Lecture 6: The \( L^1 \) Fourier Transform

By an easy rescaling argument, one knows that, for any \( L \in \mathbb{Z}^+ \) and \( f \in C^1([-L, L]; \mathbb{C}) \) satisfying \( f(-L) = f(L) \),

\[
  f(x) = \frac{1}{2L} \sum_{m \in \mathbb{Z}} \int_{-L}^{L} e^{\frac{2\pi i m (x-y)}{2L}} f(y) \, dy = \lim_{R \to \infty} \int_{-L}^{L} \left( \frac{1}{2L} \sum_{|m| \leq R} e^{\frac{2\pi i m (y-x)}{2L}} \right) f(y) \, dx.
\]

Now suppose that \( f \in C^1_c(\mathbb{R}; \mathbb{C}) \). Then

\[
  f(x) = \lim_{L \to \infty} \lim_{R \to \infty} \int_{-L}^{L} \left( \frac{1}{2L} \sum_{|m| \leq R} e^{\frac{2\pi i m (y-x)}{2L}} \right) f(y) \, dy.
\]

Thus, if one can justify reversing the order in which the limits are taken, one would have that

\[
  f(x) = \lim_{R \to \infty} \int \left( \int_{-R}^{R} e^{i\xi (x-y)} \, dy \right) f(y) \, d\xi
  = \lim_{R \to \infty} \frac{1}{2\pi} \int_{-2\pi R}^{2\pi R} e^{-i\xi x} \left( \int e^{i\xi y} f(y) \, dy \right) \, d\xi.
\]

In other words, there is reason to hope that, under suitable conditions on \( f \),

\[
  f(x) = \frac{1}{2\pi} \int e^{-i\xi x} \hat{f}(\xi) \, d\xi \quad \text{where} \quad \hat{f}(\xi) = \int e^{i\xi y} f(y) \, dy.
\]

The function \( \hat{f} \) is called the Fourier transform of \( f \), and our primary goal here will be to find out in what sense (6.1) is true when \( f \in L^1(\lambda_\mathbb{R}; \mathbb{C}) \), but we will begin with some computations involving \( f \) that don’t require it.