

LECTURE 6: THE  $L^1$  FOURIER TRANSFORM

By an easy rescaling argument, one knows that, for any  $L \in \mathbb{Z}^+$  and  $f \in C^1([-L, L]; \mathbb{C})$  satisfying  $f(-L) = f(L)$ ,

$$f(x) = \frac{1}{2L} \sum_{m \in \mathbb{Z}} \int_{-L}^L e^{i \frac{2\pi m(y-x)}{2L}} f(y) dy = \lim_{R \rightarrow \infty} \int_{-L}^L \left( \frac{1}{2L} \sum_{|m| \leq R} e^{i \frac{2\pi m(y-x)}{2L}} \right) f(y) dx.$$

Now suppose that  $f \in C_c^1(\mathbb{R}; \mathbb{C})$ . Then

$$f(x) = \lim_{L \rightarrow \infty} \lim_{R \rightarrow \infty} \int_{-L}^L \left( \frac{1}{2L} \sum_{|m| \leq R} e^{i \frac{2\pi m(y-x)}{2L}} \right) f(y) dy.$$

Thus, if one can justify reversing the order in which the limits are taken, one would have that

$$\begin{aligned} f(x) &= \lim_{R \rightarrow \infty} \int \left( \int_{-R}^R e^{i\xi 2\pi(x-y)} d\xi \right) f(y) dy \\ &= \lim_{R \rightarrow \infty} \frac{1}{2\pi} \int_{-2\pi R}^{2\pi R} e^{-i\xi x} \left( \int e^{i\xi y} f(y) dy \right) d\xi. \end{aligned}$$

In other words, there is reason to hope that, under suitable conditions on  $f$ ,

$$(6.1) \quad f(x) = \frac{1}{2\pi} \int e^{-i\xi x} \hat{f}(\xi) d\xi \text{ where } \hat{f}(\xi) \equiv \int e^{i\xi y} f(y) dy.$$

The function  $\hat{f}$  is called the *Fourier transform* of  $f$ , and our primary goal here will be to find out in what sense (6.1) is true when  $f \in L^1(\lambda_{\mathbb{R}}; \mathbb{C})$ , but we will begin with some computations involving  $\hat{f}$  that don't require it.

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RES.18-015 Topics in Fourier Analysis  
Spring 2024

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