Lecture 7: Computations and Applications of L^1 Fourier Transforms

If $f \in L^1(\lambda_{[0,1)};\mathbb{C})$, then it is clear that $\hat f$ is continuous and that

$$\|\hat f\|_{\mathrm{u}} \le \|f\|_{L^1(\lambda_{[0,1)};\mathbb{C})}. \tag{7.1}$$

Lemma 7.1. *If $f \in C^1(\mathbb{R},\mathbb{C}) \cap L^1(\lambda_{[0,1)};\mathbb{C})$ and $f' \in L^1(\lambda_{[0,1)};\mathbb{C})$, then*

$$\widehat{f'}(\xi) = -\imath\xi \hat f(\xi). \tag{7.2}$$

Proof. If f has compact support, then (7.2) is an easy application of integration by parts. To prove it under the given conditions, choose a function $\eta \in C^\infty\big(\mathbb{R};[0,1]\big)$ for which $\eta(y) = 1$ when $y \in [-1,1]$ and $\eta(y) = 0$ when $y \notin [-2,2]$, and set $f_n(y) = \eta\big(\frac{y}{n}\big)f(y)$. Then $f_n \longrightarrow f$ and $f_n' \longrightarrow f'$ in $L^1(\lambda_{[0,1)};\mathbb{C})$ and so

$$\widehat{f'}(\xi) = \lim_{n\to\infty} \widehat{f_n'}(\xi) = -\imath\xi \lim_{n\to\infty} \widehat{f_n}(\xi) = -\imath\xi\hat f(\xi).$$

□

As a consequence of Lemma 7.1, it is easy to prove the *Riemann-Lebesgue lemma* in this context. Namely, (7.2) makes it clear for compactly support $f \in C^1(\mathbb{R};\mathbb{C})$, and (7.1) makes it clear that the set of f's for which it is holds is closed in $L^1(\lambda_{[0,1)};\mathbb{C})$.

We next turn to the computation of $\hat f$ in two important cases.

Set $g_t(x) = (2\pi t)^{-\frac12} e^{-\frac{x^2}{2}}$ for $(t,x) \in (0,\infty)\times\mathbb{R}$, and check that $\partial_t g_t(x) = \frac12 \partial_x^2 g_t(x)$. Hence, for any $\zeta \in \mathbb{C}$, integration by parts leads to

$$\partial_t \int e^{\zeta x} g_t(x)\,dx = \frac12 \int e^{\zeta x}\partial_x^2 g_t(x)\,dx = \frac{\zeta^2}{2}\int e^{\zeta x} g_t(x)\,dx.$$

Since

$$\int e^{\zeta x} g_t(x)\,dx = \int e^{t^{\frac12}\zeta x} g_1(x)\,dx \longrightarrow 1$$

as $t \searrow 0$,

$$\int e^{\zeta x} g_t(x)\,dx = e^{\frac{t\zeta^2}{2}}.$$

In particular

$$\widehat{g_t}(\xi) = e^{-\frac{\xi^2}{2}} \tag{7.3}$$

Equivalently, $\widehat{g_t} = \big(\frac{2\pi}{t}\big)^{\frac12} g_{\frac1t}$ and so

$$\big(\widehat{g_t}\big)^\wedge = 2\pi g_t. \tag{7.4}$$

Set $p_y(x) = \frac{1}{\pi}\frac{y}{x^2+y^2}$ for $(y,x) \in (0,\infty)\times\mathbb{R}$, and note that

$$\int p_y(x)\,dx = \int p_1(x)\,dx = 1 \text{ for all } y > 0.$$

In addition, because $p_y(x)$ is the real part of $\frac{\imath}{\pi z}$ with $z = x + \imath y$, $(x,y) \rightsquigarrow p_y(x)$ is harmonic. Thus, $\partial_x^2 p_y = -\partial_y^2 p_y$, and so, by (7.2),

$$\partial_y^2 \widehat{p_y}(\xi) = \xi^2 \widehat{p_y}(\xi).$$

Thus, for each ξ,

$$\widehat{p_y(\xi)} = a(\xi)e^{y\xi} + b(\xi)e^{-y\xi},$$

where, since $\widehat{p_y}(0) = 1$, $a(\xi) + b(\xi) = 1$. Because $|\widehat{p_y(\xi)}| \le 1$, $\xi \ge 0 \implies a(\xi) = 0$ & $b(\xi) = 1$ and $\xi < 0 \implies a(\xi) = 1$ & $b(\xi) = 0$. Hence

$$\widehat{p_y}(\xi) = e^{-y|\xi|}. \tag{7.5}$$

Here is an interesting application of equations (7.3) and (7.5). Since

$$\frac{1}{\xi^2 + y^2} = \int_0^\infty e^{-t(\xi^2+y^2)}\,dx = \int_0^\infty e^{-ty^2}\widehat{g_{2t}}(\xi)\,dt$$

and $\left(\widehat{g_{2t}}\right)^\wedge = 2\pi g_{2t}$,

$$\frac{\pi}{y}e^{-y|x|} = 2\pi\int_0^\infty e^{-ty^2} g_{2t}(x)\,dt = \pi^{\frac{1}{2}}\int_0^\infty t^{-\frac{1}{2}}e^{-ty^2}e^{-\frac{x^2}{4t}}\,dt.$$

Thus, for $x, y \in (0,\infty)$,

$$\int_0^\infty t^{-\frac{1}{2}}e^{-ty^2}e^{-\frac{x^2}{t}}\,dt = \frac{\pi^{\frac{1}{2}}e^{-2yx}}{y}, \tag{7.6}$$

a computation which can also be done using a somewhat tricky change of variables.

Theorem 7.2. (**Poisson Sum**) *Let $f \in L^1(\lambda_{\mathbb{R}};\mathbb{C}) \cap C(\mathbb{R};\mathbb{C})$, and assume that*

$$\sum_{n\in\mathbb{Z}}\left(\sup_{x\in[0,1]}|f(x+n)| + |\hat{f}(2\pi n)|\right) < \infty.$$

Then

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}}\hat{f}(2\pi n). \tag{7.7}$$

Proof. Define $\tilde{f}(x) = \sum_{n\in\mathbb{Z}} f(x+n)$. Then $\tilde{f}$ is a continuous periodic function with period 1, and

$$\left(\tilde{f}, \mathfrak{e}_m\right)_{L^2(\lambda_{[0,1]};\mathbb{C})} = \sum_{n\in\mathbb{Z}}\int_0^1 e^{-\imath 2\pi m x} f(x+n)\,dx = \int e^{-\imath 2\pi m x} f(x)\,dx = \hat{f}(-2\pi m).$$

Thus, since $\sum_{m\in\mathbb{Z}} |(\tilde{f}, \mathfrak{e}_m)_{L^|(\lambda_{[0,1)};\mathbb{C})} < \infty$,

$$\tilde{f}(x) = \sum_{m\in\mathbb{Z}}\hat{f}(-2\pi m)\mathfrak{e}_m(x) = \sum_{m\in\mathbb{Z}}\hat{f}(2\pi m)\mathfrak{e}_{-m}(x),$$

where the convergence of the series is absolute and uniform. By taking $x = 0$, (7.7) follows. □

Equation (7.7) is known as the *Poisson summation formula.* Among its many applications is the following.

When $f = p_y$, (7.7) says that

$$\frac{y}{\pi}\sum_{n\in\mathbb{Z}}\frac{1}{y^2+n^2} = \sum_{n\in\mathbb{Z}} e^{-2\pi y|n|} = \frac{1+e^{-2\pi y}}{1-e^{-2\pi y}} = \coth \pi y,$$

and so

$$\sum_{n\in\mathbb{Z}}\frac{1}{y^2+n^2} = \frac{\pi\coth\pi y}{y} \tag{7.8}$$

for $y > 0$.

A famous application of (7.8) is *Euler's product formula*:

$$\sin \pi x = \pi x \prod_{m=1}^{\infty} \left(1 - \frac{x^2}{m^2}\right). \tag{7.9}$$

To prove it, first observe that

$$\frac{1}{x^2 + m^2} = \frac{1}{2x}\partial_x \log\bigl(x^2 + m^2\bigr) = \frac{1}{2x}\partial_x \log\left(1 + \frac{x^2}{m^2}\right) \text{ for } m \neq 0$$

and that $\pi \coth \pi y = \partial_y \log(\sinh \pi y)$. Hence, by (7.8)

$$\frac{1}{x}\partial_x \log \prod_{n=1}^{\infty} \left(1 + \frac{x^2}{n^2}\right) + \frac{1}{x^2} = \frac{1}{x}\partial_x \log(\sinh \pi x),$$

which means that

$$\partial_x \log \prod_{n=1}^{\infty} \left(1 + \frac{x^2}{n^2}\right) = \partial_x \log(x^{-1} \sinh \pi x).$$

Integrating both sides from 0 to x, one gets

$$\log x \prod_{n=1}^{\infty} \left(1 + \frac{x^2}{n^2}\right) = \log(\sinh \pi x) - \log \pi = \log \frac{\sinh \pi x}{\pi x},$$

which means that

$$\sinh \pi x = \pi x \prod_{n=1}^{\infty} \left(1 + \frac{x^2}{n^2}\right) \tag{7.10}$$

from which (7.9) follows by analytic continuation.

Another application of (7.5) is a proof[5] that

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{\sin \xi x}{x}\, dx = \operatorname{sgn}(\xi)\pi \quad \text{for } \xi \neq 0. \tag{7.11}$$

We begin with the more or less trivial observation that

$$\int_{-R}^{R} \frac{\sin \xi x}{x}\, dx = \operatorname{sgn}(\xi) \int_{-R}^{R} \frac{\sin |\xi| x}{x}\, dx = \operatorname{sgn}(\xi) \int_{-|\xi|R}^{|\xi|R} \frac{\sin x}{x}\, dx.$$

Thus, what we have to show is that

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{\sin x}{x}\, dx = \pi. \tag{$*$}$$

The first step in the proof $(*)$, is to show that if

$$g_R(\xi, y) \equiv \int_{-R}^{R} \frac{x \sin \xi x}{x^2 + y^2}\, dx \longrightarrow \pi e^{-y\xi} \quad \text{for } \xi > 0, \tag{$**$}$$

then $(*)$ holds. Indeed,

$$\left| \int_{-R}^{R} \frac{\sin \xi x}{x}\, dx - g_R(\xi, y) \right| \leq 2y^2 \left| \int_{0}^{\infty} \frac{|\sin \xi x|}{x(x^2 + y^2)}\, dx \right) \leq \xi \pi y,$$

and so $(**)$ implies $(*)$.

[5]The most commonly given proof is based on contour integration and Cauchy's theorem.

The next step is to show that for each $y > 0$ there exists a continuous $\xi \in (0,\infty) \longmapsto g(\xi,y) \in \mathbb{C}$ such that $g_R(\xi,y) \longrightarrow g(\xi,y)$ uniformly for ξ compact subsets of $(0,\infty)$. To this end, note that

$$\begin{aligned} g_R(\xi,y) &= 2\int_0^R \frac{x\sin\xi x}{x^2+y^2}\,dx = \frac{2}{\xi}\left(-\frac{R\cos\xi R}{R^2+y^2} + 2\int_0^R \frac{(y^2-x^2)\cos\xi x}{(x^2+y^2)^2}\,dx\right) \\ &\longrightarrow \frac{2}{\xi}\left(\frac{1}{y^2} + 2\int_0^\infty \frac{(y^2-x^2)\cos\xi x}{(x^2+y^2)^2}\,dx\right) \end{aligned}$$

uniformly for ξ in compacts subsets of $(0,\infty)$.

The final step is the identify $g(\xi,y)$ as $\pi e^{-y\xi}$. For this purpose, observe that

$$g_R(\xi,y) = -\imath\int_{-R}^R \frac{xe^{\imath\xi x}}{x^2+y^2}\,dx = \partial_\xi f_R(\xi,y)$$

where

$$f_R(\xi,y) = -\frac{\pi}{y}\int_{-R}^R p_y(x)e^{\imath\xi x}\,dx \longrightarrow -\frac{\pi}{y}e^{-y\xi}.$$

Hence

$$f_R(\eta) - f_R(\xi) = \int_\xi^\eta g_R(t,y)\,dt,$$

and therefore

$$\frac{\pi}{y}\left(e^{-y\xi} - e^{-y\eta}\right) = \int_\xi^\eta g(t,y)\,dt,$$

from which $g(\xi,y) = \pi e^{-y\xi}$ follows easily.

Exercise 7.3. Show that if $f \in L^1(\lambda_{[0,1)};\mathbb{C})$ and $f_t(x) = t^{-1}f\big(t^{-1}x\big)$, then $\hat{f}_t(\xi) = \hat{f}(t\xi)$.

Exercise 7.4. Show that if $f \in C^2(\mathbb{R};\mathbb{C}) \cap L^1(\lambda_{[0,1)};\mathbb{C})$ and both f' and f'' are in $L^1(\lambda_{[0,1)};\mathbb{C})$, then $\hat{f} \in L^1(\lambda_{[0,1)};\mathbb{C})$.

Exercise 7.5. Using $\cosh t = 1 + \frac{t^2}{2} + O(t^4)$ and $\sinh t = t - \frac{t^3}{6} + O(t^5)$, prove from (7.8) that $\sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{6}$.

Exercise 7.6. Show that

$$\sum_{n\in\mathbb{Z}} e^{-\frac{\pi n^2}{t}} = t^{\frac{1}{2}}\sum_{n\in\mathbb{Z}} e^{-\pi t n^2},$$

a formula that plays an important role in the theory of Theta functions.