MITOCW | Part II: Differential Equations, Lec 3: Solving the Linear Equations L(y) = 0; Constant Coefficients

HERBERT GROSS: Hi, in our lesson last time, we showed that, to find the general solution of a linear differential equation, we first looked for the general solution of the homogeneous or reduced equation-- that is the equation where the right-hand side was 0-- and then found a particular solution of the original equation. And then the sum of these two solutions would be the general solution.

Consequently, we agreed that we would now separate our study into two pots, one to find a general solution of the reduced or homogeneous equation and the other to find a particular solution of the given equation. In today's lesson, we are going to tackle the problem of finding the general solution of the homogeneous equation, but subject to a very special case, special in the sense that it's easy to handle, but also special in the sense that many practical applications live up to this particular model.

What we're going to tackle today is solving the linear equation L of y equals 0 subject to the condition that we have constant coefficients. In other words, we are going to deal with y double prime plus 2a y prime plus by equals 0 where a and b are constants. The only reason I put the 2 in here is to simplify some algebra later on. After all, if a is an arbitrary constant, so is 2a. We'll talk about that in more detail where I need it.

The thing is that, to solve an equation of this particular type where a and b are constants, we essentially try for a solution in the form y equals e to rx, recalling that, when we differentiate this, we get re to the rx. When we differentiate it a second time, we get r squared e to the rx. Consequently, substituting this into here, we get that e to the rx is a constant factor. And we're then left with a polynomial in r equal to r squared plus 2ar plus b.

In other words, mechanically, notice that the power of r replaces the given derivative. r squared replaces y double prime. r to the first power replaces y prime. And this is the r to the 0 term replaces the 0 derivative term, namely the zeroth derivative is y itself.

At any rate, since e to the rx cannot be 0, to find the values of r that satisfy this equation, then what we have to do is set this polynomial, in this case a quadratic equation. And those, of course, if I had order greater than 2, the same theory would hold. I have simply, as usual, picked second-order equations for illustrative purposes. The exercises will take care of higher orders. At any rate, the values of r are determined by the equation r squared plus 2ar plus b equals 0.

Now you see, if I use the quadratic formula and solve this equation for r, I find that r is minus a plus or minus the square root of a squared minus b. That's the reason I used 2a instead of a here. You see, if I had used a here without the 2, then the quadratic formula would have said r is equal to minus a plus or minus the square root of a squared minus 4b over 2a.

And, with blackboard space at a premium, I hate to have to keep writing in a 2a as a denominator. And, every time I do that, I lose a line of space on my board. So, for convenience sake, I chose the 2 in here so I could write r this way.

Now notice that, in general, this determines two values of r, one with the positive square root, one with the negative square root. Let me call r1 the root that I get with the positive sign, r2 the root with the negative sign. And the point that I want to mention is that the nature of the magnitude of a squared minus b determines what the values of r look like.

For example, if a squared minus b is positive, then r1 and r2 are unequal real roots. If a2 minus b-- if a squared minus b is 0, then r1 and r2 are both equal to negative a. And, if a squared minus b is negative, r1 and r2 are complex conjugates.

Now the idea is simply this. What we're going to do now is I'm going to summarize for you what these results are. I'm going to use the cookbook technique first of illustrating all three cases for you, giving you examples of each, and then I will conclude today's lesson simply by showing you why the recipes work. And, with a little bit of luck, for the first time in our course, we may actually have a short lecture. But let's see what actually goes on here. Let's not be too presumptuous.

The three cases are this. Either a squared is greater than b-- case 1, a squared is greater than b-- so r1 is unequal to r2, and they're both real. And, in this case, the general solution of the homogeneous equation, y double prime plus 2a y prime plus by equals 0, is simply a linear combination of e to the r1 x and e to the r2 x, namely, c1 e to the r1 x plus c2 e to the r2 x.

For the second case, when a squared equals b, in that case, r1 equals r2. And, because they're equal, let me just call both of them r1. Notice that, if I came back up here, this would really not be a general solution in this case because, with r1 equal to r2, e to the r1 x is a common factor.

I could then, well, amalgamate the two arbitrary constants here and write what? That y sub G, the general solution, is just c1 plus c2 times e to the r1 x. And, if c1 and c2 are arbitrary constants, then their sum is simply a arbitrary constant.

It turns out that, in this particular case, another solution of this equation, meaning what? Another one other than e to the r1 x, which is not a constant multiple of e to the r1 x, is xe to the r1 x. And we'll verify that later in the lecture. For now, just take this for granted. And, in that case, it turns out that the general solution is given by the linear combination of e to the r1 x and xe to the r1 x, c1 e to the r1 x plus c2 xe to the r1 x.

In the third case, we have that a squared is less than b. That means that r1 and r2 are complex conjugates, which means that r has to form a real number plus or minus i times a real number. Calling the real part alpha and the imaginary part beta, all we're saying is, if a squared minus b is negative, alpha is playing the role of negative a here. And beta is just the negative of what's under the radical sign here.

In other words, if a squared minus b is negative, b minus a squared is positive. In other words, the beta part is just the square root of b minus a squared, you see? And, in this case, it turns out that the general solution is a linear combination, a very interesting one.

Well, let me factor out the e to the alpha x. What it says is first take e raised to the alpha x power, where alpha is the real part of our root r, OK? And then multiply that by a linear combination of sine beta x and cosine beta x where beta is the imaginary part of r.

By the way, without going into this because the textbook does it very eloquently, and we'll have enough exercises on this to hammer this home later, notice the intrinsically different geometric nature of interpreting these various solutions. For example, in this particular case, just speaking very quickly, here we see that the general solution is the sum of two exponentials.

Coming down to here, notice that, if alpha happened to be 0, in other words, if r were purely imaginary, notice that the solution would be what we call oscillatory motion. And, if alpha is not 0, notice that e to the alpha x c1 cosine beta x plus c2 sine beta x is an oscillation that is either blown up if alpha is positive or shrunk if alpha is negative.

In other words, the oscillatory part stays here, but, if alpha happens to be negative, say, as x gets very large, e to the alpha x goes to 0. That pulls this whole expression down to 0. And that's the case where you get what's called damped oscillatory motion. In other words, you get oscillation, but the amplitude keeps shrinking. You see, physically, different things start to happen here.

And this is a very interesting subtlety that just small changes change the entire solution or the meaning of a solution. And maybe the best way to illustrate that is to show examples of all three of these cases where the examples almost look alike. Well, let me show what I mean.

For example, we already saw last time that, if y double prime minus 4y prime plus 3 y equals 0, then the general solution was c1 e to the x plus c2 e to the 3x. That's precisely this case here because, if we look at the equation that determines r, it's r squared minus 4r plus 3 equals 0.

That says that r1 is 1. r2 is 3. The roots are 1 and 3. So two solutions are e to the x, e to the 3x. r1 and r2 are real and unequal. That's exactly the type that we were talking about back here when we were talking about case one.

As far as case two is concerned, notice that y double prime minus 4y prime plus 4y equals 0 leads to the equation r squared minus 4r plus 4 equals 0. That factors into r minus 2 squared is 0. And that says that r equals 2 is a multiple root.

To illustrate case two, again, the proof coming shortly, but not now, what we're saying is that, in this case, e to the 2x and xe to the 2x will be non-constant multiples of one another, the solutions, and that, consequently, the general solution will be c1 e to the 2x plus c2 x e to the 2x.

The third case that I'll pick is y double prime minus 4y prime plus 5y equals 0, which leads to the equation r squared minus 4r plus 5 equals 0. And, solving by the quadratic formula for r, this leads to r equals 4 plus or minus the square root of 16 minus 20 over 2. The square root of 16 minus 20-- that's the square root of minus 4-- is 2i. 4 plus or minus 2i/2 just leads to 2 plus or minus i.

Since alpha is the real part of the root, alpha will be 2. Since beta is the magnitude of the imaginary part of the root, beta will be 1. In this case, alpha is 2. Beta is 1. And, therefore, the general solution is what? e to the 2x times c1 cosine x plus c2 sine x.

And, by the way, before I leave this example, here's what I meant by saying how subtle this was. Notice that all three of these equations are identical, except for the coefficient of y. In the first equation, the coefficient of y was 3, in the second equation, 4, and, in the third equation, 5. Other than that, they all looked alike, the equations. And yet, in one case, we got the two real and unequal roots. In the second case, we got the real but equal roots and, in the third case the non-real complex conjugates, all right?

That's all there is to this lesson, except for giving you some insight as to why these recipes hold. From this point on, the rest is drill that there's not much more we can do, other than to have you look to see what's happening and work on the exercises, but I thought that it might help if I give you a few little supplementary notes as to why these things work out.

And they work out easily enough. So I've decided to do the supplementary notes as part of the lecture. Namely, first of all, notice that, in the case where r1 is unequal to r2, and they're both real, that e to the r1 x and e to the r2 x are both real solutions. And, more importantly, since one cannot be a constant multiple of the other, their linear combination must be the general solution. That's what we saw last time.

Remember, the only time we were in trouble is if one was a constant multiple of the other. Why can't e to the r1 x and e to the r2 x be constant multiples of one another? The answer quite simply is this. Assume that they were. Divide both sides by e to the r2 x. We then get e to the r1 minus r2 x is a constant.

But look at this is e raised to a variable power. The only way that e raised to a variable power can be a constant is if the multiplier of the x is itself 0 so that the x doesn't appear here. In other words, in that case, notice that r1 minus r2 must be 0. And that's the same as saying that r1 equals r2. In other words, the only way that e to the r1 x can be a constant multiple of e to the r2 x is if r1 equals r2, but we're given in case one that r1 is not equal r2. That's why the solution works.

As far as case two is concerned, notice that what we have to establish is it's obvious that xe to the r1 x is not a constant multiple of e to the r1 x. After all, their ratio is x, which is not a constant. The hard thing to do is to justify how we can find out that xe to the r1 x is a solution in that case.

And the answer is look at that case occurs when a squared equals b. If a squared equals b, notice that y double prime plus 2ay prime plus by becomes replaced by y double prime plus 2ay prime plus a squared y. Now we very cleverly invoke the trick of writing 2a y prime as ay prime plus ay prime. See, we take this and write it like this.

Now we factor out an a from this expression here. We have a times y prime plus ay. Notice that these two terms are just the derivative of y prime plus ay.

In other words, if I let u equal y prime plus ay, then this being u means that, with the prime on this, this is u prime. This just becomes u prime plus au equals 0. That's a first-order differential equation in u in which the variables are separable. See, in other words, this becomes this.

From this, I can very quickly find that u is equal to ce to the minus ax. I didn't want c. I wanted-- I didn't want u. I wanted y. I remember that, by definition of my substitution, u is y prime plus ay. Consequently, this leads to y prime plus ay is ce to the minus ax.

This is a first-order linear differential equation. We learnt how to solve that in a previous unit. Leaving the details again to you, it now follows that a particular solution of this equation will be y sub p is xe to the minus ax.

Recalling that, in this case, r1 is minus a, this is xe to the r1 x. This together with e to the r1 x form your two linearly independent solutions. And, consequently, that's why the general solution in this case is c1 e to the r1 x plus c2 xe to the r1 x.

The third case where we've got the imaginary roots or the non-real roots hinges on a very interesting property of linearity plus complex functions. If u and v are real, notice that L of u plus iv, by linearity, is L of u plus iL of v because, after all, i is a constant, be it imaginary or otherwise.

Consequently, by linearity, since L of u plus iv is L of u plus iL of v, the only way that L of u plus iv can equal 0 is if L of u plus iL of v equals 0. And the only way a complex number can equal 0 is if both its real and imaginary parts are 0. In other words, then, if L of u plus iv is 0, it means that L of u and L of v must themselves be 0. See, the real part must be 0. The imaginary part must be 0.

Now how do we apply this to case three? Notice that, in case three, we solved for r and got that r was equal to alpha plus or minus i beta. Let me pick the positive root. And we said our solution was e to the alpha plus i beta x.

Now the idea is this looks quite imaginary, and it this in this form. But what we've just seen is that, since this is a solution of this homogeneous equation, the real and the imaginary part separately must be solutions. Consequently, if I write this in the traditional u plus iv form, then u and v separately will also be solutions. Well, let's see how I write e to the alpha plus i beta x in traditional form.

First of all, this is e to the alpha x times e to the i beta x by the rules of exponents. We recall from our lecture on complex functions that e to the i beta x is cosine beta x plus i sine beta x. Consequently, e to the alpha plus i beta x is just e to the alpha x times this expression here.

Or, separating it into the form u plus iv, e to the alpha plus i beta x is e to the alpha x cosine beta x, which is a real number because alpha and beta are real, plus i times e to the alpha x sine beta x. And e to the alpha x sine beta x is also a real number because alpha and beta are real.

In other words, u, the alpha x cosine beta x, that's the real part of this complex number. v is e to the alpha x sine beta x, which is the imaginary part of this. Since we've just seen that the real and the imaginary parts of this function must satisfy this equation also, that tells us that e to the alpha x cosine beta x and e to the alpha x sine beta x also satisfy this equation. And that, by the way, is exactly what we saw in case three, only we factored out the e to the alpha x.

Notice, by the way, that these two equations cannot be constant multiples of one another because, if we divide, say, the top by the bottom, the e to the alpha x cancel. Sine beta x over cosine beta x is tan beta x. And, for beta not equal to 0, tan beta x can't be a constant so that these two solutions do indeed generate the general solution.

Now that's all there is to this particular problem. The rest, as I say, is left to the exercises. What we're going to do next time is tackle the companion part of this problem. And that is, how do you find the particular solution or a linear differential equation with constant coefficients where the right-hand side is not 0, in other words, the non-homogeneous case? But more about that next time. So, until next time, goodbye.

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