Lecture 5: Some Refinements

In this section we will apply the notions of summability discussed in the previous section to Fourier series. Observe that we have already considered Abel summability in §1.

To examine further when the series is summable, introduce the function

$$D_n(x) = \sum_{|m|\le n} \mathfrak{e}_m(x) \text{ for } x \in \mathbb{R}.$$

Then D_n, which is often called the *Dirichlet kernel*, is an even, periodic function with period 1, $\int_0^1 D_n(x)\,dx = 1$, and $S_n\varphi = D_n * \varphi$. In addition

$$\begin{aligned} D_n(x) &= \mathfrak{e}_{-n}(x)\sum_{m=0}^{2n}\mathfrak{e}_m(x) = \mathfrak{e}_{-n}(x)\frac{1-\mathfrak{e}_{n+1}(x)}{1-\mathfrak{e}_1(x)} = \frac{e^{-\imath\pi(2n+1)x} - e^{\imath\pi(2n+1)x}}{e^{-\imath\pi x} - e^{\imath\pi x}} \\ &= \frac{\sin\pi(2n+1)x}{\sin\pi x}. \end{aligned}$$

Hence,

$$S_n\varphi(x) - \varphi(x) = \int_{[0,1]} \frac{\tilde\varphi(x+y) - \varphi(x)}{\sin\pi y}\sin\pi(2n+1)y\,dy.$$

Now suppose that φ is an $\mathbb{R}$-valued function for which $\varphi(0) = \varphi(1)$, and assume that φ is Hölder continuous periodic function of order $\alpha \in (0,1)$. Set

$$\psi(y) = e^{\imath\pi y}\frac{\tilde\varphi(x+y) - \varphi(x)}{\sin\pi y}.$$

Then $\psi \in L^1(\lambda_{[0,1)};\mathbb{C})$ and $D_n * \varphi(x) - \varphi(x)$ is the imaginary part of

$$\int_{[0,1]} \psi(y)\mathfrak{e}_{2n}(y)\,dt = \big(\psi, \mathfrak{e}_{2n}\big)_{L^2(\lambda_{[0,1)};\mathbb{C})},$$

and so, by the Riemann–Lebesgue lemma (cf. Exercise 1.5), $S_n\varphi(x) \longrightarrow \varphi(x)$ as $n \to \infty$. The preceding shows that if $\varphi \in C^\alpha\big([0,1];\mathbb{C}\big)$ satisfies $\varphi(0) = \varphi(1)$, then $S_n\varphi \longrightarrow \varphi$ pointwise, but it does not provide a rate of convergence or even say if the convergence is uniform.

Césaro summability of Fourier series was initiated by Fejér. Obviously,

$$\frac{1}{n}\sum_{m=0}^{n-1} S_m\varphi = F_n * \varphi,$$

where

$$F_n(x) \equiv \frac{1}{n}\sum_{m=0}^{n-1} D_n(x).$$

The function F_n is called the *Fejér kernel*, and it is clear that F_n is a continuous, even function of period 1 for which $\int_{[0,1]} F_n(x)\,dx = 1$. In addition, $nF_n(x)\sin\pi x$ is the imaginary part of

$$e^{\imath\pi x}\sum_{m=0}^{n-1}\mathfrak{e}_{2m}(x) = e^{\imath\pi x}\frac{1-e^{\imath\pi 2nx}}{1-e^{\imath 2\pi x}} = \frac{\imath(1-e^{\imath 2\pi nx})}{2\sin\pi x},$$

and so

$$F_n(x) = \frac{1-\cos 2\pi nx}{2\sin^2\pi x} = \frac{1}{n}\left(\frac{\sin\pi nx}{\sin\pi x}\right)^2. \tag{5.1}$$

Proceeding as in the proof of Theorem 1.1, one sees that

$$F_n * \varphi(x) - \varphi(x) = \int_{[0,1]} F_n(y)\big(\tilde{\varphi}(x+y) - \varphi(x)\big)\, dx \longrightarrow 0$$

uniformly if φ is continuous and satisfies $\varphi(1) = \varphi(0)$. Equivalently,

$$\lim_{n\to\infty} \left\| \frac{1}{n} \sum_{m=0}^{n-1} S_m\varphi - \varphi \right\|_{\mathrm{u}} = 0.$$

It turns out that one can do much better.

Theorem 5.1. *Let $\varphi : [-\frac{1}{2}, \frac{1}{2}] \longrightarrow \mathbb{C}$ be a measurable function, let $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$, and assume that there is a $C \in (0,\infty)$ and $\alpha \in (0,1]$ such that $|\tilde{\varphi}(x+y) - \varphi(x)| \le C|y|^\alpha$ for $y \in \left[-\frac{1}{2}, \frac{1}{2}\right]$. For $n \ge 5$*

$$\big|F_n * \varphi(x) - \varphi(x)\big| \le \begin{cases} \frac{2}{(1+\alpha)n^\alpha} + \frac{4(n^{1-\alpha} - 4^{1-\alpha})}{\pi^2(1-\alpha)n} + \frac{1-2^{-(1+\alpha)}}{2^\alpha(1+\alpha)n} & \text{if } \alpha \in (0,1) \\ \frac{19}{16n} + \frac{4\log\frac{n}{4}}{\pi^2 n(1-\alpha)} & \text{if } \alpha = 1. \end{cases} \tag{5.2}$$

Hence

$$\overline{\lim_{n\to\infty}}\, n^\alpha \big|F_n * \varphi(x) - \varphi(x)\big| \le \frac{2}{1+\alpha} + \frac{4}{\pi^2(1-\alpha)} \quad \text{if } \alpha \in (0,1)$$

and

$$\overline{\lim_{n\to\infty}}\, \frac{n}{\log n} \big|F_n * \varphi(x) - \varphi(x)\big| \le \frac{4}{\pi^2} \quad \text{if } \alpha = 1.$$

Proof. Without loss in generality, I will assume that $C = 1$.

The proof turns on the estimates

$$F_n(y) \le \begin{cases} n & \text{for all } y \in \left[-\frac{1}{2}, \frac{1}{2}\right] \\ \frac{2}{\pi^2 n y^2} & \text{when } |y| \in \left(0, \frac{1}{4}\right] \\ \frac{2}{n} & \text{when } |y| \in \left[\frac{1}{4}, \frac{1}{2}\right] \end{cases} \tag{5.3}$$

That $F_n(y) \le n$ is clear from the fact that $\|D_m\|_{\mathrm{u}} \le 1$ and therefore that $nF_n(y) \le 2\sum_{m=1}^{n-1} m + n = n^2$. To see second inequality, note that $\cos \pi t \ge 2^{-\frac{1}{2}}$ when $|y| \in \left(0, \frac{1}{4}\right]$ and therefore that

$$|\sin \pi y| = \int_0^{\pi|y|} \cos t\, dt \ge 2^{-\frac{1}{2}}\pi|y|.$$

As for $F_n(y) \le \frac{2}{n}$ when $|y| \in \left[\frac{1}{4}, \frac{1}{2}\right]$, simply remember that $|\sin \pi y| \ge 2^{-\frac{1}{2}}$ for such y's.

Assume that $\alpha \in (0,1)$. Because $\int_{-\frac{1}{2}}^{\frac{1}{2}} F_n(y) = 1$

$$\begin{aligned} \big|F_n * \varphi(x) - \varphi(x)\big| &\le \int_{-\frac{1}{2}}^{\frac{1}{2}} F_n(y) \big|\tilde{\varphi}(x+y) - \varphi(x)\big|\, dy \\ &\le n \int_0^{\frac{1}{n}} |y|^\alpha\, dy + \frac{2}{\pi^2 n} \int_{\frac{1}{n}}^{\frac{1}{4}} |y|^{\alpha-2}\, dy + \frac{2}{n} \int_{\frac{1}{4} \le |y| \le \frac{1}{2}} |y|^\alpha\, dy \\ &\le \frac{2}{(1+\alpha)n^\alpha} + \frac{4(n^{1-\alpha} - 4^{1-\alpha})}{\pi^2(1-\alpha)n} + \frac{1 - 2^{-(1+\alpha)}}{2^\alpha(1+\alpha)n}. \end{aligned}$$

If $\alpha = 1$, the top line in (5.2) holds for all $\alpha \in (0,1)$ and therefore need only examine what happens as $\alpha \nearrow 1$. Clearly $\frac{2}{(1+\alpha)n^\alpha} \searrow \frac{1}{n}$ and $\frac{1-2^{-(1+\alpha)}}{2^\alpha(1+\alpha)n} \searrow \frac{3}{16n}$ as $\alpha \nearrow 1$. To handle the remaining term, note that it can be written as

$$\frac{4^{2-\alpha}}{\pi^2 n}\frac{\left(\frac{n}{4}\right)^{1-\alpha}-1}{1-\alpha}$$

which decreases to $\frac{4\log\frac{n}{4}}{\pi^2 n}$ as $\alpha \nearrow 1$. □

One could of course have derived the estimate when $\alpha = 1$ directly by the same argument as was used when $\alpha < 1$. However, the derivation given has the advantage that it shows the estimates get stronger for all $n \geq 5$, not just asymptotically, as α increases.

Obviously, results like those in Theorem 5.1 turn on the continuity properties of φ, properties that a generic element of $L^1(\lambda_{[0,1)};\mathbb{C})$ will not possess. However, Lebesgue showed that every locally $\lambda_{\mathbb{R}}$-integrable φ does have a continuity property at almost everywhere point. Namely, he showed that

$$\lim_{r\searrow 0}\frac{1}{r}\int_0^r |\tilde{\varphi}(x\pm t)-\varphi(x)|\,dt = 0 \quad \text{for } \lambda_{\mathbb{R}}\text{-almost every } x\in\mathbb{R},$$

and he used this fact to prove the following theorem.

Theorem 5.2. *If* $\varphi \in L^1\big(\lambda_{[-\frac{1}{2},\frac{1}{2}]};\mathbb{C}\big)$, *then*

$$\lim_{n\to\infty} F_n * \varphi(x) = \varphi(x) \textit{ for } \lambda_{[-\frac{1}{2},\frac{1}{2}]}\textit{-almost every } x \in [0,1].$$

Proof. Set $\varphi_x(y) = |\tilde{\varphi}(x+y)-\varphi(x)|$ and

$$\Phi_x(y) = \frac{1}{|y|}\int_0^{|y|} \varphi_x(\operatorname{sgn}(y)t)\,dt.$$

By Lebesgue's theorem, $\lim_{|y|\searrow 0}\Phi_x(y) = 0$ for $\lambda_{[-\frac{1}{2},\frac{1}{2}]}$-almost every $x \in \left[-\frac{1}{2},\frac{1}{2}\right]$.

Let x be such a point. Then

$$\big|F_n*\varphi(x)-\varphi(x)\big| \leq \int_{-\frac{1}{2}}^0 F_n(y)\varphi_x(y)\,dy + \int_0^{\frac{1}{2}} F_n(y)\varphi_x(y)\,dy.$$

We will show only that $\lim_{n\to\infty}\int_0^{\frac{1}{2}} F_n(y)\varphi_x(y)\,dy = 0$ because the proof that $\lim_{n\to\infty}\int_{-\frac{1}{2}}^0 F_n(y)\varphi_x(y)\,dy = 0$ is essentially the same.

Using our estimates for F_n in (5.3), one has

$$\begin{aligned}\int_0^{\frac{1}{2}} F_n(y)\varphi_x(y)\,dy &= \int_0^{\frac{1}{n}} F_n(y)\varphi_x(y)\,dy + \int_{\frac{1}{n}}^{\frac{1}{2}} F_n(y)\varphi_x(y)\,dy\\ &\leq n\int_0^{\frac{1}{n}}\varphi_x(y)\,dy + \frac{1}{n}\int_{\frac{1}{n}}^{\frac{1}{2}}\frac{\varphi_x(y)}{y^2}\,dy.\end{aligned}$$

Since

$$n\int_0^{\frac{1}{n}}\varphi_x(y)\,dy = \Phi_x\big(\tfrac{1}{n}\big),$$

the first term tends to 0. As for the second, use integration by parts to see that it is dominated by

$$\frac{2\Phi_x\big(\frac{1}{2}\big)}{n} + \frac{2}{n}\int_{\frac{1}{n}}^{\frac{1}{2}}\frac{\Phi_x(y)}{y^2}\,dy.$$

Finally, given $\epsilon > 0$, choose $\delta \in \left(0, \frac{1}{2}\right)$ so that $\Phi_x(y) \le \epsilon$ for $0 \le y \le \delta$. Then, for $n > \frac{1}{\delta}$,

$$\frac{2}{n}\int_{\frac{1}{n}}^{\frac{1}{2}} \frac{\Phi_x(y)}{y^2}\,dy \le \frac{2\epsilon}{n}\int_{\frac{1}{n}}^{\delta} \frac{1}{y^2}\,dy + \frac{2}{n}\int_{\delta}^{\frac{1}{2}} \frac{\Phi_x(y)}{y^2}\,dy \le 2\epsilon + \frac{2\|\Phi_x\|_{\mathrm{u}}}{\delta n},$$

and so

$$\overline{\lim_{n\to\infty}} \int_0^{\frac{1}{2}} F_n(y)\varphi_x(y)\,dy \le 2\epsilon.$$

□

Theorem 5.2 is a stark contrast to a famous example produced in 1926 by Kolmogorov[4] of a function in $L^1\big(\lambda_{[-\frac{1}{2},\frac{1}{2}]};\mathbb{C}\big)$ for which $\{S_n\varphi(x) : n \ge 0\}$ diverges at *every* x. It is also interesting to compare it to more recent results by L. Carleson and R. Hunt. Namely, Carleson showed that $S_n\varphi \longrightarrow \varphi$ (a.e.,$\lambda_{[-\frac{1}{2},\frac{1}{2}]}$) if $\varphi \in L^2\big(\lambda_{[-\frac{1}{2},\frac{1}{2}]};\mathbb{C}\big)$, and Hunt showed that the same is true for $\varphi \in L^p\big(\lambda_{[-\frac{1}{2},\frac{1}{2}]};\mathbb{C}\big)$ for $p > 1$.

Exercise 5.3. Show that

$$\underline{\lim_{n\to\infty}}\, n^\alpha \int_{-\frac{1}{2}}^{\frac{1}{2}} F_n(y)|y|^\alpha\,dy > 0 \text{ for } \alpha \in (0,1)$$

and that

$$\underline{\lim_{n\to\infty}} \frac{n}{\log n}\int_{-\frac{1}{2}}^{\frac{1}{2}} F_n(y)|y|\,dy > 0.$$

Hence the rates given Theorem 5.1 are optimal.

Hint: If $0 \le m \le n-1$, show that

$$F_n(y) \ge \frac{1}{2\pi n y^2} \text{ if } \frac{4m+1}{n4} \le y \le \frac{2m+1}{2n}.$$

[4] A.N. Kolmogorov, *Une série de Fourier-Lebesgue divergente partout*, C.R. 183 (1926), pp. 1327-1328.

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