Lecture 3: Bernoulli Polynomials

Theorem 3.1. *Define* $\{b_\ell : \ell \geq 0\} \subseteq \mathbb{R}$ *inductively by*

$$b_0 = 1 \text{ and } b_{\ell+1} = \sum_{k=0}^{\ell} \frac{(-1)^k b_{\ell-k}}{(k+2)!},$$

and set

$$B_\ell(x) = \sum_{k=0}^{\ell} \frac{(-1)^k b_{\ell-k}}{k!} x^k \text{ for } \ell \geq 0. \tag{3.1}$$

Then $\{B_\ell : \ell \geq 0\}$ *are the one and only functions satisfying*

$$B_0 = \mathbf{1},\ B'_{\ell+1} = -B_\ell \text{ for } \ell \geq 0, \text{ and } B_\ell(1) = B_\ell(0) \text{ for } \ell \geq 2. \tag{3.2}$$

Proof. To see that there is at most one set of functions satisfying (3.2), let $\{D_\ell : \ell \geq 0\}$ be the set of differences between two solutions, and let $\ell = \inf\{\ell : D_\ell \neq \mathbf{0}\}$. Then $\ell \geq 1$, and, if $\ell < \infty$, then D_ℓ is a constant a and there is a $b \in \mathbb{R}$ such that $D_{\ell+1}(x) = -ax + b$. But $-a + b = D_{\ell+1}(1) = D_{\ell+1}(0) = b$, and therefore $a = 0$. Since this would mean that $D_\ell = -D'_{\ell+1} = \mathbf{0}$, no such ℓ can exist.

By definition, $B_0 = \mathbf{1}$, and it is easy to check that $B'_{\ell+1} = -B_\ell$. To verify the periodicity property, note that

$$\begin{aligned} B_{\ell+2}(1) - B_{\ell+2}(0) &= \sum_{k=1}^{\ell+2} \frac{(-1)^k b_{\ell+2-k}}{k!} \\ &= -b_{\ell+1} + \sum_{k=2}^{\ell+2} \frac{(-1)^k b_{\ell+2-k}}{k!} = -b_{\ell+1} + \sum_{k=0}^{\ell} \frac{(-1)^k b_{\ell-k}}{(k+2)!} = 0. \end{aligned}$$

□

The functions $\{B_\ell : \ell \geq 0\}$ in (3.1) are known as *Bernoulli polynomials*.

Theorem 3.2. *For* $\ell \geq 2$ *and* $x \in [0,1]$,

$$B_\ell(x) = \frac{-\imath^\ell}{(2\pi)^\ell} \sum_{n \neq 0} \frac{\mathfrak{e}_n(x)}{n^\ell}. \tag{3.3}$$

In particular, $b_{2\ell+1} = 0$ *and*

$$\zeta(2\ell) \equiv \sum_{m=1}^{\infty} \frac{1}{m^{2\ell}} = (-1)^{\ell+1} 2^{2\ell-1} \pi^{2\ell} b_{2\ell} \tag{3.4}$$

for $\ell \geq 1$.

Proof. First observe that, for $\ell \geq 1$,

$$\big(B_\ell, \mathfrak{e}_0\big)_{L^2(\lambda_{[0,1]};\mathbb{C})} = -\int_0^1 B'_{\ell+1}(x)\,dx = B_{\ell+1}(0) - B_{\ell+1}(1) = 0$$

and, for $\ell \geq 2$ and $n \neq 0$,

$$\big(B_\ell, \mathfrak{e}_n\big)_{L^2(\lambda_{[0,1]};\mathbb{C})} = \frac{\imath}{2\pi n}\big(B_{\ell-1}, \mathfrak{e}_n\big)_{L^2(\lambda_{[0,1]};\mathbb{C})}$$

and therefore

$$\left(\frac{2\pi n}{\imath}\right)^{\ell-1} \big(B_\ell, \mathfrak{e}_n\big)_{L^2(\lambda_{[0,1]};\mathbb{C})} = \big(B_1, \mathfrak{e}_n\big)_{L^2(\lambda_{[0,1]};\mathbb{C})} = \int_0^1 \big(\tfrac{1}{2} - x\big)\mathfrak{e}_n(x)\,dx = \frac{-\imath}{2\pi n}.$$

Hence

$$\big(B_\ell, \mathfrak{e}_n\big)_{L^2(\lambda_{[0,1]};\mathbb{C})} = \frac{-\imath^\ell}{(2\pi n)^\ell}$$

for $\ell \geq 2$ and $n \neq 0$, which completes the proof of (3.3). Finally, because $b_\ell = B_\ell(0)$, it is clear from (3.3) that $b_{2\ell+1} = 0$ and (3.4) holds. □

Besides (3.4), the Bernoulli polynomials play a critical role in what is known as the *Euler–Maclauren formula*:

$$\begin{aligned}&\int_0^n f(x)\,dx - \sum_{m=1}^n f(m)\\ &= -\sum_{k=1}^{\ell} b_k\big(f^{(k-1)}(n) - f^{k-1}(0)\big) + \int_0^n \tilde{B}_\ell(x) f^{(\ell)}(x)\,dx\end{aligned} \qquad \text{for } \ell \geq 1, \tag{3.5}$$

where $\tilde{B}_\ell$ is the periodic extension of $B_\ell \restriction [0,1)$ to $\mathbb{R}$. To prove (3.5), first note that

$$\begin{aligned}\int_0^n f(x)\,dx - \sum_{m=1}^n f(m) &= \sum_{m=1}^n \int_{m-1}^m \big(f(x) - f(m)\big)\,dx\\ &= -\sum_{m=1}^n \int_{m-1}^m \big(x-(m-1)\big) f'(x)\,dx\\ &= \sum_{m=1}^n \left(-b_1\big(f(m) - f(m-1)\big) + \int_{m-1}^m B_1\big(x-(m-1)\big) f'(x)\,dx\right)\\ &= -b_1\big(f(n) - f(0)\big) + \int_0^n \tilde{B}_1(x) f'(x)\,dx.\end{aligned}$$

Hence, (3.5) holds when $\ell = 1$. Next observe that for any $\ell \geq 1$,

$$\int_0^n \tilde{B}_\ell(x) = n\int_0^1 B_\ell(x)\,dx = n\big(B_{\ell+1}(1) - B_{\ell+1}(0)\big) = 0,$$

and therefore

$$\begin{aligned}&\int_0^n \tilde{B}_\ell(x) f^{(\ell)}(x)\,dx = \sum_{m=1}^n \int_{m-1}^m B_\ell\big(x-(m-1)\big)\big(f^{(\ell)}(x) - f^{(\ell)}(m)\big)\,dx\\ &= \sum_{m=1}^n \left(-b_{\ell+1}\big(f^{(\ell)}(m) - f^{(\ell)}(m-1)\big) + \int_{m-1}^m B_{\ell+1}\big(x-(m-1)\big) f^{(\ell+1)}(x)\,dx\right)\\ &= -b_{\ell+1}\big(f^{(\ell)}(n) - f(0)\big) + \int_0^n \tilde{B}_{\ell+1}(x) f^{(\ell+1)}(x)\,dx.\end{aligned}$$

Therefore, (3.5) for ℓ implies (3.5) for $\ell + 1$.

Theorem 3.3. *If $\ell \geq 1$ and $\varphi \in C^\ell\big([0,1];\mathbb{C}\big)$, then*

$$\begin{aligned}&\int_0^1 \varphi(x) - \frac{1}{n}\sum_{m=1}^n \varphi\big(\tfrac{m}{n}\big)\\ &= -\sum_{k=1}^{\ell} \frac{b_k}{n^k}\big(\varphi^{(k-1)}(1) - \varphi^{(k-1)}(0)\big) + \frac{1}{n^\ell}\int_0^1 \tilde{B}_\ell(nx)\varphi^{(\ell)}(x)\,dx,\end{aligned} \tag{3.6}$$

Proof. Take $f(x) = \varphi\big(\frac{x}{n}\big)$, apply (3.5) to f, and make a simple change of variables. □

By Schwarz's inequality,

$$\left|\int_0^1 \tilde{B}_\ell(nx)\varphi^{(\ell)}(x)\,dx\right| \le \left(\int_0^1 \tilde{B}_\ell(nx)^2\,dx\right)^{\frac{1}{2}} \|\varphi^{(\ell)}\|_{L^2(\lambda_{[0,1]};\mathbb{C})},$$

and

$$\int_0^1 \tilde{B}_\ell(nx)^2\,dx = \frac{1}{n}\int_0^n \tilde{B}_\ell(x)^2\,dx = \|B_\ell\|^2_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}.$$

Further, by Parseval's identity and (3.3),

$$\|B_\ell\|^2_{L^2(\lambda_{\mathbb{R}};\mathbb{C})} = \frac{1}{(2\pi)^{2\ell}}\sum_{n\ne 0}\frac{1}{n^{2\ell}}.$$

Hence, by (3.6),

$$\tag{3.7} \begin{aligned} &\left|\int_0^1 \varphi(x)\,dx - \frac{1}{n}\sum_{m=1}^n \varphi\big(\tfrac{m}{n}\big) + \sum_{k=1}^{\ell}\frac{b_k}{n^k}\big(\varphi^{(k-1)}(1) - \varphi^{(k-1)}(0)\big)\right| \\ &\quad \le \frac{\sqrt{2\zeta(2\ell)}}{(2\pi n)^\ell}\|\varphi^{(\ell)}\|_{L^2(\lambda_{[0,1]};\mathbb{C})}. \end{aligned}$$

From (3.7) one sees that if, for some $n \ge 1$,

$$\tag{3.8} \lim_{\ell\to\infty}\frac{\|\varphi^{(\ell)}\|_{L^2(\lambda_{[0,1]};\mathbb{C})}}{(2\pi n)^\ell} = 0,$$

then

$$\int_0^1 \varphi(x)\,dx - \frac{1}{n}\sum_{m=1}^n \varphi\big(\tfrac{m}{n}\big) = -\lim_{\ell\to\infty}\sum_{k=1}^{\ell}\frac{b_k}{n^k}\big(\varphi^{(k-1)}(1) - \varphi^{(k-1)}(0)\big).$$

In particular, if $\varphi \in C^\infty\big([0,1];\mathbb{C}\big)$ and $\varphi^{(k)}$ is periodic for all $k \ge 0$, then (3.8) implies that

$$\int_0^1 \varphi(x)\,dx = \frac{1}{n}\sum_{m=1}^n \varphi\big(\tfrac{m}{n}\big),$$

a result that has a much simpler derivation (cf. Exercise 3.4 below).

More generally, because $\big|\varphi^{(k-1)}(1) - \varphi^{(k-1)}(0)\big| \le \|\varphi^{(k)}\|_{L^2(\lambda_{[0,1]};\mathbb{C})}$,

$$\sum_{k=1}^{\infty}\frac{\|\varphi^{(k)}\|_{L^2(\lambda_{[0,1]};\mathbb{C})}}{(2\pi n)^k} < \infty$$

implies that

$$\tag{3.9} \int_0^1 \varphi(x)\,dx - \frac{1}{n}\sum_{m=1}^n \varphi\big(\tfrac{m}{n}\big) = -\sum_{k=1}^{\infty}\frac{b_k}{n^k}\big(\varphi^{(k-1)}(1) - \varphi^{(k-1)}(0)\big),$$

where the series is absolutely convergent.

Exercise 3.4. Suppose that φ and all its derivatives are periodic on $[0,1]$, and show that

$$\lim_{\ell\to\infty}\frac{\|\varphi^{(\ell)}\|_{L^2(\lambda_{[0,1]};\mathbb{C})}}{(2\pi n)^\ell}=0 \iff \big(\varphi,\mathfrak{e}_m\big)_{L^2(\lambda_{[0,1]};\mathbb{C})}=0 \text{ if } |m|\ge n$$
$$\iff \varphi=\sum_{|m|<n}\big(\varphi,\mathfrak{e}_m\big)_{L^2(\lambda_{[0,1]};\mathbb{C})}\mathfrak{e}_m.$$

Next, show that

$$\frac{1}{n}\sum_{j=1}^{n}\mathfrak{e}_m\big(\tfrac{j}{n}\big)=0$$

for $1\le|m|<n$, and thereby arrive at the conclusion reached above.

RES.18-015 Topics in Fourier Analysis
Spring 2024