Lecture 10: Hermite Polynomials

Define $H_n(x) = (-1)^n e^{\frac{s^2}{2}} \partial_x^n e^{-\frac{x^2}{2}}$. Then H_n is an nth order monic polynomial known as the nth *Hermite polynomial.* Define the operator $A_+ = x\mathbf{1} - \partial_x$, and note that $A_+ H_n = H_{n+1}$, for which reason it is called the *raising operator*. Using this, check that $H_n(-x) = (-1)^n H_n(x)$.

Next note that if $\varphi, \psi \in C^1(\mathbb{R};\mathbb{C})$ which together with their derivatives have at most exponential growth, then

$$(A_+\varphi, \psi)_{L^2(\gamma;\mathbb{C})} = (\varphi, \partial\psi)_{L^2(\gamma;\mathbb{C})}. \tag{10.1}$$

Hence, if $0 \le m \le n$, then

$$\big(H_n, H_m\big)_{L^2(\gamma;\mathbb{C})} = \big(H_0, \partial^n H_m\big)_{L^2(\gamma;\mathbb{C})} = \begin{cases} m! & \text{if } n = m \\ 0 & \text{if } n > m. \end{cases}$$

Next, observe that if $n \ge 1$, then $\partial H_n \in \operatorname{span}\{H_m : 0 \le m < n\}$, and so

$$\begin{aligned} \partial H_n &= \sum_{m=0}^{n-1} \frac{\big(\partial H_n, H_m\big)_{L^2(\gamma;\mathbb{C})} H_m}{m!} \\ &= \sum_{m=0}^{n-1} \frac{\big(H_n, H_{m+1}\big)_{L^2(\gamma;\mathbb{C})} H_m}{m!} = \frac{\big(H_n, H_n\big)_{L^2(\gamma;\mathbb{C})} H_{n-1}}{(n-1)!}. \end{aligned}$$

Hence $\partial H_n = nH_{n-1}$, and for this reason $A_- \equiv \partial$ is called the *lowering operator*.

Theorem 10.1. *$\|H_m\|_{L^2(\gamma;\mathbb{C})} = (m!)^{\frac{1}{2}}$ and $\{H_m : m \ge 0\}$ is an orthogonal basis in $L^2(\gamma;\mathbb{C})$. Equivalently, if $\tilde{H}_m = \frac{H_m}{\sqrt{m!}}$, then $\{\tilde{H}_m : m \ge 0\}$ is an orthonormal basis in $L^2(\gamma;\mathbb{C})$*

Proof. All that we need to show is that if $\varphi \in L^2(\gamma;\mathbb{C})$ and $(\varphi, H_m)_{L^2(\gamma;\mathbb{C})} = 0$ for all $m \ge 0$, then $\varphi = \mathbf{0}$. To this end, use Taylor's theorem to see that, for all $\zeta \in \mathbb{C}$,

$$e^{\zeta x - \frac{\zeta^2}{2}} = \sum_{m=0}^{\infty} \frac{\zeta^m}{m!} H_m(x), \tag{10.2}$$

where the series converges uniformly on compact subsets of $\mathbb{C} \times \mathbb{R}$, and, by the preceding calculation, in $L^2(\gamma;\mathbb{C})$ uniformly for ζ in compact subsets of $\mathbb{C}$. Now suppose that $\varphi \in L^2(\gamma;\mathbb{C})$, and set $\psi(x) = e^{-\frac{s^2}{2}}\varphi(x)$. Then

$$\|\psi\|_{L^1(\lambda_{\mathbb{R}},\mathbb{C})} = \int_{\mathbb{R}} e^{-\frac{x^2}{4}} \big(e^{-\frac{x^2}{4}} |\varphi(x)|\big)\, ds \le (2\pi)^{\frac{1}{2}} \|\varphi\|_{L^2(\gamma;\mathbb{C})},$$

and

$$e^{\frac{\xi^2}{2}} \hat{\psi}(\xi) = (2\pi)^{\frac{1}{2}} \int_{\mathbb{R}} e^{\imath\xi x - \frac{(\imath\xi)^2}{2}} \varphi(x)\gamma(dx) = (2\pi)^{\frac{1}{2}} \sum_{m=0}^{\infty} \frac{(\imath\xi)^m (\varphi, H_m)_{L^2(\gamma;\mathbb{C})}}{m!}.$$

Hence $\hat{\psi}$ and therefore φ vanish if $(\varphi, H_m)_{L^2(\gamma;\mathbb{C})} = 0$ for all $m \ge 0$. □

Observe that $\mathcal{L} = -\frac{A_+ A_-}{2}$, and therefore, by (10.1)

$$\big(\mathcal{L}\varphi, \psi\big)_{L^2(\gamma;\mathbb{C})} = -\big(\varphi', \psi'\big)_{L^2(\gamma;\mathbb{C})} = \big(\varphi, \mathcal{L}\psi\big)_{L^2(\gamma;\mathbb{C})}$$

for $\varphi, \psi \in C^2(\mathbb{R};\mathbb{C})$ which together with their derivatives have at most exponential growth. Thus, by (9.6) and (9.9),

$$\begin{aligned}\big(\mathcal{L}P_t\varphi,\psi\big)_{L^2(\gamma;\mathbb{C})} &= \partial_t\big(P_t\varphi,\psi\big)_{L^2(\gamma;\mathbb{C})} = \partial_t\big(\varphi,P_t\psi\big)_{L^2(\gamma;\mathbb{C})}\\ &= \big(\varphi,\mathcal{L}P_t\psi\big)_{L^2(\gamma;\mathbb{C})} = \big(P_t\mathcal{L}\varphi,\psi\big)_{L^2(\gamma;\mathbb{C})},\end{aligned}$$

and therefore $\mathcal{L}P_t = P_t\mathcal{L}$. In particular, because $-2\mathcal{L}H_n = nA_+H_{n-1} = nH_n$,

$$\partial_t P_t H_n = \mathcal{L}P_t H_n = P_t\mathcal{L}H_n = -\frac{n}{2}P_t H_n,$$

and so, because $\lim_{t\searrow 0} P_t H_n = H_n$,

$$P_t H_n = e^{-\frac{nt}{2}} H_n. \tag{10.3}$$

Exercise 10.2. Using (10.3), give another proof of (9.13), and, using $A_+H_m = H_{m+1}$, give another proof of (9.12).

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