LECTURE 6: THE L^1 FOURIER TRANSFORM

By an easy rescaling argument, one knows that, for any $L \in \mathbb{Z}^+$ and $f \in C^1([-L, L]; \mathbb{C})$ satisfying $f(-L) = f(L)$,

$$f(x) = \frac{1}{2L} \sum_{m\in\mathbb{Z}} \int_{-L}^{L} e^{\imath \frac{2\pi m(y-x)}{2L}} f(y)\, dy = \lim_{R\to\infty} \int_{-L}^{L} \left(\frac{1}{2L} \sum_{|m|\le R} e^{\imath \frac{2\pi m(y-x)}{2L}} \right) f(y)\, dx.$$

Now suppose that $f \in C^1_{\mathrm{c}}(\mathbb{R}; \mathbb{C})$. Then

$$f(x) = \lim_{L\to\infty} \lim_{R\to\infty} \int_{-L}^{L} \left(\frac{1}{2L} \sum_{|m|\le R} e^{\imath \frac{2\pi m(y-x)}{2L}} \right) f(y)\, dy.$$

Thus, if one can justify reversing the order in which the limits are taken, one would have that

$$\begin{aligned} f(x) &= \lim_{R\to\infty} \int \left(\int_{-R}^{R} e^{\imath\xi 2\pi(x-y)}\, d\xi \right) f(y)\, dy \\ &= \lim_{R\to\infty} \frac{1}{2\pi} \int_{-2\pi R}^{2\pi R} e^{-\imath\xi x} \left(\int e^{\imath\xi y} f(y)\, dy \right) d\xi. \end{aligned}$$

In other words, there is reason to hope that, under suitable conditions on f,

$$f(x) = \frac{1}{2\pi} \int e^{-\imath\xi x} \hat{f}(\xi)\, d\xi \text{ where } \hat{f}(\xi) \equiv \int e^{\imath\xi y} f(y)\, dy. \tag{6.1}$$

The function $\hat{f}$ is called the *Fourier transform* of f, and our primary goal here will be to find out in what sense (6.1) is true when $f \in L^1(\lambda_{\mathbb{R}}; \mathbb{C})$, but we will begin with some computations involving $\hat{f}$ that don't require it.