Set $\mathfrak{e}_m(x) = e^{\imath 2\pi m x}$ for $m \in \mathbb{Z}$ and $x \in \mathbb{R}$, and observe that $\{\mathfrak{e}_m : m \in \mathbb{Z}\}$ is an orthonormal family in $L^2(\lambda_{[0,1)};\mathbb{C})$. Even though it involves an abuse of notation, we will use $(\varphi, \mathfrak{e}_m)_{L^2(\lambda_{[0,1)};\mathbb{C})}$ to denote $\int_{[0,1)} \varphi(y)\mathfrak{e}_{-m}(y)\,dy$ for $\varphi \in L^1(\lambda_{[0,1)};\mathbb{C})$.

Given a function $\varphi : [0,1) \longrightarrow \mathbb{C}$, define its periodic extension $\tilde\varphi : \mathbb{R} \longrightarrow \mathbb{C}$ by $\tilde\varphi(x) = \varphi\big(x - \lfloor x \rfloor\big)$, where $\lfloor x \rfloor = \max\{n \in \mathbb{Z} : x \geq n\}$. Notice that if[1]. $\varphi \in L^1(\lambda_{[0,1)};\mathbb{C})$, then

$$\int_{[0,1)} \varphi(x)\,dx = \int_{[a,a+1)} \tilde\varphi(x)\,dx \text{ for all } a \in \mathbb{R}.$$

Similarly,

$$\int_{[0,1)} \tilde\varphi(-x)\,dx = \int_{[0,1)} \varphi(x)\,dx.$$

For bounded, continuous functions φ and ψ on $[0,1)$, define

$$\varphi * \psi(x) = \int_{[0,1)} \varphi(x-y)\psi(y)\,dy,$$

and use the preceding to check that

$$\varphi * \psi(x) = \int_{[-x,-x+1]} \tilde\varphi(y)\tilde\psi(x-y)\,dy = \psi * \varphi(x).$$

Finally, by the continuous version of Minkowski's inequality,[2]

$$\|\varphi * \psi\|_{L^p(\lambda_{[0,1)};\mathbb{C})} \leq \|\varphi\|_{L^p(\lambda_{[0,1)};\mathbb{C})}\|\psi\|_{L^1(\lambda_{[0,1)};\mathbb{C})} \wedge \|\psi\|_{L^p(\lambda_{[0,1)};\mathbb{C})}\|\varphi\|_{L^1(\lambda_{[0,1)};\mathbb{C})}$$

for any $p \in [1,\infty)$. Hence, for each $p \in [1,\infty)$, $(\varphi,\psi) \rightsquigarrow \varphi * \psi$ has a unique continuous extension as a map bilinear map from $L^1(\lambda_{[0,1)};\mathbb{C}) \times L^p(\lambda_{[0,1)};\mathbb{C})$ into $L^p(\lambda_{[0,1)};\mathbb{C})$, and

$$\|\varphi * \psi\|_{L^p(\lambda_{[0,1)};\mathbb{C})} \leq \|\varphi\|_{L^1(\lambda_{[0,1)};\mathbb{C})}\|\psi\|_{L^p(\lambda_{[0,1)};\mathbb{C})} \tag{1.1}$$

continues to hold.

Theorem 1.1. *If $\varphi \in L^p\big(\lambda_{[0,1]};\mathbb{C}\big)$ for some $p \in [1,\infty)$, then*

$$\lim_{r \nearrow 1} \Bigg\| \varphi - \sum_{m \in \mathbb{Z}} r^{|m|}\big(\varphi, \mathfrak{e}_m\big)_{L^2(\lambda_{[0,1)};\mathbb{C})}\mathfrak{e}_m \Bigg\|_{L^p(\lambda_{[0,1]};\mathbb{C})} = 0,$$

and, if $\varphi \in C\big([0,1];\mathbb{C}\big)$ satisfies $\varphi(0) = \varphi(1)$, then[3]

$$\lim_{r \nearrow 1} \Bigg\| \varphi - \sum_{m \in \mathbb{Z}} r^{|m|}\big(\varphi, \mathfrak{e}_m\big)_{L^2(\lambda_{[0,1)};\mathbb{C})}\mathfrak{e}_m \Bigg\|_{\mathrm{u}} = 0.$$

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[1] λ_S is the Lebesgue measure on a subset S of $\mathbb{R}^N$

[2] $L^p(\mu;\mathbb{C})$ is the Lebesgue space for the measure μ of $\mathbb{C}$-valued functions, and $\|\cdot\|_{L^p(\mu;\mathbb{C})}$ is the corresponding norm.

[3] $\|\cdot\|_{\mathrm{u}}$ is the uniform (i.e., supremum norm).

Proof. Define

$$p_r(x) = \sum_{m\in\mathbb{Z}} r^{|m|}\mathfrak{e}_m(x) \text{ for } r \in [0,1) \text{ and } x \in [0,1).$$

Clearly $\int_0^1 p_r(x)\,dx = 1$, $p_r(-x) = p_r(x)$, and $\widetilde{p}_r$ is continuous. In addition,

$$p_r(x) = \frac{1}{1-r\mathfrak{e}_1(x)} + \frac{r\mathfrak{e}_{-1}(x)}{1-r\mathfrak{e}_{-1}(x)} = \frac{1-r^2}{|1-r\mathfrak{e}_1(x)|^2} = \frac{1-r^2}{1-2r\cos 2\pi x + r^2} \text{ for } r \in [0,1),$$

and so $p_r \geq 0$.

Obviously,

$$\sum_{m\in\mathbb{Z}} r^{|m|}\big(\varphi, \mathfrak{e}_m\big)_{L^2(\lambda_{[0,1)};\mathbb{C})}\mathfrak{e}_m(x) = p_r * \varphi(x) = \int_{[0,1)} p_r(y)\tilde{\varphi}(x+y)\,dy$$

since p_r is even. Now suppose that $\varphi \in C\big([0,1]:\mathbb{C}\big)$ with $\varphi(0) = \varphi(1)$. Then, since $\lim_{r\nearrow 1}\int_\delta^1 p_r(y)\,dy = 0$ for each $\delta \in (0,1)$, it is easy to check that

$$\varlimsup_{r\nearrow 1}\sup_{x\in[0,1]}\left|\int_0^1 \big(\varphi(x+y)\,dy - f(x)\big)\right| \leq \omega_\varphi(\delta),$$

where ω_φ is the modulus of continuity of φ. Thus the second part of the theorem has been proved.

To prove the first part, let $\varphi \in L^p(\lambda_{[0,1)};\mathbb{C})$, and choose choose a sequence $\{\varphi_k : k \geq 1\} \subseteq C\big([0,1];\mathbb{C}\big)$ which satisfy $\varphi_k(0) = \varphi_k(1)$ and $\|\varphi - \varphi_k\|_{L^p(\lambda_{[0,1]};\mathbb{C})} \longrightarrow 0$ as $k \to \infty$. Then, for each k,

$$\begin{aligned}&\|p_r * \varphi - \varphi\|_{L^p(\lambda_{[0,1]};\mathbb{C})}\\ &\leq \|p_r * (\varphi - \varphi_k)\|_{L^p(\lambda_{[0,1]};\mathbb{C})} + \|p_r * \varphi_k - \varphi_k\|_{L^p(\lambda_{[0,1]};\mathbb{C})} + \|\varphi_k - \varphi\|_{L^p(\lambda_{[0,1]};\mathbb{C})},\end{aligned}$$

and so, for all k.

$$\varlimsup_{r\nearrow 1}\|p_r * \varphi - \varphi\|_{L^p(\lambda_{[0,1]};\mathbb{C})} \leq 2\|\varphi_k - \varphi\|_{L^p(\lambda_{[0,1]};\mathbb{C})}.$$

Finally, let $k \to \infty$. □

Theorem 1.2. *$\{\mathfrak{e}_m : m \in \mathbb{Z}\}$ is an orthonormal basis in $L^2(\lambda_{[0,1)};\mathbb{C})$, and so, for each $\varphi \in L^2(\lambda_{[0,1)};\mathbb{C})$,*

$$\sum_{m\in\mathbb{Z}}(\varphi, \mathfrak{e}_m)_{L^2(\lambda_{[0,1)};\mathbb{C})}\mathfrak{e}_m \equiv \lim_{n\to\infty}\sum_{|m|\leq n}(\varphi, \mathfrak{e}_m)_{L^2(\lambda_{[0,1)};\mathbb{C})} = \varphi,$$

where the convergence is in $L^2(\lambda_{[0,1)};\mathbb{C})$. In addition, for all $\varphi, \psi \in L^2(\lambda_{[0,1)};\mathbb{C})$,

$$(\varphi, \psi)_{L^2(\lambda_{[0,1)};\mathbb{C})} = \sum_{m\in\mathbb{Z}}(\varphi, \mathfrak{e}_m)_{L^2(\lambda_{[0,1)};\mathbb{C})}\overline{(\psi, \mathfrak{e}_m)}_{L^2(\lambda_{[0,1)};\mathbb{C})}.$$

Proof. It suffices to check the first statement, and to do so all we need to know is that $(\varphi, \mathfrak{e}_m)_{L^2(\lambda_{[0,1)};\mathbb{C})} = 0$ for all $m \in \mathbb{Z}$ implies $\varphi = 0$ for a set of φ's which is dense in $L^2(\lambda_{[0,1)};\mathbb{C})$. But, by Theorem 1.1, we know this for continuous φ's satisfying $\varphi(0) = \varphi(1)$, and these are dense in $L^2(\lambda_{[0,1)};\mathbb{C})$. □

Define the partial sum $S_n\varphi = \sum_{|m|\leq n}(\varphi, \mathfrak{e}_m)_{L^2(\lambda_{[0,1)};\mathbb{C})}\mathfrak{e}_m$.

Corollary 1.3. *If* $\varphi \in C([0,1];\mathbb{C})$ *and*

$$\sum_{m\neq 0} \big|(\varphi, \mathfrak{e}_m)_{L^2(\lambda_{[0,1)};\mathbb{C})}\big| < \infty,$$

then the series

$$\sum_{m\in\mathbb{Z}} (\varphi, \mathfrak{e}_m)_{L^2(\lambda_{[0,1)};\mathbb{C})}\mathfrak{e}_m(x)$$

is uniformly absolutely convergent to φ*. In fact,*

$$\big\|S_n(\varphi) - \varphi\big\|_{\mathrm{u}} \le \sum_{|m|>n} \big|(\varphi, \mathfrak{e}_m)_{L^2(\lambda_{[0,1)};\mathbb{C})}\big|.$$

Proof. That the series if uniformly absolutely convergent is obvious. To see that it must be converging to φ, let ψ be uniform limit of $\{S_n\varphi : n \ge 0\}$. Then ψ is continuous and, because φ is the $L^2(\lambda_{[0,1)};\mathbb{C})$ limit of this series, $\psi = \varphi$ $\lambda_{[0,1]}$-almost everywhere, which, since both are continues, means that they are equal everywhere. Given these statements, the final estimate is trivial. □

Lemma 1.4. *Let* $\ell \ge 1$ *and assume that* $\varphi \in C^\ell([0,1];\mathbb{C})$ *satisfies* $\varphi^{(k)}(0) = \varphi^{(k)}(1)$ *for* $0 \le k \le \ell - 1$*. Then*

$$(\varphi, \mathfrak{e}_m)_{L^2(\lambda_{[0,1)};\mathbb{C})} = \left(\frac{\imath}{2\pi m}\right)^\ell \left(\varphi^{(\ell)}, \mathfrak{e}_m\right)_{L^2(\lambda_{[0,1)};\mathbb{C})} \text{ for } m \neq 0.$$

Proof. Clearly it suffices that prove the result when $\ell = 1$. To do so, use integration by parts and the condition $\varphi(0) = \varphi(1)$ to check that

$$\int_0^1 \varphi(y)\mathfrak{e}_{-m}(y)\,dy = \frac{1}{-\imath 2\pi m}\int_0^1 \varphi'(y)\mathfrak{e}_{-m}(y)\,dy.$$

□

As a consequence of Lemma 1.4, we see that if $\varphi \in C^1([0,1];\mathbb{C})$ satisfies $\varphi(0) = \varphi(1)$, then

$$\begin{aligned}\sum_{|m|>n} \big|(\varphi, \mathfrak{e}_m)_{L^2(\lambda_{[0,1)};\mathbb{C})}\big| &\le \sum_{|m|>n} \frac{\big|(\varphi', \mathfrak{e}_m)_{L^2(\lambda_{[0,1)};\mathbb{C})}\big|}{2\pi|m|} \\ &\le \frac{1}{2\pi}\left(\sum_{m>n} m^{-2}\right)^{\frac12} \|\varphi'\|_{L^2(\lambda_{[0,1)};\mathbb{C})} \le \frac{\|\varphi'\|_{L^2(\lambda_{[0,1)};\mathbb{C})}}{2\pi n^{\frac12}}.\end{aligned}$$

Hence, by Corollary 1.3,

$$\|S_n\varphi - \varphi\|_{\mathrm{u}} \le \frac{\|\varphi'\|_{\mathrm{u}}}{2\pi n^{\frac12}}.$$

Exercise 1.5. Prove the *Riemann–Lebesgue lemma*, which is the statement that $\lim_{n\to\infty}(\varphi, \mathfrak{e}_n)_{L^2(\lambda_{[0,1)};\mathbb{C})} = 0$ for all $\varphi \in L^1(\lambda_{[0,1)};\mathbb{C})$.

Exercise 1.6. Let φ be a Lipschitz continuous function satisfying $\varphi(0) = \varphi(1)$, and show that

$$\big\|S_n\varphi - \varphi\|_{\mathrm{u}} \le \frac{\|\varphi\|_{\mathrm{Lip}}}{2\pi n^{\frac12}}.$$

Hint: Introduce the functions $\varphi_k = p_{\frac1k} * \varphi$.