Lecture 4: Comparing Summability Methods

In preparation for the following section, we will review here basic definitions and results for different notions of convergence of a series.

Given a sequence $\{a_m : m \geq 1\} \subseteq \mathbb{C}$, set

$$S_n = \frac{1}{n}\sum_{m=1}^{n} a_m \text{ and } A_n = \frac{1}{n}\sum_{m=1}^{n} S_m,$$

and when $\overline{\lim}_{n\to\infty} |a_m|^{\frac{1}{m}} \leq 1$, set

$$A(r) = \sum_{m=1}^{\infty} a_m r^{m-1} \text{ for } r \in [0,1).$$

The S_n's are called the *partial sums* of the corresponding series, the A_n's are its *Césaro means*, and $r \rightsquigarrow A(r)$ is its *Abel function*. The series is said to be *summable* to $s \in \mathbb{C}$ if $s = \lim_{n\to\infty} S_n$, it is *Césaro summable* to $s \in \mathbb{C}$ if $\lim_{n\to\infty} A_n = s$, and it is *Abel summable* to $s \in \mathbb{C}$ if $s = \lim_{r\nearrow 1} A(r)$

Here we will show that

$$\textbf{summable to } s \implies \textbf{Césaro summable to } s \implies$$
$$\lim_{m\to\infty} \frac{a_m}{m} = 0 \text{ and } \textbf{Abel summable to } s.$$

The Exercise below outlines a proof that neither implication can be reversed.

The first implication is trivial. To prove the second, assume Césaro summability, and note that

$$\frac{a_n}{n} = A_n - A_{n-1} + \frac{A_{n-1}}{n} \longrightarrow 0.$$

Next, write

$$a_m = \begin{cases} A_1 & \text{if } m = 1 \\ 2A_2 - A_1 & \text{if } m = 2 \\ mA_m - 2(m-1)A_{m-1} + (m-2)A_{m-2} & \text{if } m \geq 3, \end{cases}$$

and therefore

$$\begin{aligned} A(r) &= \sum_{m=1}^{\infty} mr^{m-1}A_m - 2\sum_{m=2}^{\infty}(m-1)r^{m-1}A_{m-1} + \sum_{m=3}^{\infty}(m-2)r^{m-1}A_{m-2} \\ &= \sum_{m=1}^{\infty}(r^{m-1} - 2r^m + r^{m+1})mA_m = (1-r)^2\sum_{m=1}^{\infty} mr^{m-1}A_m. \end{aligned}$$

Now observe that

$$\sum_{m=1}^{n} mr^{m-1} = \partial_r \sum_{m=0}^{n} r^m = \partial_r \frac{1-r^n}{1-r} = \frac{1 - r^n - n(1-r)r^{n-1}}{(1-r)^2}.$$

Hence,

$$(1-r)^2\sum_{1}^{n} mr^{m-1} \leq 1 - r^n \text{ and } (1-r)^2\sum_{1}^{\infty} mr^{m-1} = 1.$$

Assume that $A_n \longrightarrow s$, and, given $\epsilon > 0$, choose n so that $|A_m - s| \le \epsilon$ for $m > n$. Then

$$\begin{aligned}\big|A(r) - s\big| = (1-r)^2 \left|\sum_{m=1}^{\infty} m r^{m-1}(A_m - s)\right| &\le (1-r)^2 \sum_{m=1}^{n} m r^{m-1} |A_m - s| + \epsilon \\ &\le (1 - r^n) \max_{1 \le m \le n} |A_m - s| + \epsilon,\end{aligned}$$

and therefore $\overline{\lim}_{r \nearrow 1} |A(r) - s| \le \epsilon$.

Exercise 4.1. Show that
(**i**) the series for $\{(-1)^{m-1} : m \ge 1\}$ is Césaro summable to $\frac{1}{2}$ but not summable,
(**ii**) the series for $\{(-1)^{m-1} m : m \ge 1\}$ is Abel summable to $\frac{1}{4}$ but not Césaro summable. In fact, show that $A_{2n} = 0$ and $A_{2n+1} = \frac{n+1}{2n+1} \longrightarrow \frac{1}{2}$.

RES.18-015 Topics in Fourier Analysis
Spring 2024