[SQUEAKING] [RUSTLING] [CLICKING]

GILBERT STRANG: This lecture is-- applies to every matrix. And it comes, really, near the beginning of linear algebra.

Elimination is the powerful way to make the matrix simpler. Get a lot of zeros in it is what is actually achieved by elimination. But then it's a little question. What have I done? How do I describe the answer? And I think factorization, which you'll see are the fundamental theme in linear algebra, are-- give a great description of elimination. You can see what you've done.

And this factorization takes any matrix A, any rectangular matrix, square or rectangular, and finds a column matrix and a row matrix that multiply. So it's maybe the first of our six factorizations that really organize the linear algebra course. So we're ready to start on this. So elimination is familiar. But this factorization is more-- is newer.

So I'm going to take an example. There is a matrix A, three-by-four, pretty arbitrary matrix. And I want to do elimination to it. Let me just describe elimination, how it would start in words.

I would say, I want to turn that 3 into a 0. So I take 3 times row 1 and subtract from row 2. And that gives me a 0 there. And it gives me something else. And I take 4 times row 1 from row 3. And that gives me a 0, I think.

This is the process that you repeat, multiples-- and the subtractions can go upwards here. After I've taken three of these away from row 2, I have a new row 2. I can take some multiple of that row 2 away from row 1 and produce that 0.

I don't plan to discuss all the horrible details of elimination right here, just this-- but I'll tell you the operations. And for this matrix, I end up with that matrix. This is what elimination goes to if we go the whole way. We get a couple of columns.

So how do I understand this matrix Z, which came from A? Well, let's see. If I look at the rows, I see two nonzero rows. And that turns out-- they turn out to be a basis for the row space. The combinations of those two rows give me any-- all of the three rows of A.

And notice that there's a 0 row. So the rank is only 2 for this matrix. This matrix has only two nonzero rows in-- when-- after-- at the end of elimination. And so they're a basis for the row space, which is two-dimensional.

And the columns, the first two-- so the first two columns here are-- that's the destination. That's the goal of elimination, to get these ones and zeros. So they tell us that these two columns were-- and we'll call those-- those are the first independent columns.

These columns are independent of the previous ones. But the last two columns are combinations of the first two. So the column space of this matrix is the same as the column space is-- sorry, has the same basis, the same-- we learn about the column space of this matrix from the column space of this one.

So that's point two. The columns of A, these first two columns, are a basis for the column space of A because they became, clearly, the basis when we went to Z.

And finally, the null space-- if we want to solve Ax equals 0, we-- this makes us-- this moves us to solving Zx equals 0. That null space is-- null meaning 0 and that-- on the right-hand side.

So to solve Zx equals 0 is easy. And we'll do it in a minute. And that will be the same solutions as Ax equals 0. So that's what elimination achieves. These subspaces have-- move in a simple one-- simple form, are connected in a nice way from A to Z.

Ready for the next slide. And I make it happen. And now my new point is to see that elimination process, which has been taught in linear algebra for quite a long time. I want to see it. This step from the A that we started with to the elimination matrix that we ended with tells me that this original matrix factors into a column matrix times a row matrix, C times R.

That'll be then the first factorization in the linear algebra textbook. It comes first. And then the great factorizations of linear algebra, like eigenvalues and singular values, are-- build out of this. But this is a factorization.

Every matrix A factors by doing that elimination into the column matrix. It has two independent columns, just as A has. And it has two independent rows, just as A has. And what does that tell us?

That tells us that 2 equaling 2 there-- that the column rank and the row rank are the same, which is really the first wonderful fact in linear algebra-- that if I take any matrix-- and if you just glance at that, you can't see how many rows are independent, how many columns are independent. But the number is 2 in both cases, always equal.

Now I have to explain-- use matrix notation to say what that R matrix is. So I start with A. I'll always use little r for the rank, the number of independent columns. And you remember that those go into C.

So I'm just able to tell you what the R factor is. So that's maybe the key idea here that's not usually in textbooks. What is that R? Well, it starts with the identity matrix. And then it has a matrix F that accounts for the dependent columns.

So the identity matrix is sitting in the columns that are independent in A, the first little r columns, the independent columns, and then the F matrix. So the F matrix-- the matrices there are dependent on the columns in C. And that tells us that they're a combination.

So what's there is those columns of the matrix A have the form C times F. So F is the matrix that makes this correct. And annoyingly, there-- we may need to reorder the columns because in my nice example, the independent columns came first, columns 1 and 2. But I could-- if I moved those off to the right, then I need to get them back to the left again for my nice picture.

And that permutation P that does it-- don't pay a lot of attention to P. Here's the important part-- the independent columns and then the matrix that tells us how to get the dependent columns from the independent columns by multiplying by F.

So that's the factorization. And now I want to use it. I did an example that involves P that requires a permutation. It's a good example to look at. So there's our original matrix A. And if I look for the independent columns, well, column 1 is great. Column 2 is not so great because it's dependent. It's just 2 times column 1, nothing new in column 2.

So that's a dependent column. And then column 3, we do have an independent column. So the two independent columns go into C. And that's the matrix that multiplies R. And you see how a permutation in this form-- the identity is over on the-- fully on the left side-- that this has columns 1 and 2 as special.

But it's the permutation then that makes columns 1 and-- puts them into columns 1 and 3, which are really the special independent columns for A, that and that. So that's just a word about P. And from now on, I'll try to forget that there might be a P matrix.

The essential information-- what does elimination accomplish after all these thousands of years? It was invented in China more than 2,000 years ago. The information it is-- it tells us which columns are independent there in the identity matrix-- shows up in those columns. And it also tells us the matrix-- how the dependent ones come from the independent ones.

You multiply C by some matrix F, which elimination finds. So elimination finds all these matrices, as we saw in that example. And now I want to use that to see what that matrix, right down here with a slide or two-- just exactly what steps of elimination are allowed.

So elimination has three simple steps. And all of them are reversible, like exchange two rows because if I had a 0 in the top left corner, I wouldn't want that. So I would do a row exchange. I can divide a row by a number. If I had a 2 in the top left corner, I'd rather have a 1. So I just divide that row by 2.

And this is the main step to subtract a multiple of one row from another row. That doesn't change the row space, just changes the rows themselves. But their combinations have not changed.

All these three steps are reversible. If I exchange two rows, I could exchange them again. And I'd be back to where I started. If I divide by 7, then I could multiply by 7, or divide by 1/7.

So this is what is always-- if you get this topic in a linear algebra course, this is the matrix that you get to. And what I want to do is understand that matrix as coming-- as entering a factorization of A. So I can use elimination. But that's the picture of what I get, as we saw.

This is the key fact. We're ending the work of elimination. It will find out how many independent columns there are, how many independent rows. And that number is the same because we're looking at the identity matrix. And then we see this is the simplification of the matrix A.

So what can we do? What do we learn from that? So I put in a few slides just to say how would you code elimination. And it's really neat. How does elimination work? Well, column by column.

So suppose we have got k columns done and we're ready for the next one. We may have seven columns taken care of. We're ready for column 8. So this is what column 7 looks like because it's that same I, F, 0, 0, P picture.

Those seven columns are finished. And now here's one. And I let u be the upper part, upper half, and l be the lower half. U is the half that's or the part that's of the new column that's-- that goes up at the top. And even with the zeros goes l.

So what do we do? And what does the elimination do with this next column? So I'm describing elimination, again, in a way you would code it. I come to the new column. And I want to know-- I want something, a picture like this, with k moved up to k plus 1.

It's a lot to give you the-- all the details of elimination. But everybody catches on to them by doing a few, either on a computer or by hand. So our question is, this is-- I want a picture like this after-- with one more column in it. And the question is, is this new column going to feed into the identity?

Well, if l is the zero vector, if it's all zeros below the pivot, then this u will go along with F because we're not getting anything new. We still have k nonzero rows. So at this point, we have k nonzero rows, or let's say whatever number we have. And if l is 0, then the new column is dependent on the old columns and doesn't add anything new.

Again, I'll just repeat. If l is the zero vector, if I'm all zeros here, then that will go in with F. If it's not all zeros there, then I'm going to do some row exchanges and some pivot steps. What I can do is clear out all of l except for one nonzero. That will be the new pivot. Maybe the next-- I hope the next--

So if l is all zeros, the new column joins with F. If l is not all 0, the new column is independent. So we've hit another independent column. And we do elimination on that column to get it into the right form to be ready for column-- column k plus 2 is coming next.

The way to learn the steps of elimination is to do them on a few matrices. And I won't do more here. But I want you to know what is elimination telling us. So the result of elimination tells us which are the first independent columns. They're the ones that end up with a 1, with the identity matrix. And those first columns back in A are the columns of C.

The row space is not changed. Those operations of elimination don't change the row space. But they give us a nice picture. So this thing is-- this slide is really telling us that the result of elimination is the identity matrix, the C matrix, and zeros in the lower rows. We can see how many rows are independent by how many-- the nonzero rows are all independent because that identity is-- makes that. So that's where we are at the end of elimination.

We have little r, the rank. The dimension of the row space, the dimension of the column space, is little r. And we can see the basis for-- in C. And then we also have that matrix F. So we got to C and F. And elimination finished by giving us a picture of the original matrix much simplified, lots of zeros.

And let's just apply it. So here we go. This is the payoff. Do you remember our matrix that had 1, 3, 2, 7? That was my example at the very beginning. But it had some combination of those two rows. So its third row I've just forgotten because it turned out to be a combination of these and gave us a row of zeros.

So these are the two equations in Ax equals 0. We're looking for the null space. This is what elimination is-- tells us how to do. It's a step to simplify the equations from that messy-looking two equations to this simple pair of equations where the 1 and the 1 are in columns 1 and 2 and nice numbers. And the point is you can solve those without taking a linear algebra course. But still, take the course.

So I'm looking for solutions to that. So I could find a solution if I set x3 to be 1 and x4 to be 0. So x4 is 0. That's gone. x3 is 1. So x1 plus 3 equals 0. x2 plus 4 equals 0. So x1 and x2 are minus 3 and minus 4. And there's the 1 and the 0.

Did you see that we produced that solution? Just by eye, it jumps out. And let's see how the next one jumps out. So we suppose we take those equations. Those are the simple form of this horrible, horrible pair. We got it to a very simple form. And now I want to know another solution.

And another different, certainly different, one will be to set x3 equal to 0. So that's gone. x4 equal to 1-- so that's 5 and 6. And then I need x1 to be minus 5 and x2 to be minus 6 to cancel the 5 and 6. Golden-- solution simple, solution complete.

So I've got two solutions to Ax equals 0 and the null space as the right dimension 2. And I've found a basis for that space. And all other solutions are combinations of these simple solutions. The simple solutions have zeros and ones in the dependent columns.

I hope you can just see that after elimination, when we get to a simple form like that, it becomes a picnic to solve it. The equations are easily solved. Just let x3 equal 1 and x4 equal 0, and you get an answer. Let x3 equals 0 and x4 equal 1, and you get another answer.

So that's the point of elimination. It's good to make the-- what the point of learning this process or coding up this process. I have to admit that it's not the most super efficient way to find the solution because it involves subtraction of rows from rows above and as well as subtraction from rows below. So it could be speeded up. And it usually is for square invertible matrices.

Let me ask you this question. If I had a square four-by-four invertible matrix, what would-- over here, what would be the reduced form after I do elimination? Well, it would just be identity, the identity matrix.

If I had a four-by-four-- if I have an invertible matrix, elimination produces the identity matrix. So it really goes all the way to simplifying the matrix. And you pay a little price if-- to get all the way to the identity matrix.

But to see the theory, it's a right thing to do. We wouldn't actually do it. We would stop at a triangular matrix and start solving from that point if we had to pay for the computer time.

And now here, I've written in with matrix notation what that last slide showed you. And again, what that last slide showed you is that once you have it-- have completed elimination, it's a cinch to solve Ax equals 0. Elimination puts A into that form. And then the solutions x are in this form. And if I multiply that by that, I get 0.

So the columns of x solve Ax equals 0. I have found the null space. I've computed the null space. And what I wanted to do was to show that elimination does its job, makes it simple. It makes the null space jump out. You just need the F that comes from elimination, the possible permutations, and you've got the answer. You've got the x's that solve Ax equals 0.

Again, it's not the code that's used all the time in computing problems in physics or engineering. But it's-- for math, it-- elimination-- completed elimination, as we've done, really-- gets you to the point where you can read off the answer.

So that's the point, really-- is to see how far elimination gets you. So here's a comment that I have just been making. The method that we push all the way to get the identity matrix is not as efficient as Gauss-- Gauss, the greatest mathematician of all time. He knew enough to stop when the matrix was triangular instead of getting to a matrix that's-- could well be diagonal.

He doesn't do the subtractions from upper rows. He just goes from-- subtracts multiples of a row from lower rows. And that makes the matrix triangular. And then solving is easy. So-- just want you to-- and that's the steps that you-- really, everybody learns in linear algebra, elimination from the matrix, from a square matrix to a triangular one. And then solution is a cinch.

I'm near the end. I just want to express what I did in matrix notation. Actually, that's the fundamental idea of this lecture that I'm-- extra lecture on elimination-- is to see what's happening in matrix language. We know if you take the course and you grind through a few examples of elimination, then you know the steps.

This lecture is about sit back, don't panic with each step. But what's the big picture? And the big picture is that you started with a matrix like this. So you see it's a matrix with submatrices.

The rank-- this matrix W here is the-- is square. And it's full. If the whole matrix A has r independent columns and r independent rows, then we move them into this position, W.

So I want to start elimination again. You'll see I'll get it in one quick line. I want to start-- think about elimination. But let me suppose that I'm starting with a matrix like my example that has a square invertible matrix there.

And what does elimination do? What does a-- well, we know that this is the result. So this is what goes in. And this is what goes out, just two-by-two block matrices. I call them a block matrix because these are blocks of numbers, not just single numbers.

Of course, to understand the picture-- quite good to take just single numbers. We might have the numbers 3, 7, 6, 11, something like that. And what does it reduce to? And the point is elimination. We get the identity matrix. W goes to the identity matrix.

So elimination is really inverting-- finding the inverse of W. That's what elimination is doing. And that was my goal, to find out what the heck is elimination doing as I take all these steps. What it's doing is it's finding-- it's inverting that matrix W. But of course, it's operating on whole rows.

The subtractions from down here leave zeros. And the steps up here multiply the W by its inverse to leave the identity. This is what we recognize. That's the r matrix after elimination.

If there are dependent rows, they turn into zero rows. The independent columns turn into-- start with the identity after elimination. That's the whole point. That's the point of elimination, to get from a general matrix to that kind of a matrix. And that's what it does. That's the row-reduced echelon form that elimination does. So that's really-- this is the simplest explanation I could give.

Now, you might say, wait a minute. This is assuming that W up in that left corner is a nice, square, invertible, full-rank matrix. Then we go here. Suppose it isn't. Suppose I have a 0 up in the top left corner or something like that. Then what do I do? Well, I do some row exchanges to get-- and maybe some column exchanges to get to this W, to get to this invertible block-- pivot, you could call it, which turns into I.

So I have to-- so the next slide will show you the possible steps. I may have to use some permutation, some reordering of the rows. You got to be up for those. It can have a reordering of the columns. And that will get it into the-- into what we want. That gets it into what we want. And then elimination gets us where we really want.

So this is a-- this is-- we're at the end now. This is the picture of elimination where-- written in its simplest matrix form. I do permutations of rows. I reorder the rows and the columns to get the first columns and the first rows to be independent. And that means that they have an r-by-r matrix W that's invertible, then H and J or whatever they are.

But then the next step of elimination-- the two jobs of elimination are get things in the right order and get zeros, produce zeros. So here, I'm supposing that I get them in the right order there. And then I produce the zeros there.

And I'll also produce a bunch of zeros in that identity matrix. And this is the F. This is our friend, F, which we're giving-- we're shining a light in this lecture on this matrix F that got really ignored for 1,000 years.

So there you go. I'm not sure if we're through. I feel I'm through. I've-- see it says 14 of 16. So there must be some brilliant ideas on the last things.

So interesting point-- a question is, where is that W, that matrix of-- that invertible r-by-r matrix that is the basis for elimination, that W up in the corner? And the theorem is that if I find r independent rows, which there are in the row space, if I find r independent columns-- so suppose the rows are in a row-- in a matrix B with r rows, the columns are in a matrix C, as before, with r independent columns, then where B meets C-- can you see this on my-- so B is coming along as independent rows. C is r independent columns.

And where they meet, I claim, is an r-by-r matrix that's invertible. It has independent rows and independent columns. So for me, this was the real math principle that wasn't-- so I wasn't so sure about, but it works.

Just to tell you that the paper is in a journal called *Mathematics Magazine*-- it's almost published as I make this recording. It's the Math Association of America. And then there's also a website that has a lot of math papers put in there before publication. And they have some crazy-long number to identify them. But this is the linear algebra textbook that uses this lecture.

Actually, this lecture is not in the textbook. It's separate because the-- I didn't want to fill the textbook by this complicated thing where the idea is simple, where you simplify your matrix A by elimination. And then you can read off all four subspaces and-- all four fundamental subspaces.

And the idea is to say, what is elimination actually doing to put in a little more of the details, which are in this talk, but not appropriate for-- to-- necessarily to go into the textbook? So there it is. The paper that takes these steps is in a journal that many libraries will take.

Well, thanks for your patience with this unusual step. I didn't know it was going to happen. But it just-- I was never happy with elimination without having a matrix description. What happens with the matrix pieces of my overall matrix A? And that's what this lecture was about.

So thanks for joining. And best wishes in linear algebra. It's a beautiful subject. Thanks.