Lecture 9: The Ornstein–Uhlenbeck Semigroup

Set $g_t(x) = (2\pi t)^{-\frac{1}{2}} e^{-\frac{x^2}{t}}$, and note that

$$\int g_s(x-\xi)g_t(\xi - y)\,d\xi = g_{s+t}(y-x) \text{ and } \partial_t g_t(x) = \tfrac{1}{2}\partial_x^2 g_t(x). \tag{9.1}$$

For $(t, x, y) \in (0, \infty) \times \mathbb{R} \times \mathbb{R}$, define

$$\begin{aligned} p(t,x,y) &= g_{1-e^{-t}}\big(y - e^{-\frac{t}{2}}x\big) \\ &= \big(2\pi(1-e^{-t})\big)^{-\frac{1}{2}} \exp\left(-\frac{(y-e^{-\frac{t}{2}}x)^2}{2(1-e^{-t})}\right) = e^{\frac{t}{2}} g_{e^t-1}\big(x - e^{\frac{t}{2}}y\big). \end{aligned} \tag{9.2}$$

From the first part of (9.1) and the third equality in (9.2), we see that

$$\begin{aligned} \int p(s,x,\xi)p(t,\xi,y)\,d\xi &= e^{\frac{t}{2}} \int g_{1-e^{-s}}\big(\xi - e^{-\frac{s}{2}}y\big) g_{e^t-1}\big(\xi - e^{\frac{t}{2}}x\big)\,d\xi \\ &= e^{\frac{t}{2}} g_{e^t - e^{-s}}\big(e^{\frac{t}{2}}y - e^{-\frac{s}{2}}x\big) = p(s+t,x,y). \end{aligned}$$

Hence $p(t,x,y)$ satisfies the *Chapman–Kolmogorov equation*

$$p(s+t,x,y) = \int p(s,x,\xi)p(t,\xi,y)\,d\xi. \tag{9.3}$$

In addition, using the second part of (9.1), one sees that

$$\partial_t p(t,x,y) = \mathcal{L}_x p(t,x,y) \text{ where } \mathcal{L}_x = \tfrac{1}{2}\big(\partial_x^2 - x\partial_x\big). \tag{9.4}$$

Next define

$$P_t\varphi(x) = \int \varphi(y)p(t,x,y)\,dy \tag{9.5}$$

for $\varphi \in C(\mathbb{R};\mathbb{C})$ with at most exponential growth at ∞, and use (9.3) to see that $\{P_t : t > 0\}$ is a semigroup (i.e., $P_{s+t} = P_s \circ P_t$). In addition, use (9.4) to show that

$$\partial_t P_t\varphi = \mathcal{L}P_t\varphi. \tag{9.6}$$

After making the change of variable $y \to e^{\frac{t}{2}}y$, one sees that another expression for $P_t\varphi$ is

$$P_t\varphi(x) = \int \varphi\big(e^{-\frac{t}{2}}y\big) g_{e^t-1}(y-x)\,dy = \int g_1(y)\varphi\big((1-e^{-t})^{\frac{1}{2}}y + x\big)\,dy, \tag{9.7}$$

from which it is easy to see that $P_t\varphi \longrightarrow \varphi$ uniformly on compact subsets as $t \searrow 0$. Further, if $p \in [1,\infty)$, then, by Minkowski's inequality,

$$\|P_t f\|_{L^p(\lambda_{\mathbb{R}};\mathbb{C})} \le \int g_1(y)\left(\int \big|f\big((1-e^{-t})^{\frac{1}{2}}y + x\big)\big|^p\,dy\right)^{\frac{1}{p}} = \|f\|_{L^p(\lambda_{\mathbb{R}};\mathbb{C})},$$

and, as $t \searrow 0$,

$$\|P_t f - f\|_{L^p(\lambda_{\mathbb{R}};\mathbb{C})} \le \int g_1(y)\left(\int \big|f\big((1-e^{-t})^{\frac{1}{2}}y + x\big) - f(x)\big|^p\,dy\right)^{\frac{1}{p}}\,dy \longrightarrow 0$$

since

$$2\|f\|_{L^p(\lambda_{\mathbb{R}};\mathbb{C})} \ge \left(\int \big|f\big((1-e^{-t})^{\frac{1}{2}}y + x\big) - f(x)\big|^p\,dy\right)^{\frac{1}{p}} \longrightarrow 0.$$

Therefore we know that

$$\|P_t f\|_{L^p(\lambda_{\mathbb{R}};\mathbb{C})} \le \|f\|_{L^p(\lambda_{\mathbb{R}};\mathbb{C})} \text{ and } \lim_{t\searrow 0} \|P_t f - f\|_{L^p(\lambda_{\mathbb{R}};\mathbb{C})} = 0. \tag{9.8}$$

In particular, we have now shown that $\{P_t : t > 0\}$ is a continuous contraction semigroup, known as the *Ornstein–Uhlenbeck* semigroup, on $L^p(\lambda_{\mathbb{R}};\mathbb{C})$ for each $p \in [1,\infty)$.

Although $\{P_t : t > 0\}$ is a continuous semigroup on the Lebesgue spaces $L^p(\lambda_{\mathbb{R}};\mathbb{C})$, these are not the Lebesgue spaces on which it acts most naturally. Instead, one should consider its action on the spaces $L^p(\gamma;\mathbb{C})$, where γ is the standard Gauss measure $\gamma(dx) = (2\pi)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} \lambda_{\mathbb{R}}(dx)$. The reason why is that

$$e^{-\frac{x^2}{2}} p(t,x,y) = p(t,y,x) e^{-\frac{y^2}{2}},$$

which means that

$$\big(\varphi, P_t\psi\big)_{L^2(\gamma;\mathbb{C})} = \big(P_t\varphi, \psi\big)_{L^2(\gamma;\mathbb{C})}. \tag{9.9}$$

Hence, since $P_t\mathbf{1} = \mathbf{1}$,

$$\int P_t\varphi\, d\gamma = \big(\varphi, P_t\mathbf{1}\big)_{L^2(\gamma;\mathbb{C})} = \int \varphi\, d\gamma.$$

At the same time, by Jensen's inequality, $|P_t\varphi|^p \le P_t|\varphi|^P$, and so,

$$\int |P_t\varphi|^p\, d\gamma \le \int P_t|\varphi|^p\, d\gamma = \int |\varphi|^p\, d\gamma.$$

Thus,

$$\|P_t\varphi\|_{L^p(\gamma;\mathbb{C})} \le \|\varphi\|_{L^p(\gamma;\mathbb{C})} \text{ for all } p \in [1,\infty). \tag{9.10}$$

In addition, if $\varphi \in C_{\mathrm{b}}(\mathbb{R};\mathbb{C})$, then $\|P_t\varphi\|_{\mathrm{u}} \le \|\varphi\|_{\mathrm{u}}$ and $P_t\varphi \longrightarrow \varphi$ pointwise as $t \searrow 0$, and therefore, for each $p \in [1,\infty)$, $\|P_t\varphi - \varphi\|_{L^p(\gamma;\mathbb{C})} \longrightarrow 0$ as $t \searrow 0$. Finally, if $\varphi \in L^p(\mathbb{R};\mathbb{C})$, then there exists a sequence $\{\varphi_n : n \ge 1\} \subseteq C_{\mathrm{b}}(\mathbb{R};\mathbb{C})$ such that $\lim_{n\to\infty} \|f_n - f\|_{L^p(\gamma;\mathbb{C})} = 0$, and

$$\begin{aligned}\|P_t\varphi - \varphi\|_{L^p(\gamma;\mathbb{C})} &\le \|P_t(\varphi - \varphi_n)\|_{L^p(\gamma;\mathbb{C})} + \|P_t\varphi_n - \varphi_n\|_{L^p(\gamma;\mathbb{C})} + \|\varphi_n - \varphi\|_{L^p(\gamma;\mathbb{C})} \\ &\le 2\|\varphi_n - \varphi\|_{L^p(\gamma;\mathbb{C})} + \|P_t\varphi_n - \varphi_n\|_{L^p(\gamma;\mathbb{C})}.\end{aligned}$$

Thus, after first letting $t \searrow 0$ and then $n \to \infty$, we see that

$$\lim_{t\searrow 0} \|P_t\varphi - \varphi\|_{L^p(\gamma;\mathbb{C})} = 0 \text{ for all } p \in [1,\infty). \tag{9.11}$$

Summarizing, $\{P_t : t > 0\}$ is a continuous contraction semigroup on $L^p(\gamma;\mathbb{C})$ for each $p \in [1,\infty)$ and P_t is self-adjoint on $L^2(\gamma;\mathbb{C})$.

Exercise 9.1. Show that

$$\big\|\varphi - (\varphi,\mathbf{1})_{L^2(\gamma;\mathbb{C})}\big\|^2_{L^2(\gamma;\mathbb{C})} \le \|\varphi'\|^2_{L^2(\gamma;\mathbb{C})} \text{ for } \varphi \in C^1_{\mathrm{b}}(\mathbb{R};\mathbb{C}) \tag{9.12}$$

and that

$$\big\|P_t\varphi - (\varphi,\mathbf{1})_{L^2(\gamma;\mathbb{C})}\big\|^2_{L^2(\gamma;\mathbb{C})} \le e^{-t}\big\|\varphi - (\varphi,\mathbf{1})_{L^2(\gamma;\mathbb{C})}\big\|^2_{L^2(\gamma;\mathbb{C})} \tag{9.13}$$

for $\varphi \in L^2(\gamma;\mathbb{C})$. The inequality in (9.12) is the *Poincaré inequality* for γ.

Hint: Note that if suffices to handle $\varphi \in C_{\mathrm{b}}^2(\mathbb{R};\mathbb{C})$ for which $(\varphi, \mathbf{1})_{L^2(\gamma;\mathbb{C})} = 0$. Next, given such a φ, show that

$$(P_t\varphi)' = e^{-\frac{t}{2}} P_t\varphi' \text{ and } -\partial_t \|P_t\varphi\|^2_{L^2(\gamma;\mathbb{C})} = \|(P_t\varphi)'\|^2_{L^2(\gamma;\mathbb{C})}.$$

Use these to show that

$$\partial_t \|P_t\varphi\|^2_{L^2(\gamma;\mathbb{C})} \le e^{-t} \big\|(P_t\varphi)'\big\|^2_{L^2(\gamma;\mathbb{C})} \le e^{-t} \|\varphi'\|^2_{L^2(\gamma;\mathbb{C})}.$$