Lecture 8: The L^1 Fourier Inversion Formula

Here we will see to what extent (6.1) can be justified, and the idea is to use the fact that we already know (cf. (7.4)) that it holds for g_t. With this in mind, we have, by Fubini's theorem,

$$2\pi g_t * f(x) = 2\pi \int g_t(y) f(x-y)\,dy = 2\pi \int g_t(y) f(x+y)\,dy$$
$$= \int (\widehat{g_t})^\wedge(y) f(x+y)\,dy = \int \widehat{g_t}(\xi)\left(\int e^{\imath\xi y} f(x+y)\,dy\right)d\xi = \int e^{-\frac{t\xi^2}{2}} e^{-\imath\xi x}\hat{f}(\xi)\,d\xi,$$

and so

$$g_t * f(x) = \frac{1}{2\pi}\int e^{-\frac{t\xi^2}{2}} e^{-\imath\xi x}\hat{f}(\xi)\,d\xi.$$

Let $f \in L^1(\lambda_{[0,1)};\mathbb{C})$. If f is continuous at x, then $\lim_{t\searrow 0} g_t * f(x) = f(x)$, and so

$$f(x) = \frac{1}{2\pi}\lim_{t\searrow 0}\int e^{-\frac{t\xi^2}{2}} e^{-\imath\xi x}\hat{f}(\xi)\,d\xi \quad \text{if } f \text{ is continuous at } x. \tag{8.1}$$

In particular

$$f(x) = \frac{1}{2\pi}\int e^{-\imath\xi x}\hat{f}(\xi)\,d\xi \quad \text{if } \hat{f} \in L^1(\lambda_{[0,1)};\mathbb{C}). \tag{8.2}$$

More generally, for any $f \in L^1(\lambda_{[0,1)};\mathbb{C})$, $g_t * f \longrightarrow f$ in $L^1(\lambda_{[0,1)};\mathbb{C})$, and so

$$\frac{1}{2\pi}\int e^{-\frac{t\xi^2}{2}} e^{-\imath\xi x}\hat{f}(\xi)\,d\xi \longrightarrow f(x) \text{ in } L^1(\lambda_{[0,1)};\mathbb{C}), \tag{8.3}$$

which can be thought of the Abel versions of (6.1). As an immediate consequence, we know that $f = \mathbf{0} \iff \hat{f} = \mathbf{0}$.

Exercise 8.1. Show that if $f \in C^2(\mathbb{R};\mathbb{C}) \cap L^1(\lambda_{\mathbb{R}};\mathbb{C})$ and both f' and f'' are in $L^1(\lambda_{\mathbb{R}};\mathbb{C})$, then $\hat{f} \in L^1(\lambda_{\mathbb{R}};\mathbb{C})$ and therefore $f = (2\pi)^{-1}\int e^{-\imath\xi x}\hat{f}(\xi)\,d\xi$.

Exercise 8.2. Using Exercise 8.1, give another proof that $\widehat{p_t}(\xi) = e^{-t|\xi|}$.

Exercise 8.3. There is nothing sacrosanct about g_t in producing formulas like (8.1) and (8.3). Indeed, give a $\rho \in C\big(\mathbb{R},[0,\infty)\big)$ for which $\int \rho(x)\,dx = 1$, set $\rho_t(x) = t^{-1}\rho(t^{-1}x)$. Then it is well known that, as $t \searrow 0$, $\rho_t * f(x) \longrightarrow f(x)$ if $f \in L^1(\lambda_{[0,1)};\mathbb{C})$ is continuous at x and that $\rho_t * f \longrightarrow f$ in $L^1(\lambda_{[0,1)};\mathbb{C})$ for any $f \in L^1(\lambda_{[0,1)};\mathbb{C})$. Now suppose that $\rho \in C^2\big(\mathbb{R},[0,\infty)\big)$ and that ρ' and ρ'' are in $L^1(\lambda_{[0,1)};\mathbb{C})$, and show that

$$\frac{1}{2\pi}\int \hat{\rho}(-t\xi)e^{-\imath\xi x}\hat{f}(\xi)\,d\xi \longrightarrow f(x) \quad \text{if } f \in L^1(\lambda_{[0,1)};\mathbb{C}) \text{ is continuous at } x$$

and

$$\frac{1}{2\pi}\int \hat{\rho}(-t\xi)e^{-\imath\xi x}\hat{f}(\xi)\,d\xi \longrightarrow f(x) \text{ in } L^1(\lambda_{[0,1)};\mathbb{C}) \text{ for any } f \in L^1(\lambda_{[0,1)};\mathbb{C}).$$

RES.18-015 Topics in Fourier Analysis
Spring 2024