A Dimensions of Irreducible Characters

In this section, we provide a proof of the final part of the main theorem of representation theory:

Theorem A.1
If $\rho : G \to \mathrm{GL}(V)$ is an irreducible representation of dimension d, then d divides $|G|$.

These notes are based on a writeup by Professor Bezrukavnikov posted to Canvas.

Recall that we extended the definition of ρ to all *linear combinations* of elements in G, or equivalently functions $f : G \to \mathbb{C}$, using the natural formula

$$\rho(f) = \sum_{g \in G} f(g)\rho(g).$$

Then $\rho(f)$ is in $\mathrm{End}(V)$ for any function f.

To start with, we find a natural construction in which $|G|/d$ arises.

Proposition A.2
For any irreducible representation $\rho : G \to \mathrm{GL}(V)$ of dimension d, we have

$$\rho(\overline{\chi_\rho}) = \frac{|G|}{d} \cdot \mathrm{Id}\,.$$

Proof. Since $\overline{\chi_\rho}$ is a class function, then $\rho(\overline{\chi_\rho})$ is G-equivariant. But by Schur's Lemma, since ρ is scalar, the only G-equivariant endomorphisms are scalar maps; so $\rho(\overline{\chi_\rho})$ must be of the form $\lambda \cdot \mathrm{Id}$ for some $\lambda \in \mathbb{C}$. Now we can compute λ by taking the trace: we saw earlier that $\mathrm{Tr}\,\rho(f) = |G|\langle \chi_\rho, \overline{f}\rangle$, so

$$\mathrm{Tr}\,\rho(\overline{\chi_\rho}) = |G|\langle \chi_\rho, \chi_\rho\rangle = |G|,$$

using the fact that the irreducible characters are orthonormal and therefore $\langle \chi_\rho, \chi_\rho\rangle = 1$. But this trace must also be $d\lambda$, so $\lambda = |G|/d$. (The properties used in this proof are discussed in more detail in Lecture 7.) □

Now in order to prove that $|G|/d$ is an integer from here, we use a bit of theory about algebraic integers.

Definition A.3
A complex number is a **algebraic integer** if it is the root of a monic polynomial with integer coefficients.

Lemma A.4
Algebraic integers have the following standard properties:

(a) If α and β are algebraic integers, so are $\alpha + \beta$ and $\alpha\beta$.

(b) If $\alpha \in \mathbb{Q}$ is an algebraic integer, then $\alpha \in \mathbb{Z}$.

The course discusses algebraic integers in more detail in future lectures; the two properties listed here are proved in Lectures 14 and 25, respectively.

It is now enough to prove the following proposition:

Proposition A.5
Let $\rho : G \to \mathrm{GL}(V)$ be any representation of G. Then if $f : G \to \mathbb{C}$ is a function such that $f(g)$ is an algebraic integer for every g, and $\rho(f) = r \cdot \mathrm{Id}$ for a rational number r, then r must be an integer.

It's clear that the two propositions together imply our theorem — by the first proposition, we have that $\rho(\overline{\chi_\rho}) = |G|/d \cdot \mathrm{Id}$, and we know that $\overline{\chi_\rho}(g)$ is an algebraic integer for all g, since $\chi_\rho(g)$ is a sum of roots of unity (and roots of unity are all algebraic integers). So by the second proposition, $|G|/d$ must be an integer.

In fact, a stronger statement is true — if f is *any* function on G such that $f(g)$ is an algebraic integer for all $g \in G$, then every eigenvalue of $\rho(f)$ is an algebraic integer. But this is much harder to prove, so we will only prove the special case necessary for our theorem.

Proof. We will show that $\operatorname{Tr}\rho(f)^n$ is an integer for all n, which suffices — this is because $\rho(f)^n = r^n \cdot \mathrm{Id}$, so dr^n is an integer for all n, and therefore r must be an integer (if a prime p divided its denominator, then for sufficiently large n the power of p in the denominator of r^n would be greater than the power of p dividing d).

When $n = 1$, we have

$$\operatorname{Tr}\rho(f) = \sum_{g \in G} f(g)\chi_\rho(g),$$

and $f(g)$ and $\chi_\rho(g)$ are both algebraic integers. So $\operatorname{Tr}\rho(f)$ is an algebraic integer. But this trace is also rational, as it is equal to dr; therefore $\operatorname{Tr}\rho(f)$ is an integer.

Now for the case of general n, it is enough to find a function f_n such that $\rho(f)^n = \rho(f_n)$ and $f_n(g)$ is again an algebraic integer for all $g \in G$ — then we can apply the above reasoning to f_n instead. To find such a function, we use the following construction:

Definition A.6

Given two functions $\phi : G \to \mathbb{C}$ and $\psi : G \to \mathbb{C}$, their **convolution** is the function $\phi * \psi$ defined as

$$(\phi * \psi)(g) = \sum_{h \in G} \phi(h)\psi(h^{-1}g).$$

Lemma A.7

For any two functions ϕ and ψ, we have

$$\rho(\phi * \psi) = \rho(\phi)\rho(\psi).$$

Proof. The space of functions on G has a basis consisting of the functions δ_g which map g to 1 and all other elements to 0, where $\rho(\delta_g) = \rho(g)$ for each $g \in G$. Then convolution is defined by setting $\delta_g * \delta_h = \delta_g\delta_h$ for all $g, h \in G$ and extending to all functions using linearity. So we have

$$\rho(\delta_g * \delta_h) = \rho_{gh} = \rho_g\rho_h = \rho(\delta_g)\rho(\delta_h),$$

and the statement for general functions ϕ and ψ then follows from linearity. ☐

Then we can take

$$f_n = \underbrace{f * f * \cdots * f}_{n \text{ times}}.$$

This satisfies $\rho(f_n) = \rho(f)^n$, and since f_n is constructed by repeatedly taking sums and products of algebraic integers, $f_n(g)$ must be an algebraic integer for all g as well.

So then $\operatorname{Tr}\rho(f)^n = \operatorname{Tr}\rho(f_n)$ is an integer for all n, as desired. ☐

This concludes the proof of the theorem.

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