HERBERT GROSS: Hi. Today we're going to begin our study of higher order differential equations. In particular, we're going to talk about a special equation called the linear differential equation, which came up, by the way, you'll notice, in first order equations as a special type. But we're going to generalize that. And the only liberty I'm going to take during the lecture is to restrict our study to the case of differential equations of order two so that we won't have unwieldy expressions all over the board.

The idea is that what happens in the case of n equals 2 happens for all orders n, except for a little modification in the algebra. But we'll talk about that more in the exercises. At any rate, for the time being we simply call today's lecture Linear Differential Equations. And now I must define for you what I mean by a linear differential equation.

And I'll motivate this more as we go along. For the time being, a linear differential equation is simply one which has this very special form. What you have is a y double prime term appearing, then some function of x alone multiplying y prime, plus some function of x alone multiplying y, and on the right some function of x alone. And an analogous result would hold for higher order.

In other words, you have the various derivatives of y appearing, y by itself. The coefficients are always functions of x alone. And no power is-- or no derivative or y is raised to any power. In particular, notice here-- it's a very small point-- but notice here that I assume that the leading coefficient here is 1.

For example, somebody might have said, couldn't you have had some function of x times y double prime? The answer is yes, I could have. But I'm assuming that if I had a function, say, r of x multiplying y double prime, I could have multiplied-- divided both sides of this equation through by that coefficient and wound up with an equation of this particular type. In other words, without loss of generality-- and this is very important, though. Many of my theorems are going to be messed up a little bit if the coefficients of y double prime isn't 1. In other words, the algebra will become a little bit tougher. I'll speak about that later in the lecture if I remember to.

But the idea is, for the time being this is all we mean by a linear differential equation. And perhaps the best way to emphasize what we really mean by linear differential equation is to show what we mean by a nonlinear differential equation-- a differential equation which is not linear. For example, y double prime plus y prime squared plus y equals sine x would not be a linear equation, because the y prime term is being raised to a power other than the first power. See, it's squared.

y double prime plus y times y prime equals e to the x is not a linear equation, because the multiplier of y prime is y as opposed to just a function of x, which is what the definition calls for. y double prime plus x y prime plus y squared equals x cubed is not linear. And the reason that it's not linear is because y appears to a power higher than the first.

Maybe I should circle the things that prevent these things from being linear so that you can see what's happening here. y double prime plus e to the x y prime plus x cubed y equals tan y is not linear, because even though the left-hand side is fine, notice that the function on the right hand side is a function of y-- depends on y rather than just x alone.

And to finish this off, an example of a linear equation would be what? Well, almost this one-- y double prime plus e to the x y prime plus x cubed y equals tangent x. You see a y double prime term, y prime to the first power being multiplied by a function of x alone, y to the first power being multiplied by a function of x alone, and the right-hand side being a function of x explicitly by itself. And notice, by the way, don't get caught in a psychological hangup here. When I say linear, it's modifying the equation, not what the coefficients look like.

In other words, certainly we don't think of e to the x as a linear function. We don't think of x cubed as being linear. We don't think of tangent x as being linear. The coefficients do not have to be linear functions of x. They have to be functions of x alone.

It's the equation that's called linear. And maybe the best way to explain that is in the following way. Let's look at y double prime plus p of xy prime plus q of xy. My claim is I can think of this as a function machine where the input is y, and the machine is told, look it-- whatever y comes in, differentiate it twice, add on p of x times the first derivative plus q of x times the function, and that will be the output. In other words, I could think of a machine where the input of the machine is y-- and let me call the machine the L machine to emphasize the word Linear-- see, what L does is this.

The domain of L are functions which are twice differentiable. After all, for this to make sense I have to be able to differentiate y at least twice. So the input of the L machine is a twice differentiable function of x, and the output is some function of x. You see y prime, y double prime are functions of x, p and q are functions of x, y is a function of x. All these things combine, then, to give you a function of x.

Let me show how this machine works for example. If the L machine is y double prime plus e to the x y prime plus x cubed y-- [? see, ?] L of y equals this-- if I feed in sine x to the L machine, what does the L machine do? It takes sine x, it differentiates it twice, adds on e to the x times the first derivative, and x cubed times the function itself, which is sine x. Going through this operation, which is trivial, L of sine x would be x cubed minus 1 sine x plus e to the x cosine x.

By the way, to help you understand what we mean by a solution, notice that what we're saying is that if we were to refer back to this particular equation, y equals sine x would not be a solution of this equation, because when I feed sine x into the L machine, I do not get tangent x, which is what a solution would mean. In other words, in terms of our new notation, this says L of y equals tan x. So anything which is a solution means if I shove in y to the machine, the output would have to be tan x.

To look at this from a different perspective, if the right-hand side of my equation had been this-- in other words, if tan x had been replaced by x cubed minus 1 sine x plus e to the x cosine x, then y equals sine x would have been a solution of this particular equation. But enough about that. And we'll emphasize that more in the exercises.

The key point as to why the word "linear" is used is that linear modifies our L machine, not the coefficients of the differential equation. See, what did "linear" mean when we were dealing with ordinary linear change of variables back in block four of the course, when we talked about u equals ax plus by and talked about linear approximations?

"Linear" meant, if the mapping was linear L of a constant times the input would be the constant times L of the input. In other words, you could factor the constant out. Notice, by the way, if I feed cu into my L machine, what will the output be? See, be very careful here. Don't be blinded by the y. y stands for the placeholder, the input. In other words, L of anything is the second derivative of that anything plus p times the first derivative plus q times that input, the anything that I put in here.

So notice that L of cu would be the quantity cu double prime plus p of x times cu prime plus q of x times cu. In other words, L of cu is equal to this expression here. Notice that differentiating a constant times a function of x means that we skip over the constants and just differentiate the function of x. In other words, differentiating this twice would just be the constant times u double prime. I can take the constants out here, I can take the constants out here.

In other words, c factors out. That's where linearity comes in. This is linear in c. I factor the c out, what's left is what? u double prime plus p u prime plus qu. That's what we're calling L of u. So L of c times u is c times L of u, which is a linear property.

And by the way, again let me emphasize why linearity definition was given the way it was. For example-- I just said here that it's not true of nonlinear-- but for example, suppose I had an equation that had y prime multiplied by a function of y rather than a function of x. In other words, let's look at L of y equals y times y prime.

If I now replace y by c times u, this becomes what? y is replaced by c times u, y prime is replaced by the derivative of cu. Well, what is the derivative of cu if c is a constant? It's c times u prime. In other words, the right-hand side here is just c squared u times u prime. u times u prime is L of u. In other words, in this example L of cu would not be c L of u. It would be c squared L of u, which is not the linear property that we're talking about.

Finally, the other property of linearity is that if I replace my input by the sum of two differentiable functions, u1 and u2, then L of u1 plus u2 turns out to be L of u1 plus L of u2. And the proof, again, is just by looking at the definition. If I feed u1 plus u2 into my L machine, I have what? u1 want plus u2 double prime plus p of x times u1 plus u2 prime plus q of x times u1 plus u2. The derivative of the sum is the sum of the derivatives, so I can split this up into u1 double prime plus u2 double prime.

Similarly, this is p u1 prime p u2 prime. This is q u1, q u2. I break down the u1 terms together, the u2 terms together. This by definition is L of u1. This, by definition of L, is L of u2. In other words, L of u1 plus u2 is L of u1 plus L of u2.

By the way, conditions one and two can be stated equivalently by the one statement L of c1u1 plus c2u2 is c1 L of u1 plus c2 L of u2. The proof will be left as an exercise. It's a fairly trivial observation.

And I think, as you may remember, that we did something at least similar to this in block four when we talked about linear mappings when we were mapping the xy plane into the uv plane, OK? But at any rate, let's see what all this means. Why are these properties so important?

And so let's call our subtopic Properties of Linear Equations. And the first thing that I'd like to point out is that if the right-hand side of our linear differential equation is 0, the interesting fact is that any linear combination of solutions of that equation is again a solution. In other words, if I find some function u1 of x that satisfies the equation L of y equals 0-- in other words L of u1 is 0-- and I also know that L of u2 is 0, then the amazing thing is that L of c1u1 plus c2u2 is also 0.

In other words, any linear combination of u1 and u2, where u1 and u2 are solutions of L of y equals 0, will also be a solution of L of y equals 0. And the proof, again, is trivial. Namely, by the properties of linearity,

L of c1u1 plus c2u2 is what? It's c1 L of u1 plus c2 L of u2. L of u1 is 0. L of u2 is 0-- that was given, you see. Consequently, this is 0. See, a constant times 0 plus a constant times 0 is 0. And that's exactly what we wanted to show over here.

A second important factor is that if I have a solution of L of y equals 0-- say, L of u is 0-- and I also have a solution v of the equation L of y equals f of x-- in other words, if L of u is 0 and L of v is f of x-- the amazing thing is, that if I add these two solutions together, the resulting function will be a solution of the equation L of y equals f of x. In other words, if L of u is 0 and L of v is f of x, then L of u plus v is also equal to f of x.

What this means is that, whenever I add onto a solution of this equation, any solution of this equation I again get back a solution of this equation. The proof, again, by definition of linearity is trivial. Namely, by linearity L of u plus v is L of u plus L of v.

But we're given that L of u is 0. We're given that L of v is f of x. And consequently, 0 plus f of x is f of x. So L of u plus v is f of x, as asserted.

By the way, let me point out here-- do not overlook the power of linearity. There is a very big danger that what you might say here is, wasn't this result true just by adding equals to equals? In other words, couldn't I just add these two results?

And the answer is yes, you can. But when you add equals to equals here, notice that what you get is what? L of u plus L of v is equal to 0 plus f of x. In other words, what you can prove by equals added to equals is that L of u plus L of v is f of x.

It was linearity that allowed us to say that L of u plus L of v was the same as L of u plus v. And to show you that in terms of a simple analogy, let's suppose I take the function f to be defined by f of x is x minus 2 times x minus 3. Well, trivially, when x is 2 or x is 3, f of x is 0, right? In other words, f of 2 is 0, f of 3 is 0.

Now, by equals added to equals, you can certainly say that f of 2 plus f of 3 is zero. But you can't say that f of the quantity 2 plus 3 is 0. In fact, 2 plus 3 is 5. If I put 5 in here, I get what? f of 5 is 5 minus 2 times 5 minus 3. That's 3 times 2, or 6. In other words, f of 2 plus f of 3 is 0, but f of 2 plus 3 is not 0, it's 6.

You see, equals added to equals gives you this result. But it's linearity that you need to be able to get from here to here. This was not a linear function, you see.

Well, let me give you an example of how to use all this stuff. Let's find all solutions of the differential equation y double prime minus 4 y prime plus 3y equals 0. Special case where the right-hand side is 0. My coefficients of y and y prime, notice, are still functions of x. They happen to be constant, but the requirement that p of x and q of x be constants is certainly not outlawed.

In other words, one special case of a function of x is the function of x, which is identically a constant. So this qualifies as a linear equation. Let me show you, then, how I tried to find all solutions of this equation.

As we've mentioned before both in part one of our course and as a motivation for defining e to the ix, a trial solution of this equation involves using e to the rx. Because we differentiate e to the rx, you get e to the rx back again. If I differentiate e to the rx, I get r e to the rx. Differentiate again, I get r squared e to the rx.

I plug that into here, factor out the e to the rx, and I wind up with e to the rx times r squared minus 4r plus 3 must equal 0. Since e to the rx is not 0, it must be that r squared minus 4r plus 3 is 0. And that says that either r is 1 or r is 3.

Remembering what r is, it means that my trial solutions should be correct when r is 1 or r is 3. Leaving the details for you to verify, it does turn out that L of e to the x is 0-- see, r is 1-- L of e to the 3x-- r is 3-- is 0. In other words, if I were to replace y by e to the x or by e to the 3x, this equation is satisfied. And in terms of our definition of L, this is the abbreviation for writing this.

Well, by our first property, the fact that this satisfies L of y equals 0 and this satisfies L of y equals 0, we had what? Any linear combination of these two must satisfy L of y equals 0. In other words, I now know in one fell swoop that every function of the form c1 e to the x plus c2 e to the 3x must also be a solution of this equation. OK?

By the way, that's another motivation-- and why we'll be doing this in block eight-- for going further into the study of vector spaces. Notice, in a sense what you're saying is, you have found a whole family of solutions which are linear combinations of e to the x and e to the 3x that somehow or other e to the 3x and e to the x behave like i and j did in the plane. Namely, every solution of this type can be written as what? A constant times e to the x plus a constant times e to the 3x.

This is like a two-dimensional vector space. But I'm not going to pursue that any further right now. I just wanted to give you a preview of coming attractions.

But at any rate, to summarize what we've done so far is that we have now found that one family of solutions of that equation is y equals c1 e to the x plus c to e to the 3x. I emphasize "one family," because so far, these are the solutions I found by assuming that the solution had to have the form y is e to the rx.

I don't know as yet whether there are other types of solutions. I'll worry about that in a little while. At any rate, the next most natural question to ask here is this-- look it. I have two arbitrary constants. And because I have two arbitrary constants, it seems to me that, just like in the first order case, not only should I be able to find a solution that passes through a particular point, but I have another degree of freedom to play around with. Maybe I can require, not only if the curve passes through a given point, but that it have a particular slope when it passes through that point.

In other words, with one constant I could make it pass through a point. With two, maybe I could make it pass through a point having a given slope. So the question now is, can I determine c1 and c2, such that for a given point x0, y0 I can find a curve, a member of this family, which passes through the point x0, y0, and has slope equal to z0. Why I use z0 instead of m here will become clear, I hope, in a few moments.

But the point is, to see if I can satisfy this system of equations, let's see what happens if I replace x and y by x0 and y0. One of my equations that must be satisfied by c1 and c2 is this. The derivative of this, which is a slope, is c1 e to the x plus 3 c2 e to the 3x. So if the slope is going to be z0, when x is equal to x0 this equation must also be obeyed.

Consequently, I must now solve-- see if I can solve these two equations for c1 and c2. By the way, this is very important to notice. Notice that this is two equations and two unknowns, and that my unknowns are c1 and c2-- that once I specify x0, y0, and z0, everything else is a known number in this problem.

To see whether this has a unique solution, the determinant of coefficients must be unequal to 0. But look at what that determinant of coefficients is. It's e x0 e 3x0, e x0, 3e 3x0. That determinant is what? 3e to the 4x0 minus e to the 4x0. That's twice e to the 4x0. And since the exponential can never be negative-- can never be 0, this determinant cannot be 0.

In other words, there is a unique member of the family y equals c1 e to the x plus c2 e to the 3x that passes through the point x0, y0 with a given slope z0. That's what we prove over here.

The only question that comes up is, is that we have now shown what? That at every point in space there is one solution from this family that passes through the given point with any given slope that you wanted to have. The question is that before we can call this the general solution, we have to be sure that there are no other solutions to the equation y double prime minus 4 y prime plus 3y equals 0. In other words, the question is are there other types of solutions-- solutions that aren't of the form e to the x, or e to the 3x, or linear combinations thereof?

And the crucial theorem is this. Suppose we can write our second order equation in the form y double prime is some function of x, y, and y prime. And notice how analogous this is to the key theorem of our last lecture. Suppose when we treat F as a function of the three independent variables x, y, and z. It turns out that F, F sub y, F sub z, are all continuous in some region R.

Then the amazing result is that for each triplet-- x0, y0, z0, in R-- see, R is three-dimensional here, because F is defined on three space-- there is a unique solution curve which passes through the point x0, y0 with slope equal to z0. Again, notice here the coding system. We are not talking about a solution passing through x0, y0, z0.

The solution curve is in the plane. See, dy dx is a slope of a curve on the plane. We have one independent variable. What we're saying is what? That there is a unique solution that passes through the point x0, y0 with slope equal to z0.

Under those conditions, the solution would be unique. Well, let's accept the truth of this theorem and apply it to a linear differential equation. Given the most general linear differential equation, to put it into the form of the crucial theorem I transpose everything but y double prime onto the right-hand side of the equation. I get y double prime is f of x minus p of x y prime minus q of xy.

Therefore, my capital F of x, y, z is obtained simply by replacing y prime by z in here. See, I simply replace y prime by z, like I did over here. I wind up with what? Capital F of x, y, z is little f of x minus pz minus qy.

Well look it. This is certainly a continuous function as soon as f, p, and q are continuous. The partial of capital F with respect to y-- remember, I'm treating x, y, and z as independent variables. If I differentiate with respect to y, notice that this drops out. The partial of capital F with respect to y is just minus q of x. The partial of capital f with respect to z is just minus p of x.

Consequently, if little f, p, and q are all continuous functions of x, automatically capital F, capital F sub y, capital F sub z will be continuous functions. Consequently, according to that crucial theorem, any solution that I find will be a unique solution. And even more to the point, even if I can't find the solution, the theorem tells me that there is a unique solution.

Well, I think that the p's and the q's sort of gets you a little bit messed up. Let's do a specific illustration of this again. Let's find now all solutions of the equation y double prime minus 4 y prime plus 3y equals e to the 2x. The same equation as before, only the right-hand side is e to the 2x rather than 0.

The key result is going to be this. I already know how to solve this equation when the right-hand side is 0. Remember, one of my properties of linearity was that if I could find any solution of this equation, then by adding onto it any solution of the equation where the right-hand side is 0-- by the way, when the right hand side is 0, the equation is called homogeneous. And I don't think that's too important now, then, to know the language when it's mentioned, but this is called homogeneous if the right-hand side is 0.

What our key theorem says is, look it-- we have solved this problem. We have found the general solution when the right-hand side is 0. Consequently, if I could find just one solution of this equation, by hook or by crook, just steal one, sneak one in. If I can find one solution of that equation, if I add that onto the general solution of the homogeneous equation, that will be the general solution of this equation.

What does that mean? Let's see how we'll tackle this. What I'm going to do is, I look at this equation. And here's how I get sneaky. I say, look it. When I differentiate, and I'm all through, what I want to wind up with is e to the 2x. Well, the only function whose derivative gives you a factor of e to the 2x, more or less, is some constant times e to the 2x itself.

So I say to myself, let me try for a solution in the form y equals some constant times e to the 2x. See, my trial solution will have this form. Well, yT prime would be this, yT double prime would be this. If I now substitute back into the original equation, yT double prime minus 4 yT prime plus 3yT must be identically equal to e to the x. That leads to the fact that minus A e to the 2x must be the same as e to the 2x.

And that tells me that if there is a solution, A had better be minus 1. You see, if x is 0 here, this is minus A equals 1. A quick check will show that minus e to the 2x is a solution of the equation. So a particular solution of this equation is minus e to the 2x. I've found one solution.

Now, here's the power of all this theory. With this one solution, I now go back to the homogeneous equation, which had as its general solution c1 e to the x plus c2 e to the 3x. I add these two together, and that is the general solution. That is the general solution of the equation that I started with.

By the way, notice just as a check-- if this were the general solution, I should be able to find unique values for c1 and c2 that allow me to make the curve-- a solution curve pass through x0, y0 with slope equal to z0. Notice that that would result in this system of equations. And I hope that you notice that since x0, y0, and z0 are constants, to determine whether c1 and c2 are uniquely determined or not hinges on the fact that the coefficient matrix, the determinant of coefficients, is still the same determinant of coefficients that I had in the homogeneous case.

By the way, let me just stress one more point that I forgot to mention over here. When I wrote down this equation, notice that for the fundamental theorem to be true, all that we required was that p, q, and little f be continuous. Notice that p in this problem is minus 4, q is 3, and f is e to the 2x. Certainly, the function f of-- the function which is identically minus 4, the function which is identically 3, and the function which is e to the 2x are all continuous functions.

So what that told me was that once I find one family of solutions, I've found them all, so in the linear case, there is a general solution. There are no singular solutions in a problem of this type. There can't be any [? mongrel ?] solutions, because this particular equation meets the requirements of the crucial theorem. At any rate, the exercises will take care of this in giving you drill.

Let me simply summarize what we've done today in terms of linear differential equation. So summary is this. Let's suppose that we're given a general-- second order is what I've been dealing with, but it's true for higher orders, too-- linear equation of the form y double prime plus p y prime plus qy equals f of x, where p and q are arbitrary functions of x. What I'm saying is to find a general solution of this equation-- and by the crucial theorem, this general solution will exist if f, p, and q are continuous-- first of all what I do is I find the general solution y sub h of the homogeneous equation L of y equals 0. In other words, I simply replace the right-hand side of this equation by 0 and find the general solution of this equation.

In terms of the theory, what we're saying also is-- and this I'll say this as an aside-- if I can find two solutions of this equation which are not scalar multiples of one another-- in other words, if I can find two functions u1 and u2 such that L of u1 and L of u2 are 0-- but that u2 is not a constant times u1, then that homogeneous solution-- in other words, the solution of this equation-- will simply be c1u1 plus c2u2. See, that was all in your combinations of these two.

And the proof of that is quite simple. Namely, what we want is what? Given the point x0, y0, we want to be sure that we can determine c1 and c2 such that a curve passes through x0, y0 with slope z0. That means I must be able to do what? I must be able to solve this pair of equations. Namely, if I replace x by x0 and y by y0, and if I differentiate the equation and then replace x by x0 and dy dx by z0, because that's the derivative, the slope I want at this point, I again wind up with what? Two equations with two unknowns.

Remember, once I specify x0, y0, and z0-- these are given constants, c1 and c2 are my only variables-- to see whether there is a unique solution here, it means that this determinant of coefficients must be unequal to 0. The determinant of coefficients is just u1 u2 prime minus u1 prime u2. That must be unequal to 0.

This suggests the quotient rule again. This expression to be unequal to 0 is the same as this divided by u1 squared to be unequal to 0. This expression here is nothing more than a derivative of u2 divided by u1 with respect to x. And for that to be unequal to 0 simply says what? That u2/u1 is not a constant. And that says that u2 is not a constant times u1.

In other words, this is just an aside to show you how you find the general solution of the homogeneous equation. But the point is what? First of all, you find the general solution of the homogeneous equation, meaning you take L of y equals f of x, replace f of x by 0, and find the general solution of L of y equals 0. After you've done that, you find any old solution, y sub p, of the given equation L of y equals f as x. Just one solution, by hook or by crook.

Then finally, when you have the general solution of the reduced-- the homogeneous equation and the particular solution of this equation, the general solution of this equation will just be the sum of these two. All right? And that's what we're going to be talking about in the next few lessons, see.

What we're going to drill on is what this stuff means. And I think you can now guess what the next lectures are going to be about. After all, since to find the general solution of the linear differential equation I need the general solution of the homogeneous equation and a particular solution of the original equation, the two separate topics I now have to tackle are, one, how do I find the general solution of the reduced equation; and two, how do I find a particular solution of the original equation?

That, by the way, comes under the heading of [? cookbook ?] again. That's drill. But this is the underlying theory, the underlying philosophy. We will talk more about how to handle the techniques in our subsequent lectures. At any rate, then, until next time, goodbye.

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