Lecture 2: Gibbs Phenomenon

Here we will examine what can be said for a $\varphi \in C([0,1];\mathbb{C})$ that is not periodic.
Consider the function $\varphi(x) = x$ for $x \in [0,1]$. Clearly

$$(\varphi, \mathfrak{e}_m)_{L^2(\lambda_{[0,1)};\mathbb{C})} = \frac{\imath}{2\pi m} \text{ for } m \neq 0,$$

and so

$$S_n(x) = \frac{1}{2} - \frac{1}{\pi}\sum_{m=1}^{n} \frac{\sin 2\pi m x}{m},$$

where $S_n \equiv S_n\varphi$. Now set

$$\Phi_m(x) = \sum_{k=1}^{m} \sin 2\pi k x.$$

Then $\Phi_m(x)$ is the imaginary part of

$$\begin{aligned}\sum_{k=1}^{m} \mathfrak{e}_k(x) = \mathfrak{e}_1(x)\frac{1-\mathfrak{e}_m(x)}{1-\mathfrak{e}_1(x)} &= \frac{\big(\mathfrak{e}_1(x) - \mathfrak{e}_{m+1}(x)\big)\big(1-\mathfrak{e}_{-1}(x)\big)}{2(1-\cos 2\pi x)} \\ &= \frac{\mathfrak{e}_1(x) - 1 - \mathfrak{e}_{m+1} + \mathfrak{e}_m(x)}{2(1-\cos 2\pi x)},\end{aligned}$$

which is

$$\frac{\sin 2\pi x - \sin 2\pi(m+1)x + \sin 2\pi x}{2(1-\cos 2\pi x)}.$$

After using a lot of trigonometric identities, one sees that

$$\Phi_m(x) = \frac{\cos \pi x \sin^2 \pi m}{\sin \pi x} + \sin \pi m x \cos \pi m x. \tag{2.1}$$

In particular, $|\Phi_m(x)| \leq 3\big(\frac{1}{x} \wedge \frac{1}{1-x}\big)$.

Summing by parts, one sees that

$$S_n(x) = \frac{1}{2} - \frac{\Phi_n(x)}{\pi n} - \sum_{m=1}^{n-1} \frac{\Phi_m(x)}{\pi m(m+1)},$$

which means that

$$\big|S_n(x) - x\big| \leq \big(\tfrac{1}{x} \vee \tfrac{1}{1-x}\big)\frac{6}{\pi n}. \tag{2.2}$$

In particular, $S_n(x)$ is converging to x uniformly on compact subsets of $(0,1)$.

To see what happens for x near to 0, consider $x = \frac{k}{2n}$ for $k \geq 1$, and observe that

$$\sum_{m=1}^{n} \frac{\sin \frac{\pi k m}{n}}{m} = \frac{1}{n}\sum_{m=1}^{n} \frac{\sin \frac{\pi k m}{n}}{\frac{m}{n}} \longrightarrow \int_{[0,1]} \frac{\sin \pi k x}{x}\, dx \longrightarrow \int_{[0,\pi k]} \frac{\sin x}{x}\, dx.$$

Hence, since (cf. (7.11) in §7)

$$\lim_{R\to\infty} \int_{[0,R]} \frac{\sin x}{x}\, dx = \frac{\pi}{2},$$

$$S_n\left(\tfrac{k}{2n}\right) = -\frac{1}{\pi}\lim_{R\to\infty}\int_{[\pi k,R]}\frac{\sin x}{x}\,dx$$
$$= \frac{(-1)^{k+1}}{\pi^2 k} - \frac{1}{\pi}\int_{[\pi k,\infty)}\frac{\cos x}{x^2}\,dx = \frac{(-1)^{k+1}}{\pi^2 k} + \frac{2}{\pi}\int_{[\pi k,\infty)}\frac{\sin x}{x^3}\,dx$$

as $n\to\infty$. Therefore

$$S_n\left(\tfrac{k}{2n}\right) = \frac{(-1)^{k+1}}{\pi^2 k}\left(1 - \frac{a_k}{\pi k}\right) + \epsilon_n(k),$$

where $\lim_{n\to\infty}\epsilon_n(k) = 0$ and

$$a_k = 2(\pi k)^2\int_{\pi k}^{\infty}\frac{\sin x}{x^3}\,dx \in (0,1).$$

This shows that, for large n, $S_n\left(\frac{k}{2n}\right)$ is at least $\frac{1}{2\pi k}$ if k is odd and at most $-\frac{1}{2\pi k}$ if k is even. This sort of oscillatory behavior is known as *Gibbs's phenomenon*, although Gibbs seems not to have been the first to discover it.

Exercise 2.1. By considering $S_n\left(\frac{1}{4}\right)$ and using equations (2.1) and (2.2), show that

$$\pi = 4\sum_{\ell=0}^{\infty}\frac{1}{(4\ell+1)(4\ell+3)}.$$

Exercise 2.2. Show that if $\varphi \in C^1\big([0,1];\mathbb{C}\big)$ then, for each $\delta\in(0,1)$,

$$\sup_{x\in[\delta,1-\delta]}\big|S_n\varphi(x) - \varphi(x)\big| \le \frac{\|\varphi'\|_{L^2(\lambda_{[0,1)};\mathbb{C})}}{\pi n^{\frac{1}{2}}} + \left(\tfrac{1}{\delta}\vee\tfrac{1}{1-\delta}\right)\frac{6|\varphi(1)-\varphi(0)|}{\pi n}.$$

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