is of i and j. And, therefore, I prefer to use different letters here to indicate that a given vector in V might require two numbers c1 and c2 to be coefficients of i and j, but, with respect to alpha and beta, two different numbers. OK? That's one point I wanted to make about this.

The other point I wanted to make about this, you see, notice that alpha and beta don't look like i and j, and there's a danger-- that you may not even realize when you see alpha and beta-- that alpha and beta determine the same plane as i and j. Now, you see, that leads to the problem that we were talking about before, namely, given the same plane, the representation of the plane looks different if you use one set of representative vectors than if you use another set of representative vectors.

But, more importantly, let me point out that there was nothing sacred about the xy plane here. Just to change this problem ever so slightly, let me add on a plus k, say, to both alpha and beta. That makes this a new example, you see. Let alpha be i plus j plus k. Let beta be 3i plus 2j plus k.

Notice that alpha and beta are-- neither is in the xy plane. They are not parallel to one another, because this is not a constant multiple of this. Therefore, notice that alpha and beta are still two vectors. W is still the set of all linear combinations of these two vectors. See, this part doesn't change. And W now is still the plane spanned by alpha and beta, but now notice that the plane spanned by alpha and beta is the plane which has alpha and beta as consecutive edges, the parallelogram determined by the plane that has alpha and beta as consecutive edges. And this parallelogram clearly is no longer in the xy plane. All right.

And now, for a finale for today, what I'd like to do now is to point out, going back ever so briefly to our first key point, that there are vector spaces which aren't viewed as n-tuples, namely, for example, the set of all functions defined on the closed interval from a to b. Let me show you what subspaces mean in terms of this interpretation. My next example, again let V be, as earlier in the lecture, the set of all functions having a common domain, say the closed interval from a to b.

Let's look now at the set W1 of all functions which not only are defined on the closed interval from a to b, but they happen to be continuous on the closed interval from a to b. You see, notice that, originally, I never had to assume continuity here. As long as the function is defined on the closed interval from a to b, I can add any two members of this family. I can scalar multiply any member of this family. So I do not need continuity to have this be a vector space.

However, suppose I now look at all functions which are continuous on the closed interval from a to b. Well, in particular, it should be trivial to see that this set W sub 1 is a subset of V, namely every function which is continuous on the closed interval from a to b, in particular, must be defined on the closed interval from a to b. Consequently, every function in this set certainly belongs to this set.

On the other hand, notice that we already know that the sum of continuous functions is again a continuous function. A scalar multiple of a continuous function is again a continuous function. Consequently, according to our definition of subspace, W1 is not only a subset of V. It's a subspace of V because it's closed with respect to-- excuse me-- addition and scalar multiplication.

As another example, let V be the same as it was in the previous example, namely the set of all functions defined on the closed interval from a to b. And now let's take that subset consisting of all functions which are differentiable on the closed interval from a to b. In other words, this is an even stronger condition than being continuous. We say not only must the functions be continuous, but now we want it to be differentiable as well. Remember, differentiability implies continuity, but continuity does not imply differentiability.

All I'm saying now is that, since the sum of two differentiable functions is again a differentiable function, and since a scalar multiple, a constant times a differentiable function, is still differentiable, not only is W2 a subset of V. It's a subspace of V. In fact, W2 is also a subspace of W1. Recall, W1 was what? The set of continuous functions on the interval from a to b. That was also a vector space. And now what we're saying is that the differentiable functions are a subspace of the continuous functions.

Again, I'm going to emphasize all of this in the exercises, and we'll pick this up in more detail next time, but what I wanted to close with is simply the following remark. If you have studied calculus in the past and have been away from it for a while, and you pick up an ultra-modern calculus book, you may be surprised to find that even elementary calculus begins with a preface to emphasizing what's called linear algebra or vector spaces.

And the reason for this is the fact that all of calculus, going back to part 1 of our course, does deal with continuous and differentiable functions. And, in particular, continuous and differentiable functions, as we've just shown, obey the structure of a vector space. And, consequently, that's why many modern authors prefer to unify the approach to calculus and introduce vector spaces right at the very outset. But we'll talk about this more gradually as we go along. And until next time, then, goodbye.

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